SUPERSYMMETRIES OF MODULES OF DIFFERENTIAL OPERATORS

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Let $\mathfrak{F}_\lambda$ be the space of tensor densities of degree λ on the supercircle $S^{1|1}$. We consider the space $\mathfrak{D}^k_{\lambda,\mu}$ of k-th order linear differential operators from $\mathfrak{F}_\lambda$ to $\mathfrak{F}_\mu$ as a module over the Lie superalgebra $\mathcal{K}(1)$ of contact vector fields on $S^{1|1}$ and we determine the superalgebra $\mathfrak{C}^k_{\lambda,\mu}$ of supersymmetries, *i.e.*, of linear maps on $\mathfrak{D}^k_{\lambda,\mu}$ commuting with the $\mathcal{K}(1)$-action.

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1. INTRODUCTION

By invariant operators, we will mean operators acting in the spaces of tensor fields (or sections of other types of vector bundles) which have the same form in any (curvilinear) coordinate system on a fixed manifold. The importance of this topic initiated by Veblen [15] became manifest after discovery of the relativity theory. Indeed, according to equivalence principle, the motion of a body in the gravitational field is equivalent to the motion in the absence of the field but in a non-inertial coordinate system, with curvilinear coordinates if the gravitational field is non-homogeneous. Similarly, invariant operators should always appear whenever there exists either a relation between tensor fields (or sections of vector bundles depending on higher jets of the diffeomorphism group), or a condition on a tensor field, or an algebraic structure, etc., that do not vary under the changes of coordinates.

Let M be a n-dimensional manifold and $\mathrm{Vect}(M)$ be the Lie algebra of vector fields on M. For every $\lambda \in \mathbb{C}$, we consider the space $\mathcal{F}_\lambda(M)$ of tensor densities of degree λ on M:

$$\varphi = f(x^1, \cdots, x^n)|\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n|^\lambda,$$

that is, of sections of the line bundle $\Delta_\lambda(M) = |\Lambda^n(T^*M)|^{\otimes\lambda}$ over M. This provides a one-parameter family of representations of $\mathrm{Vect}(M)$. Any differential operator on M can be viewed as a linear mapping from $\mathcal{F}_\lambda(M)$ to $\mathcal{F}_\mu(M)$ ($\lambda, \mu \in \mathbb{C}$). Thus, the space of differential operators is a $\mathrm{Vect}(M)$-module, denoted $\mathcal{D}_{\lambda,\mu}(M) := \mathrm{Hom}_{\mathrm{diff}}\big(\mathcal{F}_\lambda(M), \mathcal{F}_\mu(M)\big)$, *i.e.*, it is in turn a representation of the Lie algebra $\mathrm{Vect}(M)$. We thus have a two-parameter family of representations of $\mathrm{Vect}(M)$. These families of representations were studied in a series of papers, starting from the first work on the classification of these representations, by C. Duval and V. Ovsienko [3] in which this problem on a smooth manifold was posed, for $\lambda = \mu$, and solved for modules of second-order operators, then it was solved on $\mathbb{R}$ in [7] and in general for $\lambda \neq \mu$ in [8]. In the multidimensional case, this classification problem was the subject of the papers [11, 12]. In the framework of supergeometry, namely over the $(1,1)$-dimensional real superspace, the authors of [1] gave a complete description of these modules and a partial one for the dimension $(1, n)$. Obviously, the classification $\mathrm{Vect}(M)$-modules $\mathcal{D}_{\lambda,\mu}(M)$ is obtained through the study of the existence of isomorphisms between distinct modules $\mathcal{D}^k_{\lambda,\mu}(M)$, *i.e.*, of linear bijective maps invariant under the $\mathrm{Vect}(M)$-action on these spaces.

Usually, one considers differential operators acting on various spaces of tensor fields on a smooth manifold, the main difference of our work from the classic literature is that we consider linear operators acting on differential operators (instead of tensor fields). More precisely, we are interested in the classification of linear maps

$$T : \mathcal{D}^k_{\lambda,\mu}(M) \to \mathcal{D}^k_{\lambda,\mu}(M) \tag{1.1}$$

commuting with the $\mathrm{Vect}(M)$-action. The set of such operators, denoted by $\mathcal{I}^k_{\lambda,\mu}(M)$, is an associative algebra called the *algebra of symmetries of the modules* $\mathcal{D}^k_{\lambda,\mu}(M)$. If $M = \mathbb{R}$ (or S^1), a well-known example of a map 1.1 is the conjugation of differential operators. This map associates to an operator A the adjoint operator $A^\star$. If $A \in \mathcal{D}^k_{\lambda,\mu}(\mathbb{R})$, then $A^\star \in \mathcal{D}^k_{1-\mu,1-\lambda}(\mathbb{R})$, so this map defines a symmetry if and only if $\lambda + \mu = 1$. In [5], the algebra of symmetries $\mathcal{I}^k_{\lambda,\mu}(S^1)$ were investigated, a complete description and classification for all integer k were supplied in this paper.

In [13], we were interested in the study of the analogue super structures. Namely, we considered the superspace $\mathfrak{D}_{\lambda,\mu}$ of differential linear operators $A : \mathfrak{F}_\lambda \to \mathfrak{F}_\mu$, where $\mathfrak{F}_\lambda$ and $\mathfrak{F}_\mu$ are the spaces of tensor densities on the supercircle $S^{1|1}$ of degree λ and μ respectively. Naturally, the Lie superalgebra $\mathrm{Vect}_{\mathbb{C}}(S^{1|1})$ acts on $\mathfrak{D}_{\lambda,\mu}$, but we restricted ourselves in [13] to the orthosymlectic superalgebra $\mathfrak{osp}(1|2)$ which can be realized as a subalgebra of $\mathrm{Vect}_{\mathbb{C}}(S^{1|1})$. We also studied, which we called the algebra of *Orthosymplectic supersymmetries* of the module $\mathfrak{D}^k_{\lambda,\mu}$, denoted by $\mathfrak{I}^k_{\lambda,\mu}$, *i.e.*, the algebra of endomorphisms of $\mathfrak{D}^k_{\lambda,\mu}$ that commute with the $\mathfrak{osp}(1|2)$-action. Of course, in [13], only the results obtained for particular values of the weights called *resonant* *i.e.*, the values

$$\mu - \lambda = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots \tag{1.2}$$

are interesting. We proved that only situations where the algebra $\mathfrak{I}^k_{\lambda,\mu}$ has a dimension greater than $2k+1$ are resonant situations. This result is expected since, firstly, for the *non resonant* values, there exists a unique (up to normalization) $\mathfrak{osp}(1|2)$-equivariant symbol map between the module $\mathfrak{D}_{\lambda,\mu}$ and the associated graded module $\mathrm{gr}\mathfrak{D}_{\lambda,\mu}$ called the *space of symbols* (see [6]), which is a direct sum of density modules. Secondly, the only case where a non trivial $\mathfrak{osp}(1|2)$-equivariant map between spaces of densities can exist is the case of a map between $\mathfrak{F}_{\frac{1-k}{4}}$ and $\mathfrak{F}_{\frac{1+k}{4}}$, where k is an odd natural number (see [6]).

Our motivation in this paper is the study of the most interesting setting, the algebra $\mathfrak{C}^k_{\lambda,\mu}$ of *contact supersymmetries* or quite simply *supersymmetries.* We consider the space $\mathfrak{D}_{\lambda,\mu}$ as a module over the superalgebra $\mathcal{K}(1)$ of contact vector fields on $S^{1|1}$. In this context, we compute the space $\mathfrak{C}^k_{\lambda,\mu}$ of linear maps on $\mathfrak{D}^k_{\lambda,\mu}$ commuting with the $\mathcal{K}(1)$-action, we establish several results similar to the S^1-case. The slightly more interesting result, unlike the case of orthosymlectic supersymmetries, is the stability of the dimension of $\mathfrak{C}^k_{\lambda,\mu}$ for $k \geq 3$, this result is due to the fact that any contact supersymmetry is completely determined by its restriction to the subspace of second-order operators. The particular values $k = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, k = 3$ are investigated, for all (λ, μ), a complete description of the algebra $\mathfrak{C}^k_{\lambda,\mu}$ for these values of k is provided; the method to construct generators for these algebras is inspired from [5], namely the composition of $\mathcal{K}(1)$-invariant bilinear operators on tensor densities and $\mathcal{K}(1)$-invariant linear projections from differential operators to tensor densities. Note finally that, in the S^1-case (see [5], section 6.4), a "non differential" (or non local) symmetry can exist only for the generic case $(\lambda, \mu) = (0, 1)$, thus we focus our study in this work to the large class of differential supersymmetries.

2. BASIC DEFINITIONS AND TOOLS

2.1. THE LIE SUPERALGEBRA OF VECTOR FIELDS ON $S^{1|1}$

We define the supercircle $S^{1|1}$ in terms of its superalgebra of functions $C^\infty_{\mathbb{C}}(S^{1|1})$ consisting of elements of the form:

$$F(x, \theta) = f_0(x) + f_1(x)\theta, \tag{2.3}$$

where x is an arbitrary parameter on S^1 (the even variable) and θ is a formal Grassmann coordinate (the odd variable) such that $\theta^2 = 0$. Even elements in $C^\infty_{\mathbb{C}}(S^{1|1})$ are $F(x,\theta) = f_0(x)$, odd elements $F(x,\theta) = f_1(x)\theta$. A vector field on $S^{1|1}$ is a superderivation of $C^\infty_{\mathbb{C}}(S^{1|1})$, we can express such vector field in coordinates in term of partial derivatives:

$$X = F_0\partial_x + F_1\partial_\theta \ ; \ F_i \in C^\infty_{\mathbb{C}}(S^{1|1}) \tag{2.4}$$

where ∂_θ and ∂_x stand for $\dfrac{\partial}{\partial\theta}$ and $\dfrac{\partial}{\partial x}$ respectively. The space $\mathrm{Vect}_{\mathbb{C}}(S^{1|1})$ of vector fields on $S^{1|1}$ is a Lie superalgebra.

Let denote by D and $\overline{D}$ the vector fields (see [14] for the interpretation of these fields):

$$D = \partial_\theta + \theta\partial_x, \ \overline{D} = \partial_\theta - \theta\partial_x. \tag{2.5}$$

The subbundle in $S^{1|1}$ generated by $\overline{D}$ defines a codimension 1 non-integrable distribution on $S^{1|1}$ called the *standard contact structure* on $S^{1|1}$ which is equivalently the kernel of differential 1-form

$$\alpha = \mathrm{d}x + \theta d\theta. \tag{2.6}$$

A vector field X is said to be contact if it preserves the contact distribution, *i.e.*,

$$[X, \overline{D}] = F_X\overline{D}, \tag{2.7}$$

where $F_X \in C^\infty_{\mathbb{C}}(S^{1|1})$ is a function depending on X. We denote by $\mathcal{K}(1)$ the Lie superalgebra of *contact vector fields* on $S^{1|1}$. An element in $\mathcal{K}(1)$ can be expressed for any $f \in C^\infty_{\mathbb{C}}(S^{1|1})$ as [6]:

$$X_f = -f\overline{D}^2 + \frac{1}{2}D(f)\overline{D}. \tag{2.8}$$

The contact bracket is defined by

$$[X_f, X_g] = X_{\{f,g\}}, \tag{2.9}$$

the space $C^\infty_{\mathbb{C}}(S^{1|1})$ is thus equipped with a Lie superalgebra (isomorphic to $\mathcal{K}(1)$) thanks to the bracket:

$$\{f, g\} = fg' - f'g + \frac{1}{2}(-1)^{|f|(|g|+1)}D(f)D(g), \tag{2.10}$$

where $|\ |$ stands for the parity function. The action of $\mathcal{K}(1)$ on $C^\infty_{\mathbb{C}}(S^{1|1})$ is defined by:

$$\mathfrak{L}_{X_f}(g) = fg' + \frac{1}{2}(-1)^{|f|+1}\overline{D}(f)\cdot\overline{D}(g). \tag{2.11}$$

2.2. THE SPACE OF WEIGHTED DENSITIES ON $S^{1|1}$

In the super setting, by replacing $\mathrm{d}x$ by the 1-form α, we get an analogous definition for weighted densities, *i.e.*, we define the space of λ-densities as

$$\mathfrak{F}_\lambda = \left\{ F\alpha^\lambda \;\mid\; F \in C^\infty_{\mathbb{C}}(S^{1|1}) \right\}. \tag{2.12}$$

As a vector space, $\mathfrak{F}_\lambda$ is isomorphic to $C^\infty_{\mathbb{C}}(S^{1|1})$.

Let X_F a contact vector field, we define a one-parameter family of first order differential operators on $C^\infty_{\mathbb{C}}(S^{1|1})$

$$\mathfrak{L}^\lambda_{X_F} = X_F + \lambda F', \lambda \in \mathbb{C}. \tag{2.13}$$

One easily checks that the map $X_F \mapsto \mathfrak{L}^\lambda_{X_F}$ is a homomorphism of Lie superalgebra, that is, $[\mathfrak{L}^\lambda_{X_F}, \mathfrak{L}^\lambda_{X_G}] = \mathfrak{L}^\lambda_{[X_F, X_G]}$, for every λ. Thus $\mathfrak{F}_\lambda$ becomes a $\mathcal{K}(1)$-module on $C^\infty_{\mathbb{C}}(S^{1|1})$. Evidently, the Lie derivative of the density $G\alpha^\lambda$ along the vector field X_F in $\mathcal{K}(1)$ is given by:

$$\mathfrak{L}_{X_F}(G\alpha^\lambda) = \mathfrak{L}^\lambda_{X_F}(G)\alpha^\lambda = \mathfrak{L}_{X_F}(G) + \lambda F'G. \tag{2.14}$$

One can easily see that:

(1) The adjoint $\mathcal{K}(1)$-module, is isomorphic to $\mathfrak{F}_{-1}$.
(2) As a $\mathrm{Vect}(S^1)$-module, $\mathfrak{F}_\lambda \simeq \mathcal{F}_\lambda \oplus \Pi(\mathcal{F}_{\lambda+\frac{1}{2}})$.

2.3. DIFFERENTIAL OPERATORS ON WEIGHTED DENSITIES

We consider a family of $\mathcal{K}(1)$-actions on the superspace of differential operators
$\mathfrak{D}_{\lambda,\mu} := \mathrm{Hom}_{\mathrm{diff}}(\mathfrak{F}_\lambda, \mathfrak{F}_\mu)$:

$$\mathfrak{L}^{\lambda,\mu}_{X_F}(A) = \mathfrak{L}^\mu_{X_F} \circ A - (-1)^{|A||F|} A \circ \mathfrak{L}^\lambda_{X_F}. \tag{2.15}$$

Since $\overline{D}^2 = -\partial_x$, every differential operator $A \in \mathfrak{D}_{\lambda,\mu}$ can be expressed in the form (see [6])

$$A(f\alpha^\lambda) = \sum_{i=0}^{\ell} a_i(x,\theta)\overline{D}^i(f)\alpha^\mu,\ f \in C^\infty_{\mathbb{C}}(S^{1|1}), \tag{2.16}$$

where the coefficients $a_i(x,\theta)$ are arbitrary functions and $\ell \in \mathbb{N}$. Moreover, if $A \in \mathfrak{D}^k_{\lambda,\mu}$, then $\ell = 2k$. For short, we will write the operator A as:

$$A = \sum_{i=0}^{2k} a_i \overline{D}^i. \tag{2.17}$$

Thus, we have a $\mathcal{K}(1)$-invariant *finer filtration*:

$$\mathfrak{D}^0_{\lambda,\mu} \subset \mathfrak{D}^{\frac{1}{2}}_{\lambda,\mu} \subset \mathfrak{D}^1_{\lambda,\mu} \subset \mathfrak{D}^{\frac{3}{2}}_{\lambda,\mu} \subset \cdots \mathfrak{D}^{k-\frac{1}{2}}_{\lambda,\mu} \subset \mathfrak{D}^k_{\lambda,\mu} \subset \cdots \tag{2.18}$$

2.4. SPACE OF SYMBOLS OF DIFFERENTIAL OPERATORS

For a differential operator $A \in \mathfrak{D}^\ell_{\lambda,\mu}$ given by (2.16), one easily checks that the expression

$$\sigma_{\rm pr}(A) := a_\ell(x,\theta)\,\alpha^{\mu-\lambda-\ell} \tag{2.19}$$

is a well-defined $(\mu-\lambda-\ell)$-density. The $\mathcal{K}(1)$-invariant projection

$$\sigma_{\rm pr} : \mathfrak{D}^\ell_{\lambda,\mu} \to \mathfrak{F}_{\mu-\lambda-\ell} \tag{2.20}$$

is called the *principal symbol map*. Therefore, the quotient module $\mathfrak{D}^\ell_{\lambda,\mu}/\mathfrak{D}^{\ell-\frac{1}{2}}_{\lambda,\mu}$ depends only on the shift $\mu-\lambda$ of the weights and is isomorphic to the module of weighted densities $\mathfrak{F}_{\mu-\lambda-\ell}$. Thus, the graded $\mathcal{K}(1)$-module $\mathrm{gr}\mathfrak{D}_{\lambda,\mu}$ associated with the filtration (2.18) is isomorphic to the *space of symbols of differential operators*

$$\mathcal{S}_{\mu-\lambda} = \bigoplus_{i=0}^{\infty} \mathfrak{F}_{\mu-\lambda-\frac{i}{2}}. \tag{2.21}$$

The space of symbols of order $\leq k$ is

$$\mathcal{S}^k_{\mu-\lambda} = \bigoplus_{i=0}^{2k} \mathfrak{F}_{\mu-\lambda-\frac{i}{2}}. \tag{2.22}$$

2.5. THE IDENTIFICATION

The $\mathcal{K}(1)$-modules $\mathfrak{D}_{\lambda,\mu}$ and $\mathcal{S}_{\mu-\lambda}$ are not isomorphic, there exists, for some values of (λ,μ), an $\mathfrak{osp}(1|2)$-isomorphism between the two modules (see [6]). For our computations, we will need to identify the spaces $\mathfrak{D}_{\lambda,\mu}$ and $\mathcal{S}_{\mu-\lambda}$ via the map $\sigma_{\rm tot}$ called the *total symbol map*: Let $A = \sum_{\ell=0}^{2k} a_\ell \overline{D}^\ell \in \mathfrak{D}^k_{\lambda,\mu}$, we set

$$\sigma_{\rm tot}(A) = \alpha^{\mu-\lambda} \sum_{\ell=0}^{2k} a_\ell\ \alpha^{-\frac{\ell}{2}}. \tag{2.23}$$

The map $\sigma_{\rm tot}$ is just an isomorphism of vector spaces but not an isomorphism of $\mathcal{K}(1)$-modules. This leads us to define the $\mathcal{K}(1)$-action $\mathcal{L}^{\lambda,\mu}$ on the space $\mathcal{S}_{\mu-\lambda}$ by the rule

$$\sigma_{\rm tot}\Big(\mathfrak{L}^{\lambda,\mu}_{X_F}(A)\Big) = \alpha^{\mu-\lambda}\mathcal{L}^{\lambda,\mu}_{X_F}\Big(\sigma_{\rm tot}(A)\Big),\ X_F \in \mathcal{K}(1). \tag{2.24}$$

3. THE ALGEBRA $\mathfrak{C}^k_{\lambda,\mu}$ OF SUPERSYMMETRIES

In this subsection, we introduce the main object of our study. As mentioned in the introduction, we restrict ourselves to differential mapping $T : \mathfrak{D}^k_{\lambda,\mu} \to \mathfrak{D}^k_{\lambda,\mu}$ where $k \in \frac{1}{2}\mathbb{N}$. As proved in [13], for all ℓ in $\{0, 1, \cdots, 2k\}$, there exist an integer m and some functions $T^\ell_{ij} \in C^\infty_{\mathbb{C}}(S^{1|1})$ such that

$$T(a\overline{D}^\ell) = \sum_{j=0}^{\ell}\sum_{i=0}^{m} T^\ell_{i,j} D^i(a)\overline{D}^j. \tag{3.25}$$

Definition 3.1. If $k \in \frac{1}{2}\mathbb{N}$, the supercommutant $\mathfrak{C}^k_{\lambda,\mu}$ is the space of linear differential mapping:

$$T : \mathfrak{D}^k_{\lambda,\mu} \to \mathfrak{D}^k_{\lambda,\mu} \tag{3.26}$$

commuting with the $\mathcal{K}(1)$-action:

$$[\mathfrak{L}^{\lambda,\mu}_{X_F}, T] := \mathfrak{L}^{\lambda,\mu}_{X_F} \circ T - (-1)^{|T||F|} T \circ \mathfrak{L}^{\lambda,\mu}_{X_F} = 0, \quad X_F \in \mathcal{K}(1). \tag{3.27}$$

The space $\mathfrak{C}^k_{\lambda,\mu}$ is an associative superalgebra called the superalgebra of *s*upersymmetries of the modules $\mathfrak{D}^k_{\lambda,\mu}$.

4. CONSTRUCTION OF SUPERSYMMETRIES

Let's start first by mentioning that, $\forall k \in \frac{1}{2}\mathbb{N}$ and $(\lambda, \mu) \in \mathbb{C}^2$, the algebra $\mathfrak{C}^k_{\lambda,\mu}$ contains the identity map Id. We will say that the algebra $\mathfrak{C}^k_{\lambda,\mu}$ is *trivial* if it is generated by the identity map.

4.1. THE CONJUGATION

The best known invariant map between the spaces of differential operators is the *conjugation*: Let us denote by $\mathcal{B}$ the Berezin integral $\mathcal{B} : \mathfrak{F}_{\frac{1}{2}} \to \mathbb{C}$ given, for any $f = f_0 + \theta f_1$, by the formula [2]

$$\mathcal{B}(f\alpha^{\frac{1}{2}}) = \int_{S^1} f_1(x)\mathrm{d}x. \tag{4.28}$$

It is well known that the Berezin integral $\mathcal{B}$ is $\mathcal{K}(1)$-invariant, that is

$$\mathcal{B}\left(\mathfrak{L}^{\frac{1}{2}}_{X_F}(f\alpha^{\frac{1}{2}})\right) = 0, \ \forall F, f \in C^\infty_{\mathbb{C}}(S^{1|1}). \tag{4.29}$$

So, the product of densities composed with $\mathcal{B}$ yields a bilinear $\mathcal{K}(1)$-invariant form:

$$< . \,, \, . >: \mathfrak{F}_\lambda \otimes \mathfrak{F}_{\frac{1}{2}-\lambda} \to \mathbb{C}, \ \lambda \in \mathbb{C}. \tag{4.30}$$

given by

$$< f\alpha^{\lambda}, g\alpha^{\frac{1}{2}-\lambda} >= \int_{S^1} (f_0 g_1 + f_1 g_0)(x)\mathrm{d}x. \tag{4.31}$$

where $f = f_0 + \theta f_1 \in \mathfrak{F}_\lambda$ and $g = g_0 + \theta g_1 \in \mathfrak{F}_{\frac{1}{2}-\lambda}$. Thus, we get the conjugation map $C : \mathfrak{D}_{\lambda,\mu} \to \mathfrak{D}_{\frac{1}{2}-\mu,\frac{1}{2}-\lambda}$ defined by

$$< A\phi, \psi >= (-1)^{|A||\phi|} < \phi, C(A)\psi >, \tag{4.32}$$

for any $A \in \mathfrak{D}_{\lambda,\mu}$, $\phi \in \mathfrak{F}_\lambda$ and $\psi \in \mathfrak{F}_{\frac{1}{2}-\mu}$. By a direct computation, one can easily check that the map C is $\mathcal{K}(1)$-invariant, *i.e.*, it satisfies

$$\mathfrak{L}_{X_F}^{\frac{1}{2}-\mu,\frac{1}{2}-\lambda}\Big(C(A)\Big) = C\left(\mathfrak{L}_{X_F}^{\lambda,\mu}(A)\right), \forall F \in C_{\mathbb{C}}^{\infty}(S^{1|1}), \forall A \in \mathfrak{D}_{\lambda,\mu}. \tag{4.33}$$

Accordingly, in the particular case $\lambda + \mu = \dfrac{1}{2}$, the conjugation map leads an element in $\mathfrak{C}^k_{\lambda,\frac{1}{2}-\lambda}$, for any $\lambda \in \mathbb{C}$. An explicit formula for the operator C was given in [6], for any $k \in \frac{1}{2}\mathbb{N}$, by:

$$C\left(\sum_{i=0}^{2k} a_i \overline{D}^i\right) = \sum_{i=0}^{2k} (-1)^{[\frac{i+1}{2}]+i|a_i|} \overline{D}^i \circ a_i, \tag{4.34}$$

where for a real number x, $[x]$ means its integer part.

Remark 4.1. For all $k \in \frac{1}{2}\mathbb{N}$ and $(\lambda,\mu) \in \mathbb{C}^2$, the algebras $\mathfrak{C}^k_{\lambda,\mu}$ and $\mathfrak{C}^k_{\frac{1}{2}-\mu,\frac{1}{2}-\lambda}$ are isomorphic. Indeed,

$$T \in \mathfrak{C}^k_{\lambda,\mu} \Longleftrightarrow T^* := C \circ T \circ C \in \mathfrak{C}^k_{\frac{1}{2}-\mu,\frac{1}{2}-\lambda}. \tag{4.35}$$

4.2. THE PROJECTION P_0 AND THE CONJUGATE P_0^*

The space $\mathfrak{D}^0_{\lambda,\mu}$ is nothing but $\mathfrak{F}_{\mu-\lambda}$. Any zeroth-order differential operator is the operator of multiplication by $(\mu-\lambda)$-density. Moreover, when $\lambda = 0$, we get the $\mathcal{K}(1)$- invariant projection map

$$P_0 : \mathfrak{D}^k_{0,\mu} \to \mathfrak{F}_\mu,\ A = \sum_{\ell=0}^{2k} a_\ell \overline{D}^\ell \mapsto a_0\alpha^\mu \tag{4.36}$$

As $\mathfrak{F}_\mu \subset \mathfrak{D}^k_{0,\mu}$, one obtains a non trivial element of the algebra $\mathfrak{C}^k_{0,\mu}$, for any $k \in \frac{1}{2}\mathbb{N}$ and $\mu \in \mathbb{C}$. For short we write

$$P_0\left(\sum_{\ell=0}^{2k} a_\ell \overline{D}^\ell\right) = a_0. \tag{4.37}$$

Obviously $P_0^* = C \circ P_0 \circ C \in \mathfrak{C}^k_{\lambda,\frac{1}{2}}, \forall \lambda \in \mathbb{C}$. By a direct computation, one can prove that

$$P_0^* \left(\sum_{\ell=0}^{2k} a_\ell \overline{D}^\ell \right) = \sum_{\ell=0}^{2k} D^\ell(a_\ell). \tag{4.38}$$

4.3. AN ADDITIONAL ELEMENT OF $\mathfrak{C}^k_{0,\frac{1}{2}}$

Of course, we have $P_0^* \in \mathfrak{C}^k_{0,\frac{1}{2}}$ for any $k \in \frac{1}{2}\mathbb{N}$. We can easily show that a non trivial element P_1 of the algebra $\mathfrak{C}^k_{0,\frac{1}{2}}$ is given by the expression

$$P_1 \left(\sum_{\ell=0}^{2k} a_\ell \overline{D}^\ell \right) = \left(\sum_{\ell=1}^{2k} D^{\ell-1}(a_\ell) \right) \overline{D}. \tag{4.39}$$

Following [5] (section 4.3), an intrinsic form for P_1 can also be written. First, we see the vector field $\overline{D}$ as an element of $\mathfrak{D}^1_{0,\frac{1}{2}}$, then we consider the $\mathcal{K}(1)$-invariant operator

$$\begin{aligned} \delta : \quad & \mathfrak{D}^k_{0,\frac{1}{2}} \to \mathfrak{D}^{k+\frac{1}{2}}_{0,\frac{1}{2}} \\ & A \mapsto A \circ \overline{D} \end{aligned}$$

which is a bijection between $\mathfrak{D}^k_{0,\frac{1}{2}}$ and $Ker(P_0) \subset \mathfrak{D}^{k+\frac{1}{2}}_{0,\frac{1}{2}}$. Thus,

$$P_1 = \delta \circ P_0 \circ C \circ \delta^{-1} \circ (Id - P_0).$$

4.4. CONSTRUCTION OF SUPERSYMMETRIES USING BILINEAR OPERATORS ON WEIGHTED DENSITIES

Many classification results for invariant differential operators are available now, and it was shown that there are quite few invariant differential operators and most of them are of a great importance. For instance, bilinear invariant differential operators on tensor fields were classified by Grozman [9]. A complete description of linear projections from differential operators to tensor densities invariant under the group of diffeomorphisms of S^1 was given in [16]. In the super setting, in [10], it was given a description of the set of $\mathcal{K}(1)$-invariant bilinear differential operators $\mathbb{T} : \mathfrak{F}_\nu \otimes \mathfrak{F}_\lambda \to \mathfrak{F}_\mu$ acting on tensor densities, *i.e.*, satisfying

$$\mathfrak{L}^\mu_{X_H} \circ \mathbb{T} - (-1)^{|\mathbb{T}||H|} \mathbb{T} \circ \mathfrak{L}^{\nu,\lambda}_{X_H} = 0, \forall H \in C^\infty_{\mathbb{C}}(S^{1|1}), \tag{4.40}$$

where $\mathfrak{L}^{\nu,\lambda}_{X_H}$ is the Lie derivative on $\mathfrak{F}_\nu \otimes \mathfrak{F}_\lambda$ defined by the Leibnitz rule

$$\mathfrak{L}^{\nu,\lambda}_{X_H}(F \otimes G) = \mathfrak{L}^{\nu}_{X_H}(F) \otimes G + (-1)^{|H||F|} F \otimes \mathfrak{L}^{\lambda}_{X_H}(G). \tag{4.41}$$

The list of these operators is the following: Let

$$\begin{aligned} \mathbb{T} : \mathfrak{F}_\nu \otimes \mathfrak{F}_\lambda &\to \mathfrak{F}_{\mu_k} \\ (f\alpha^\nu) \otimes (g\alpha^\lambda) &\mapsto \mathbb{T}_{\nu,\lambda,\mu_k}(f,g)\alpha^{\mu_k}, \end{aligned} \tag{4.42}$$

where $\frac{k}{2}$ $(k = 0, 1, 2, 3, 4)$ is the order of $\mathbb{T}$ and $\mu_k = \nu + \lambda + \frac{k}{2}$.

- Zeroth-order operators:

$$\mathbb{T}_{\nu,\lambda,\mu_0}(f,g) = fg. \tag{4.43}$$

- Operators of order $\frac{1}{2}$:

$$\mathbb{T}^{a,b}_{0,0,\frac{1}{2}}(f,g) = a(-1)^{|f|} f D(g) + bD(f)g;\ a,\, b \in \mathbb{R}, \tag{4.44}$$

and

$$\mathbb{T}_{\nu,\lambda,\mu_1}(f,g) = \lambda\, D(f)g - \nu(-1)^{|f|} f D(g). \tag{4.45}$$

- First-order operators:

$$\mathbb{T}_{\nu,\lambda,\mu_2}(f,g) = \lambda f' g - \frac{1}{2}(-1)^{|f|}\overline{D}(f)\overline{D}(g) - \nu f g'. \tag{4.46}$$

- Operators of order $\frac{3}{2}$:

$$\mathbb{T}_{0,\lambda,\mu_3}(f,g) = S(f,g) - 2\lambda D(f')g, \tag{4.47}$$

$$\mathbb{T}_{\nu,0,\mu_3}(f,g) = S(f,g) - 2\nu(-1)^{|f|} f D(g') \tag{4.48}$$

and

$$\mathbb{T}_{\nu,-\nu-1,\frac{1}{2}}(f,g) = \nu(-1)^{|f|} f D(g') + (\nu+1)D(f')g + (\nu + \frac{1}{2})S(f,g). \tag{4.49}$$

- Second-order operators:

$$\mathbb{T}_{0,0,2}(f,g) = f'g' + (-1)^{|f|}\left(D(f')D(g) - D(f)D(g')\right), \tag{4.50}$$

$$\mathbb{T}_{-\frac{3}{2},0,\frac{1}{2}}(f,g) = 3fg'' - (-1)^{|f|} M(f,g) + 2f'g' \tag{4.51}$$

and

$$\mathbb{T}_{0,-\frac{3}{2},\frac{1}{2}}(f,g) = 3f''g + (-1)^{|f|} M(g,f) + 2f'g', \tag{4.52}$$

where

$$M(f,g) = 2D(f)D(g') + D(f')D(g) \text{ and } S(f,g) = D(f)g' + (-1)^{|f|} f' D(g). \tag{4.53}$$

However, in the super case, there is no work dealing with the description of $\mathcal{K}(1)$-invariant projections from differential operators to tensor densities. Let $k \in \frac{1}{2}\mathbb{N}$. We will call a projection of order p ($p \in \frac{1}{2}\mathbb{N}$) a $\mathcal{K}(1)$-invariant projection

$$\sigma_p : \mathfrak{D}^k_{\lambda,\mu} \to \mathfrak{F}_{\mu-\lambda-k+p}, \tag{4.54}$$

i.e., such that

$$\sigma_p\Big(\mathfrak{L}^{\lambda,\mu}_{X_H}(A)\Big) = \mathfrak{L}^{\mu-\lambda-k+p}_{X_H}\Big(\sigma_p(A)\Big), \forall H \in C^\infty_{\mathbb{C}}(S^{1|1}). \tag{4.55}$$

Obviously, as a "zeroth-order" projection σ_0, we quote the renowned *principal symbol map*

$$\begin{aligned} \sigma_0 = \sigma_{\mathrm{pr}} : \mathfrak{D}^k_{\lambda,\mu} &\to \mathfrak{F}_{\mu-\lambda-k} \\ A = \sum_{\ell=0}^{2k} a_\ell(x,\theta)\overline{D}^\ell &\mapsto a_{2k}(x,\theta)\,\alpha^{\mu-\lambda-k} \end{aligned} \tag{4.56}$$

Of course, σ_{pr} is a $\mathcal{K}(1)$-invariant projection.

Remark 4.2. When (λ,μ) is non resonant, *i.e.*, $(\lambda,\mu) \neq (\frac{1-m}{4}, \frac{1+m}{4})$ where m is odd, there is a tight link between these projections and the $\mathfrak{osp}(1|2)$-equivariant symbol map given by Gargoubi at all in [6]. Indeed the symbol map gives an $\mathfrak{osp}(1|2)$-equivariant map from the space of differential operator $\mathfrak{D}^k_{\lambda,\mu}$ and its associated graded space $\mathcal{S}^k_{\mu-\lambda} = \bigoplus_{i=0}^{2k} \mathfrak{F}_{\mu-\lambda-\frac{i}{2}}$, which is a direct sum of density spaces. Applying theorem 6.1 of [6] to an operator of the form $A = \sum_{\ell=0}^{2k} a_\ell \overline{D}^\ell$ and gathering the terms of (density) weight $\mu-\lambda-k+p$ gives the expression of σ_p up to multiplication by a constant factor for suitable values of (λ,μ). Surely, we must check the invariance of the resulting expression with respect to vector fields that generate $K(1)$, together with $\mathfrak{osp}(1|2)$. Unfortunately, one must seek other means to check such projections for resonant values of (λ,μ).

In the sequel, we prove in the following theorems the existence of the projections $\sigma_{\frac{1}{2}}$, σ_1 and $\sigma_{\frac{3}{2}}$:

Theorem 4.3. *For all k in $\frac{1}{2}\mathbb{N}$, or all (λ,μ) in $\mathbb{C}^2$, the map*

$$\sigma_{\frac{1}{2}} : \mathfrak{D}^k_{\lambda,\mu} \to \mathfrak{F}_{\mu-\lambda-k+\frac{1}{2}} \tag{4.57}$$

defined by

(4.58)
$$\sigma_{\frac{1}{2}}\Big(\sum_{\ell=0}^{2k}a_\ell(x,\theta)\overline{D}^\ell\Big)=\Big[\Big((1-(-1)^{2k})(\lambda-\frac{1}{4})+k\Big)D(a_{2k})-(2\mu-2\lambda-2k)a_{2k-1}\Big]\alpha^{\mu-\lambda-k+\frac{1}{2}}$$

is $\mathcal{K}(1)$-invariant.

Proof. Let $X_{\theta f}$, $f\in C^\infty(S^1)$, be an odd contact vector field and $\ell\in\mathbb{N}$. By a direct computation, the action 2.15 is given by the rules

$$\begin{aligned}\mathfrak{L}^{\lambda,\mu}_{X_{\theta f}}\Big(a_{2\ell}\overline{D}^{2\ell}\Big)&=\Big(\theta f a'_{2\ell}+\frac{1}{2}f\overline{D}(a_{2\ell})+(\mu-\lambda)\theta a_{2\ell}f'\Big)\overline{D}^{2\ell}\\&+\sum_{s=1}^{\ell}(-1)^sC^s_\ell(-1)^{|a_{2\ell}|}a_{2\ell}\Big(\theta f^{(s)}\overline{D}^{2\ell-2s+2}-\frac{1}{2}f^{(s)}\overline{D}^{2\ell-2s+1}\\&-\lambda\theta f^{(s+1)}\overline{D}^{2\ell-2s}\Big)\end{aligned}\tag{4.59}$$

and

(4.60)
$$\begin{aligned}\mathfrak{L}^{\lambda,\mu}_{X_{\theta f}}\Big(a_{2\ell+1}\overline{D}^{2\ell+1}\Big)&=\Big(\theta f a'_{2\ell+1}+\frac{1}{2}f\overline{D}(a_{2\ell+1})+(\mu-\lambda-\frac{1}{2})\theta f'a_{2\ell+1}\Big)\overline{D}^{2\ell+1}\\&+\lambda f'(-1)^{|a_{2\ell+1}|}a_{2\ell+1}\overline{D}^{2\ell}+\sum_{s=1}^{\ell}(-1)^sC^s_\ell\Big(\theta f^{(s)}a_{2\ell+1}\overline{D}^{2\ell-2s+3}\\&-\frac{1}{2}f^{(s)}(-1)^{|a_{2\ell+1}|}a_{2\ell+1}\overline{D}^{2\ell-2s+2}-(\lambda+\frac{1}{2})\theta f^{(s+1)}a_{2\ell+1}\overline{D}^{2\ell-2s+1}\\&+\lambda f^{(s+1)}(-1)^{|a_{2\ell+1}|}a_{2\ell+1}\overline{D}^{2\ell-2s}\Big).\end{aligned}$$

Thus, for $k\in\mathbb{N}$ and $A=\sum_{\ell=0}^{2k}a_\ell\overline{D}^\ell=\sum_{\ell=0}^{k}a_{2\ell}\overline{D}^{2\ell}+\sum_{\ell=0}^{k-1}a_{2\ell+1}\overline{D}^{2\ell+1}\in D^k_{\lambda,\mu}$, we get

$$\begin{aligned}\mathfrak{L}^{\lambda,\mu}_{X_{\theta f}}(A)&=\Big(\theta f a'_{2k}+\frac{1}{2}f\overline{D}(a_{2k})+(\mu-\lambda-k)\theta f'a_{2k}\Big)\overline{D}^{2k}\\&+\Big(\theta f a'_{2k-1}+\frac{1}{2}f\overline{D}(a_{2k-1})+(\mu-\lambda-k+\frac{1}{2})\theta f'a_{2k-1}\\&+\frac{1}{2}kf'(-1)^{|a_{2k}|}a_{2k}\Big)\overline{D}^{2k-1}+\text{ terms in }\overline{D}^s, s\le 2k-2.\end{aligned}\tag{4.61}$$

Therefore,

(4.62)
$$\sigma_{\frac{1}{2}}\Big(\mathfrak{L}^{\lambda,\mu}_{X_{\theta f}}(A)\Big) = \Big[k\Big(\theta f a'_{2k} + \tfrac{1}{2} f\overline{D}(a_{2k}) + (\mu-\lambda-k)\theta f' a_{2k}\Big)$$
$$-(2\mu-2\lambda-2k)\Big(\theta f a'_{2k-1} + \tfrac{1}{2} f\overline{D}(a_{2k-1}) + (\mu-\lambda-k+\tfrac{1}{2})\theta f' a_{2k-1}+$$
$$\tfrac{1}{2} k f'(-1)^{|a_{2k}|}a_{2k}\Big)\Big]\alpha^{\mu-\lambda-k+\frac{1}{2}}.$$

On the other hand, due to (2.14), we obtain

(4.63)
$$\mathfrak{L}^{\mu-\lambda-k+\frac{1}{2}}_{X_{\theta f}}\Big(\sigma_{\frac{1}{2}}(A)\Big) = \mathfrak{L}^{\mu-\lambda-k+\frac{1}{2}}_{X_{\theta f}}\Big[\Big(kD(a_{2k})-(2\mu-2\lambda-2k)a_{2k-1}\Big)\alpha^{\mu-\lambda-k+\frac{1}{2}}\Big]$$
$$= \Big[\theta f\Big(kD(a_{2k}) - (2\mu-2\lambda-2k)a_{2k-1}\Big)'$$
$$+\tfrac{1}{2}D(\theta f)\overline{D}\Big(kD(a_{2k}) - (2\mu-2\lambda-2k)a_{2k-1}\Big)$$
$$+(\mu-\lambda-k+\tfrac{1}{2})\theta f'\Big(kD(a_{2k}) - (2\mu-2\lambda-2k)a_{2k-1}\Big)\Big]\alpha^{\mu-\lambda-k+\frac{1}{2}}.$$

Now by substituting in equations (4.62) and (4.63) $a_{2k,0}+\theta a_{2k,1}$ and $a_{2k-1,0}+\theta a_{2k-1,1}$ to a_{2k} and a_{2k-1} respectively, we easily check

$$\sigma_{\frac{1}{2}}\Big(\mathfrak{L}^{\lambda,\mu}_{X_{\theta f}}(A)\Big) = \mathfrak{L}^{\mu-\lambda-k+\frac{1}{2}}_{X_{\theta f}}\Big(\sigma_{\frac{1}{2}}(A)\Big). \tag{4.64}$$

A similar reasoning can be made, first for $k\in(\mathbb{N}+\dfrac{1}{2})$, and then globally in the case where X_f, $f\in C^\infty(S^1)$, is an even contact vector field. The $\mathcal{K}(1)$-invariance of the map $\sigma_{\frac{1}{2}}$ is thus proved. $\square$

Remark 4.4. In the particular case, where $k\in(\mathbb{N}+\dfrac{1}{2})$ and $(\lambda,\mu)=(\dfrac{1-2k}{4},\dfrac{1+2k}{4})$, the map $\sigma_{\frac{1}{2}}$ vanishes. In this case, there are two independent projections in $\mathfrak{F}^{\frac{1}{2}}$ given by

$$\sum_{\ell=0}^{2k} a_\ell(x,\theta)\overline{D}^\ell \mapsto D(a_{2k})\alpha^{\frac{1}{2}}, \text{ and } \sum_{\ell=0}^{2k} a_\ell(x,\theta)\overline{D}^\ell \mapsto a_{2k-1}\alpha^{\frac{1}{2}}. \tag{4.65}$$

THEOREM 4.5. *The map*

$$\sigma_1 : \mathfrak{D}^k_{\lambda,\mu} \to \mathfrak{F}_{\mu-\lambda-k+1} \tag{4.66}$$

defined for $A=\displaystyle\sum_{\ell=0}^{2k} a_\ell\overline{D}^\ell$ *by*

$$\sigma_1(A) = \Big[k(2\lambda+k-1)a'_{2k} + (2\lambda+k-1)D(a_{2k-1}) \tag{4.67}$$

$$- (2\mu - 2\lambda - 2k + 1)a_{2k-2}\Big]\alpha^{\mu-\lambda-k+1}$$

when $k \in \mathbb{N}^*$, *and when* $k \in (\mathbb{N}^* + \frac{1}{2})$ *by*

$$\sigma_1(A) = \Big[(k - \frac{1}{2})(2\lambda + k - \frac{1}{2})a'_{2k} + (k - \frac{1}{2})D(a_{2k-1}) - (2\mu - 2\lambda - 2k + 1)a_{2k-2}\Big]\alpha^{\mu-\lambda-k+1} \tag{4.68}$$

is $\mathcal{K}(1)$*-invariant.*

Proof. Similar to the proof of Theorem 4.3. □

Remark 4.5. If $k \in \mathbb{N}^*$ and $(\lambda, \mu) = (\frac{1-k}{2}, \frac{k}{2})$, the map σ_1 vanishes. In this case, we have two independent projections in $\mathfrak{F}^{\frac{1}{2}}$ given by

$$\sum_{\ell=0}^{2k} a_\ell \overline{D}^\ell \mapsto \Big(ka'_{2k} + D(a_{2k-1})\Big)\alpha^{\frac{1}{2}}, \text{ and } \sum_{\ell=0}^{2k} a_\ell \overline{D}^\ell \mapsto a_{2k-2}\alpha^{\frac{1}{2}}. \tag{4.69}$$

Theorem 4.6. *The map* $\sigma_{\frac{3}{2}} : \mathfrak{D}^k_{\lambda,\mu} \to \mathfrak{F}_{\mu-\lambda-k+\frac{3}{2}}$ *given by*

$$\begin{aligned}\sigma^{\frac{3}{2}}\Big(\sum_{\ell=0}^{2k} a_\ell(x,\theta)\overline{D}^\ell\Big) = \Big[&k(k-1)(2\lambda+k-1)D^3(a_{2k})\\ &-(k-1)(2\lambda+k-1)(2\mu-2\lambda-2k+1)a'_{2k-1}\\ &-(k-1)(2\mu-2\lambda-2k+1)D(a_{2k-2})\\ &+(2\mu-2\lambda-2k+1)(2\mu-2\lambda-2k+2)a_{2k-3}\Big]\alpha^{\mu-\lambda-k+\frac{3}{2}},\end{aligned} \tag{4.70}$$

where k *in* $\mathbb{N}$, $k \geq 2$, *and* (λ, μ) *such that* $2\mu + 2\lambda - 1 = 0$, *and by*

$$\begin{aligned}\sigma_{\frac{3}{2}}\Big(\sum_{\ell=0}^{2k} a_\ell(x,\theta)\overline{D}^\ell\Big) = \Big[&(k-\tfrac{1}{2})(2\lambda+k-\tfrac{1}{2})(2\lambda+k-\tfrac{3}{2})D^3(a_{2k})\\ &-(k-\tfrac{1}{2})(2\lambda+k-\tfrac{3}{2})(2\mu-2\lambda-2k+1)a'_{2k-1}\\ &-(2\lambda+k-\tfrac{3}{2})(2\mu-2\lambda-2k+1)D(a_{2k-2})\\ &+(2\mu-2\lambda-2k+1)(2\mu-2\lambda-2k+2)a_{2k-3}\Big]\alpha^{\mu-\lambda-k+\frac{3}{2}},\end{aligned} \tag{4.71}$$

where k *in* $(\mathbb{N} + \frac{1}{2})$, $k \geq \frac{3}{2}$, *and* (λ, μ) *such that*

$$(2\lambda + k - \frac{1}{2})(2\lambda + k - \frac{3}{2}) + (2\lambda + \frac{2k-3}{4})(2\mu - 2\lambda - 2k + 1) = 0, \tag{4.72}$$

is $\mathcal{K}(1)$*-invariant.*

Proof. Similar to the proof of Theorem 4.3. □

Following the idea of [5], we can produce supersymmetries in the following way: Let $B : \mathfrak{F}_\nu \otimes \mathfrak{F}_\lambda \to \mathfrak{F}_\mu$ a $\mathcal{K}(1)$-invariant bilinear differential operator and $\pi : \mathfrak{D}^k_{\lambda,\mu} \to \mathfrak{F}_\nu$ a $\mathcal{K}(1)$-invariant linear projection, then the linear map $B \circ \pi : \mathfrak{D}^k_{\lambda,\mu} \to \mathfrak{D}^k_{\lambda,\mu}$ given by the rule

$$(B \circ \pi)(A)(\cdot) = B(\pi(A),\ \cdot),\ A \in \mathfrak{D}^k_{\lambda,\mu} \tag{4.73}$$

is still $\mathcal{K}(1)$-invariant. The $\mathcal{K}(1)$-invariant bilinear and linear projections listed above will be used in the next section to specify the generators of the algebras $\mathfrak{C}^k_{\lambda,\mu}$ that can be obtained by this procedure.

5. THE ALGEBRAS $\mathfrak{C}^k_{\lambda,\mu}$, FOR $k \leq 3$

In this section, we compute the algebras of supersymmetries for lower dimensions, namely $k = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$. Let us first state the main theorem of the paper in which we give the dimension of the algebra $\mathfrak{C}^k_{\lambda,\mu}$ for all (λ, μ) in $\mathbb{C}^2$ and k in $\frac{1}{2}\mathbb{N}$. The proof of this theorem will be given in Section 6. The upshot is:

THEOREM 5.1. *If $k \geq 3$, the algebras $\mathfrak{C}^k_{\lambda,\mu}$ do not depend on k. The dimensions of $\mathfrak{C}^k_{\lambda,\mu}$ are given in the following table footnotesize*

k	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	≥ 3
$(0, \frac{1}{2})$	1	3	4	5	6	5	5
$(0,2), (-\frac{1}{2},1), (-\frac{3}{2},\frac{1}{2}), (-\frac{1+\sqrt{3}}{4}, \frac{3+\sqrt{3}}{4}), (\frac{-1+\sqrt{3}}{4}, \frac{3-\sqrt{3}}{4})$	1	2	3	4	4	3	2
$(0,\mu), \mu \neq \frac{1}{2}, 2$	1	2	3	4	4	2	2
$(\lambda, \frac{1}{2}), \lambda \neq 0, -\frac{3}{2}$	1	2	3	4	4	2	2
$\lambda + \mu = \frac{1}{2}, \lambda \neq 0, -\frac{1}{2}, -\frac{1+\sqrt{3}}{4}, \frac{-1+\sqrt{3}}{4}$	1	2	3	4	4	2	2
$\mu - \lambda = \frac{3}{2}, \lambda \neq 0, -\frac{1}{2}, -1$	1	2	3	3	2	2	1
$(2\lambda+2)(2\lambda+1) + (2\lambda+\frac{1}{2})(2\mu-2\lambda-4) = 0$, $\lambda \neq -\frac{1+\sqrt{3}}{4}, \frac{-1+\sqrt{3}}{4}$	1	2	3	3	2	2	1
(λ,μ) *otherwise*	1	2	3	3	2	1	1

Now, the point of Theorems 5.2 to 5.7 is to provide enough linearly independent generators to span the algebras $\mathfrak{C}^k_{\lambda,\mu}$, for $k \leq 3$, according to the dimensions of these algebras given above and using the method stated in the previous section.

THEOREM 5.2. *We have* $\dim(\mathfrak{C}^{\frac{1}{2}}_{\lambda,\mu}) = \begin{cases} 3 & \textit{if } (\lambda,\mu) = (0, \frac{1}{2}) \\ 2 & \textit{otherwise} \end{cases}$.

More over,

$\mathfrak{C}^{\frac{1}{2}}_{0,\frac{1}{2}} = \mathrm{Span}\{\mathrm{Id}, P_0, C\}$
$\mathfrak{C}^{\frac{1}{2}}_{\lambda,\frac{1}{2}-\lambda} = \mathrm{Span}\{\mathrm{Id}, C\}$
$\mathfrak{C}^{\frac{1}{2}}_{0,\mu} = \mathrm{Span}\{\mathrm{Id}, P_0\},\ \ \mu \neq \frac{1}{2}$
$\mathfrak{C}^{\frac{1}{2}}_{\lambda,\frac{1}{2}} = \mathrm{Span}\{\mathrm{Id}, P_0^*, \},\ \ \lambda \neq 0$
$\mathfrak{C}^{\frac{1}{2}}_{\lambda,\mu} = \mathrm{Span}\{\mathrm{Id}, T^{\frac{1}{2}}_{\lambda,\mu}\}$ *otherwise,*

where

$$T^{\frac{1}{2}}_{\lambda,\mu} : a_0 + a_1\overline{D} \mapsto \lambda D(a_1) - (\mu - \lambda - \frac{1}{2})(-1)^{|a_1|}a_1 D.$$

Proof. We mention that algebras $\mathfrak{C}^{\frac{1}{2}}_{\lambda,\frac{1}{2}}$ and $\mathfrak{C}^{\frac{1}{2}}_{0,\frac{1}{2}-\lambda}$, $\lambda \neq 0$ are isomorphic and that the module $\mathfrak{D}^{\frac{1}{2}}_{\lambda,\frac{1}{2}-\lambda}$ is self-adjoint. Generators of $\mathfrak{C}^{\frac{1}{2}}_{0,\frac{1}{2}}$ and $\mathfrak{C}^{\frac{1}{2}}_{0,\mu}$, $\mu \neq 0$, are immediately obtained. The map $T^{\frac{1}{2}}_{\lambda,\mu}$ is obtained using the bilinear operator (4.45) and the principal symbol map (4.56) following (4.73). □

Theorem 5.3. *We have* $\dim(\mathfrak{C}^1_{\lambda,\mu}) = \begin{cases} 4 & \textit{if } (\lambda,\mu) = (0,\frac{1}{2}) \\ 3 & \textit{otherwise.} \end{cases}$

More over,

$\mathfrak{C}^1_{0,\frac{1}{2}} = \mathrm{Span}\{\mathrm{Id}, P_0, P_1, C\}$
$\mathfrak{C}^1_{\lambda,\frac{1}{2}-\lambda} = \mathrm{Span}\{\mathrm{Id}, C, T^1_{\lambda,\frac{1}{2}-\lambda}\}$
$\mathfrak{C}^1_{0,\mu} = \mathrm{Span}\{\mathrm{Id}, P_0, T^1_{0,\mu}\},\ \ \mu \neq \frac{1}{2}$
$\mathfrak{C}^1_{\lambda,\frac{1}{2}} = \mathrm{Span}\{\mathrm{Id}, P_0^*, \left(T^1_{0,\frac{1}{2}-\lambda}\right)^*\},\ \ \lambda \neq 0$
$\mathfrak{C}^1_{\lambda,\mu} = \mathrm{Span}\{\mathrm{Id}, T^1_{\lambda,\mu}, S^1_{\lambda,\mu}\}$ *otherwise,*

where for $A = \sum_{\ell=0}^{2} a_\ell \overline{D}^\ell$,

$$\begin{cases} T^1_{\lambda,\mu}(A) = \lambda D^2(a_2) + \frac{1}{2}D(a_2)\overline{D} + (\mu-\lambda-1)a_2\overline{D}^2 \\ S^1_{\lambda,\mu}(A) = \lambda D^2(a_2) - \lambda(2\mu-2\lambda-2)D(a_1) + (\mu-\lambda-\frac{1}{2})\Big((-1)^{|a_2|}D(a_2) \\ +(2\mu-2\lambda-2)(-1)^{|a_1|}a_1\Big)D. \end{cases}$$

Proof. The generator $T^1_{\lambda,\mu}$ is obtained by the composition of the contact bracket (4.46) and the principal symbol map (4.56). The generator $S^1_{\lambda,\mu}$ is obtained by the composition of the bilinear operator (4.45) and the linear projection (4.57). □

THEOREM 5.4. *We have* $\dim(\mathfrak{C}^{\frac{3}{2}}_{\lambda,\mu}) = \begin{cases} 5 & \textit{if } (\lambda,\mu) = (0,\frac{1}{2}) \\ 4 & \textit{if } \lambda = 0, \mu \neq \frac{1}{2} \\ 4 & \textit{if } \lambda \neq 0, \mu = \frac{1}{2} \\ 4 & \textit{if } \mu = \frac{1}{2}-\lambda, \lambda \neq 0 \\ 3 & \textit{otherwise.} \end{cases}$

More over,

$$\begin{aligned}
&\mathfrak{C}^{\frac{3}{2}}_{0,\frac{1}{2}} = \mathrm{Span}\{\mathrm{Id}, P_0, P_0^*, P_1, C\}\\
&\mathfrak{C}^{\frac{3}{2}}_{0,\mu} = \mathrm{Span}\{\mathrm{Id}, P_0, T^{\frac{3}{2}}_{0,\mu}, S^{\frac{3}{2}}_{0,\mu}\},\ \ \mu \neq \tfrac{1}{2}\\
&\mathfrak{C}^{\frac{3}{2}}_{\lambda,\frac{1}{2}} = \mathrm{Span}\{\mathrm{Id}, P_0^*, \Big(T^{\frac{3}{2}}_{0,\frac{1}{2}-\lambda}\Big)^*, \Big(S^{\frac{3}{2}}_{0,\frac{1}{2}-\lambda}\Big)^*\},\ \ \lambda \neq 0\\
&\mathfrak{C}^{\frac{3}{2}}_{-\frac{1}{2},1} = \mathrm{Span}\{\mathrm{Id}, C, J^{\frac{3}{2}}_{-\frac{1}{2},1}, K^{\frac{3}{2}}_{-\frac{1}{2},1}\}\\
&\mathfrak{C}^{\frac{3}{2}}_{\lambda,\frac{1}{2}-\lambda} = \mathrm{Span}\{\mathrm{Id}, C, T^{\frac{3}{2}}_{\lambda,\frac{1}{2}-\lambda}, E^{\frac{3}{2}}_{\lambda,\frac{1}{2}-\lambda}\}\\
&\mathfrak{C}^{\frac{3}{2}}_{\lambda,\mu} = \mathrm{Span}\{\mathrm{Id}, T^{\frac{3}{2}}_{\lambda,\mu}, E^{\frac{3}{2}}_{\lambda,\mu}\}\ \ \textit{otherwise,}
\end{aligned}$$

where for $A = \displaystyle\sum_{\ell=0}^{3} a_\ell \overline{D}^\ell$,

$$\begin{cases}
T^{\frac{3}{2}}_{\lambda,\mu}(A) = \lambda\Big((2\lambda+1)D^3(a_3) - (2\mu-2\lambda-3)D^2(a_2)\Big)\\
-\frac{1}{2}\Big((2\lambda+1)D^2(a_3) - (2\mu-2\lambda-3)D(a_2)\Big)\overline{D}+\\
(\mu-\lambda-1)\Big((2\lambda+1)(D(a_3) - (2\mu-2\lambda-3)a_2\Big)\overline{D}^2\\
S^{\frac{3}{2}}_{0,\mu}(A) = -D(a_3)\overline{D}^2 + (-1)^{|a_3|}D^2(a_3)D - (2\mu-3)(-1)^{|a_3|}a_3D^3\\
J^{\frac{3}{2}}_{-\frac{1}{2},1}(A) = D^3(a_3) - D^2(a_3)\overline{D} - D(a_3)\overline{D}^2\\
K^{\frac{3}{2}}_{-\frac{1}{2},1}(A) = D^2(a_2) - D(a_2)\overline{D} - a_2\overline{D}^2\\
E^{\frac{3}{2}}_{\lambda,\mu}(A) = \lambda(k-\frac{1}{2})(2\lambda+k-\frac{1}{2})D^3(a_3) + \lambda(k-\frac{1}{2})D^2(a_2)\\
-\lambda(2\mu-2\lambda-2)D(a_1) - (\mu-\lambda-\frac{1}{2})\Big((k-\frac{1}{2})(2\lambda+k-\frac{1}{2})(-1)^{|a_3|}D^2(a_3)\\
-(k-\frac{1}{2})(-1)^{|a_2|}D(a_2) - (2\mu-2\lambda-2)(-1)^{|a_1|}a_1\Big)D.
\end{cases}$$

Proof. The generator $T^{\frac{3}{2}}_{\lambda,\mu}$ is obtained by the composition of the bilinear operator (4.46) and the linear projection (4.57), the generator $S^{\frac{3}{2}}_{0,\mu}$ is obtained by the composition of the bilinear operator (4.48) and the linear projection (4.56). Generators $J^{\frac{3}{2}}_{-\frac{1}{2},1}$ and $K^{\frac{3}{2}}_{-\frac{1}{2},1}$ are respectively obtained by the composition of the bilinear operator (4.46) and the linear projections (4.65). Finally,

$E^{\frac{3}{2}}_{\lambda,\mu}$ is obtained by the composition of the bilinear operator (4.45) and the linear projection (4.68). $\square$

THEOREM 5.5. *We have* $\dim(\mathfrak{C}^2_{\lambda,\mu}) = \begin{cases} 6 \ \ \textit{if } (\lambda,\mu)=(0,\frac{1}{2}) \\ 4 \ \ \textit{if } (\lambda,\mu)=(-\frac{1}{2},1) \\ 4 \ \textit{if } \lambda=0, \mu\neq\frac{1}{2} \\ 4 \ \textit{if } \lambda\neq 0, \mu=\frac{1}{2} \\ 4 \ \textit{if } \lambda+\mu=\frac{1}{2}, \lambda\neq 0 \\ 3 \ \textit{if } \mu-\lambda=\frac{3}{2} \\ 2 \ \ \textit{otherwise.} \end{cases}$

More over

$\mathfrak{C}^2_{0,\frac{1}{2}} = \mathrm{Span}\{\mathrm{Id}, P_0, P_0^*, P_1, C, Q\}$

$\mathfrak{C}^2_{0,\mu} = \mathrm{Span}\{\mathrm{Id}, P_0, T^2_{0,\mu}, S^2_{0,\mu}\},\ \ \mu\neq\frac{1}{2}$

$\mathfrak{C}^2_{\lambda,\frac{1}{2}} = \mathrm{Span}\{\mathrm{Id}, P_0^*, \left(T^2_{0,\frac{1}{2}-\lambda}\right)^*, \left(S^2_{0,\frac{1}{2}-\lambda}\right)^*\},\ \ \lambda\neq 0$

$\mathfrak{C}^2_{-\frac{1}{2},1} = \mathrm{Span}\{\mathrm{Id}, C, V^2_{-\frac{1}{2},1}, W^2_{-\frac{1}{2},1}\}$

$\mathfrak{C}^2_{\lambda,\frac{1}{2}-\lambda} = \mathrm{Span}\{\mathrm{Id}, C, T^2_{\lambda,\frac{1}{2}-\lambda},\ L^2_{\lambda,\frac{3}{2}+\lambda}\}, \lambda\neq 0, -\frac{1}{2}$

$\mathfrak{C}^2_{\lambda,\frac{3}{2}+\lambda} = \mathrm{Span}\{\mathrm{Id}, T^2_{\lambda,\frac{3}{2}+\lambda},\ R^2_{\lambda,\frac{3}{2}+\lambda}\}, \lambda\neq 0, -\frac{1}{2}$

$\mathfrak{C}^2_{\lambda,\mu} = \mathrm{Span}\{\mathrm{Id}, T^2_{\lambda,\mu}\}$ *otherwise. Where for* $A=\displaystyle\sum_{\ell=0}^{4} a_\ell \overline{D}^\ell$,

$$\begin{cases} Q(A) = 3a_4\overline{D}^4 + (-1)^{|a_4|}\Big(2\overline{D}(a_4)\overline{D}^3 + 2a_4'\overline{D}^2 + \overline{D}^3(a_4)\overline{D}\Big) \\ T^2_{\lambda,\mu}(A) = \lambda(2\lambda+1)(2a_4'' + D^3(a_3)) - (2\mu-2\lambda-3)a_2' \\ +\frac{1}{2}\Big(2(2\lambda+1)(-1)^{|a_4|}\overline{D}^3(a_4) - (2\lambda+1)a_3' + (2\mu-2\lambda-3)(-1)^{|a_2|}\overline{D}(a_2)\Big)\overline{D} \\ +(\mu-\lambda-1)\Big(2(2\lambda+1)a_4' + (2\lambda+1)D(a_3) - (2\mu-2\lambda-3)a_2\Big)\overline{D}^2 \\ V^2_{-\frac{1}{2},1}(A) = -a_4'' - \frac{1}{2}D^3(a_3) \\ +\frac{1}{2}\Big(2(-1)^{|a_4|}\overline{D}^3(a_4) - a_3'))\Big)\overline{D} + \frac{1}{2}\big(2a_4' + D(a_3)\big)\overline{D}^2 \\ W^2_{-\frac{1}{2},1}(A) = \frac{1}{2}\Big(a_2\overline{D}^2 - (-1)^{|a_2|}\overline{D}(a_2)\overline{D} + a_2'\Big) \\ S^2_{0,\mu}(A) = \Big(-2(-1)^{|a_4|}D^3(a_4) - (2\mu-4)(-1)^{|a_3|}a_3'\Big)D + \Big(2a_4' \\ -(2\mu-4)D(a_3)\Big)D^2 + (2\mu-3)\Big(2(-1)^{|a_4|}D(a_4) + (2\mu-4)(-1)^{|a_3|}a_3\Big)D^3 \\ L^2_{\lambda,\frac{1}{2}-\lambda}(A) = 2\lambda(2\lambda+1)\Big(D^4(a_4) + (2\lambda-1)D^3(a_3) + D^2(a_2) \\ +(4\lambda+1)D(a_1)\Big) + 4\lambda(2\lambda+1)\Big((-1)^{|a_4|}D^3(a_4) + (2\lambda-1)(-1)^{|a_3|}D^2(a_3) \\ +(-1)^{|a_2|}D(a_2) + (4\lambda+1)(-1)^{|a_1|}a_1\Big)D \end{cases}$$

$$\left\{\begin{array}{l} R^2_{\lambda,\frac{3}{2}+\lambda}(A) = \Big(2a_4' + D(a_3)\Big)D^2 + \Big(-(-1)^{|a_4|}D^3(a_4) + (-1)^{|a_3|}a_3'\Big)D \\ -2\lambda\Big(2a_4'' + D^3(a_3)\Big). \end{array}\right.$$

Proof. The generator Q is obtained by the composition of the bilinear operator (4.51) and the linear projection (4.56). The generator $T^2_{\lambda,\mu}$ (and so $T^2_{0,\mu}$) is obtained by the composition of the bilinear operator (4.46) and the linear projections (4.67), $V^2_{-\frac{1}{2},1}$ and $W^2_{-\frac{1}{2},1}$ are obtained by the composition of the bilinear operator (4.46) and linear projections (4.69). Generators $S^2_{0,\mu}$ and $R^2_{\lambda,\frac{3}{2}+\lambda}$ are obtained by the composition of the linear projection (4.57) and bilinear operators (4.48) and (4.47) respectively. Finally, $L^2_{\lambda,\frac{1}{2}-\lambda}$ is given by composing the bilinear operator (4.45) and the projection (4.70). $\square$

THEOREM 5.6. *The algebra* $\mathfrak{C}^{\frac{5}{2}}_{\lambda,\mu}$ *is trivial except*

$\mathfrak{C}^{\frac{5}{2}}_{0,\frac{1}{2}} = \mathrm{Span}\{\mathrm{Id}, P_0, P_0^*, P_1, C\}$

$\mathfrak{C}^{\frac{5}{2}}_{0,2} = \mathrm{Span}\{\mathrm{Id}, P_0, X^{\frac{5}{2}}_{0,2}\}$

$\mathfrak{C}^{\frac{5}{2}}_{-\frac{1}{2},1} = \mathrm{Span}\{\mathrm{Id}, C, T^{\frac{5}{2}}_{-\frac{1}{2},1}\}$

$\mathfrak{C}^{\frac{5}{2}}_{-\frac{3}{2},\frac{1}{2}} = \mathrm{Span}\{\mathrm{Id}, P_0^*, Y^{\frac{5}{2}}_{-\frac{3}{2},\frac{1}{2}}\}$

$\mathfrak{C}^{\frac{5}{2}}_{-\frac{1+\sqrt{3}}{4},\frac{3+\sqrt{3}}{4}} = \mathrm{Span}\{\mathrm{Id}, C, Z^{\frac{5}{2}}_{-\frac{1+\sqrt{3}}{4},\frac{3+\sqrt{3}}{4}}\}$

$\mathfrak{C}^{\frac{5}{2}}_{\frac{-1+\sqrt{3}}{4},\frac{3-\sqrt{3}}{4}} = \mathrm{Span}\{\mathrm{Id}, C, Z^{\frac{5}{2}}_{\frac{-1+\sqrt{3}}{4},\frac{3-\sqrt{3}}{4}}\}$

$\mathfrak{C}^{\frac{5}{2}}_{0,\mu} = \mathrm{Span}\{\mathrm{Id}, P_0\}, \mu \neq \frac{1}{2}, 2$

$\mathfrak{C}^{\frac{5}{2}}_{\lambda,\frac{1}{2}} = \mathrm{Span}\{\mathrm{Id}, P_0^*\}, \lambda \neq 0, -\frac{3}{2}$

$\mathfrak{C}^{\frac{5}{2}}_{\lambda,\frac{1}{2}-\lambda} = \mathrm{Span}\{\mathrm{Id}, C\}, \lambda \neq 0, -\frac{1+\sqrt{3}}{4}, \frac{-1+\sqrt{3}}{4}$

$\mathfrak{C}^{2}_{\lambda,\frac{3}{2}+\lambda} = \mathrm{Span}\{\mathrm{Id}, F^{\frac{5}{2}}_{\lambda,\frac{3}{2}+\lambda}\}, \lambda \neq 0, -1$

$\mathfrak{C}^{\frac{5}{2}}_{\lambda,\mu} = \mathrm{Span}\{\mathrm{Id}, Z^{\frac{5}{2}}_{\lambda,\mu}\}, (2\lambda+2)(2\lambda+1) + (2\lambda+\frac{1}{2})(2\mu-2\lambda-4) = 0,\ \lambda \neq -\frac{1+\sqrt{3}}{4}, \frac{-1+\sqrt{3}}{4}$

where for $A = \sum_{\ell=0}^{5} a_\ell \overline{D}^\ell$,

$$\left\{\begin{array}{l} X^{\frac{5}{2}}_{0,2}(A) = \Big(-2(-1)^{|a_5|}D^4(a_5) + (-1)^{|a_4|}D^3(a_4)\Big)D \\ +\Big(2D^3(a_5) + D^2(a_4)\Big)D^2 + \Big(2(-1)^{|a_5|}D^2(a_5) - (-1)^{|a_4|}D(a_4)\Big)D^3 \end{array}\right.$$

$$\begin{cases} Y^{\frac{5}{2}}_{-\frac{3}{2},\frac{1}{2}} = -3\Big(D^5(a_5)+D^4(a_4)\Big)+2\Big((-1)^{|a_5|}D^4(a_5)-(-1)^{|a_4|}D^3(a_4)\Big)D \\ -2\Big(D^3(a_5)+D^2(a_4)\Big)D^2+\Big((-1)^{|a_5|}D^2(a_5)-(-1)^{|a_4|}D(a_4)\Big)D^3 \\ Z^{\frac{5}{2}}_{\lambda,\mu} = 2\lambda(2\lambda+1)(2\lambda+2)D^5(a_5)+2\lambda(2\lambda+1)(2\mu-2\lambda-4)D^4(a_4) \\ -\lambda(2\lambda+1)(2\mu-2\lambda-4)D^3(a_3)+\lambda(2\mu-2\lambda-4)(2\mu-2\lambda-3)D^2(a_2) \\ +\frac{1}{2}\Big(2(2\lambda+1)(2\lambda+2)D^4(a_5)+2(2\lambda+1)(2\mu-2\lambda-4)D^3(a_4) \\ -(2\lambda+1)(2\mu-2\lambda-4)D^2(a_3)+(2\mu-2\lambda-4)(2\mu-2\lambda-3)D^4D(a_2)\Big)\overline{D} \\ +(\mu-\lambda-1)\Big(2(2\lambda+1)(2\lambda+2)D^3(a_5)+2(2\lambda+1)(2\mu-2\lambda-4)D^2(a_4) \\ -(2\lambda+1)(2\mu-2\lambda-4)D(a_3)+(2\mu-2\lambda-4)(2\mu-2\lambda-3)a_2\Big)\overline{D}^2 \\ F^{\frac{5}{2}}_{\lambda,\frac{3}{2}+\lambda} = \Big(2(\lambda+1)(-1)^{|a_5|}a_5''+(-1)^{|a_4|}D^3(a_4)+\frac{1}{2}(-1)^{|a_3|}a_3'\Big)D \end{cases}$$

Proof. Generators $X^{\frac{5}{2}}_{0,2}$ and $Y^{\frac{5}{2}}_{-\frac{3}{2},\frac{1}{2}}$ are respectively obtained by the composition of the linear projection (4.57) and the bilinear operators (4.50) and (4.52). The generator $Z^{\frac{5}{2}}_{\lambda,\mu}$ (and so $Z^{\frac{5}{2}}_{-\frac{1+\sqrt{3}}{4},\frac{3+\sqrt{3}}{4}}$, $Z^{\frac{5}{2}}_{\frac{-1+\sqrt{3}}{4},\frac{3-\sqrt{3}}{4}}$) is given by the composition of the bilinear operator (4.46) and the projection (4.71) for suitable (λ,μ) and k. $F^{\frac{5}{2}}_{\lambda,\frac{3}{2}+\lambda}$ is given by the composition of the projection (4.68) and the bilinear operator (4.47). $\square$

THEOREM 5.7. *The algebra* $\mathfrak{C}^3_{\lambda,\mu}$ *is trivial except*

$$\mathfrak{C}^3_{0,\frac{1}{2}} = \mathrm{Span}\{\mathrm{Id}, P_0, P_0^*, P_1, C\}$$
$$\mathfrak{C}^3_{0,\mu} = \mathrm{Span}\{\mathrm{Id}, P_0\}, \mu \neq \frac{1}{2}$$
$$\mathfrak{C}^3_{\lambda,\frac{1}{2}} = \mathrm{Span}\{\mathrm{Id}, P_0^*\}, \lambda \neq 0$$
$$\mathfrak{C}^3_{\lambda,\frac{1}{2}-\lambda} = \mathrm{Span}\{\mathrm{Id}, C\}, \lambda \neq 0.$$

Proof. Straightforward. $\square$

6. PROOF OF THE MAIN THEOREM

This section is devoted to the proof of the main theorem, *i.e.*, Theorem 5.1. We will express the invariance of a supersymmetry with respect to the algebra $\mathcal{K}(1)$ starting with the most simple vector fields, the affine subalgebra, and then expressing the invariance with respect to an additional field, namely $X_{x^2\theta}$.

6.1. Aff-INVARIANT OPERATORS

We consider the affine subalgebra of the contact superalgebra $\mathcal{K}(1)$

$$\mathrm{Aff} := \mathrm{Span}\{X_1, X_x, X_\theta\}. \tag{6.74}$$

Obviously, the affine algebra acts on the module $\mathfrak{D}^k_{\lambda,\mu}$. In [13], we gave a characterization of Aff-supersymmetries, *i.e.*, linear differential mapping $T : \mathfrak{D}^k_{\lambda,\mu} \to \mathfrak{D}^k_{\lambda,\mu}$ commuting with the Aff-action on $\mathfrak{D}^k_{\lambda,\mu}$. Let us first recall this result.

THEOREM 6.1 ([13]). *The algebra of* Aff*-supersymmetries is given by the set of linear operators $T : \mathfrak{D}^k_{\lambda,\mu} \longrightarrow \mathfrak{D}^k_{\lambda,\mu}$ such that*

$$T\left(a\overline{D}^{\ell}\right) = \sum_{i=0}^{\ell} T^{\ell}_i D^i(a)\overline{D}^{\ell-i}, \forall \ell \in \{0, 1, \cdots, 2k\}, \tag{6.75}$$

where T^{ℓ}_i are arbitrary constants.

The following corollary is then immediate.

COROLLARY 6.2. *A differential mapping $T : \mathfrak{D}^k_{\lambda,\mu} \to \mathfrak{D}^k_{\lambda,\mu}$ is* Aff*-invariant if and only if, for all ℓ in $\{0, 1, \cdots, 2k\}$, after identification (2.23), the restriction of T to the homogeneous component $\mathfrak{F}_{\mu-\lambda-\frac{\ell}{2}}$ is given by the rule*

$$T(a\alpha^{-\frac{\ell}{2}}) = \alpha^{\mu-\lambda} \sum_{i=0}^{\ell} T^{\ell}_i D^i(a)\alpha^{-\frac{\ell-i}{2}}. \tag{6.76}$$

Remark 6.3. It is easy to see that the operator T given in (6.1) (or 6.2) is even. So, in the sequel, the $\mathcal{K}(1)$-invariance of T will be stated as

$$[\mathfrak{L}^{\lambda,\mu}_X, T] = \mathfrak{L}^{\lambda,\mu}_X \circ T - T \circ \mathfrak{L}^{\lambda,\mu}_X = 0,\ \forall X \in \mathcal{K}(1). \tag{6.77}$$

6.2. PROOF OF THEOREM 5.1

Now, to achieve the proof of Theorem 5.1, we shall provide several steps. Note that, in the sequel, we brought in our calculations to distinguish even and odd cases since the expression of $\overline{D}^i, i \in \mathbb{N}$, depends on the parity of i.

LEMMA 6.4. *The action $\mathcal{L}^{\lambda,\mu}_{X_{x^2\theta}}$ defined in (2.24) is given by the rule*

$\forall \ell \in \mathbb{N}$,
(6.78)
$$\begin{cases} \mathcal{L}^{\lambda,\mu}_{X_{x^2\theta}}(a\alpha^{-\ell}) = \Big(\theta x^2 a' + \frac{1}{2}x^2\overline{D}(a) + (2\mu - 2\lambda - 2\ell)\theta x a\Big)\alpha^{-\ell} \\ +\ell x(-1)^{|a|}a\alpha^{-\ell+\frac{1}{2}} + \ell\theta a(2\lambda+\ell-1)\alpha^{-\ell+1} - \frac{\ell(\ell-1)}{2}(-1)^{|a|}a\alpha^{-\ell+\frac{3}{2}} \\ \mathcal{L}^{\lambda,\mu}_{X_{x^2\theta}}(a\alpha^{-(\ell+\frac{1}{2})}) = \Big(\theta x^2 a' + \frac{1}{2}x^2\overline{D}(a) + (2\mu - 2\lambda - 2\ell - 1)\theta x a\Big)\alpha^{-\ell-\frac{1}{2}} \\ +(2\lambda+\ell)x(-1)^{|a|}a\alpha^{-\ell} + \ell(2\lambda+\ell)\theta a\alpha^{-\ell+\frac{1}{2}} - \ell(2\lambda + \frac{\ell-1}{2})(-1)^{|a|}a\alpha^{-\ell+1}. \end{cases}$$

Proof. By a direct computation using (2.15), (2.23) and (2.24). □

LEMMA 6.5. *Let T as in (6.2). Then T is invariant under the action of the vector field $X_{x^2\theta}$ if and only if the scalars T_i^ℓ satisfy the following relationships: $\forall (s,p) \in \mathbb{N}^2, s \le p \le k$,*

(6.79)
$$\begin{cases} 0 = sT^{2p}_{2s} - (2\lambda+p-s)T^{2p}_{2s-1} - pT^{2p-1}_{2s-1} \\ 0 = s(2\mu - 2\lambda - 2p + \frac{s-1}{2})T^{2p}_{2s} - (p-s+1)(2\lambda+p-s)T^{2p}_{2s-2} \\ -(p-s+1)(2\lambda+\frac{p-s}{2})T^{2p}_{2s-3} + psT^{2p-1}_{2s-1} - p(2\lambda+p-1)T^{2p-2}_{2s-2} - \frac{p(p-1)}{2}T^{2p-3}_{2s-3} \\ 0 = \frac{s(s-1)}{2}T^{2p}_{2s} + (p-s+1)(2\lambda+\frac{p-s}{2})T^{2p}_{2s-3} - p(s-1)T^{2p-1}_{2s-1} + \frac{p(p-1)}{2}T^{2p-3}_{2s-3} \\ 0 = (2\mu - 2\lambda - 2k + s - 1)T^{2p}_{2s-1} - (p-s+1)T^{2p}_{2s-2} + pT^{2p-1}_{2s-2} \\ 0 = (2\mu - 2\lambda - 2k + \frac{s}{2} - 1)(s-1)T^{2p}_{2s-1} + \frac{(p-s+2)(p-s+1)}{2}T^{2p}_{2s-4} \\ +p(s-1)T^{2p-1}_{2s-2} + p(2\lambda+p-1)T^{2p-2}_{2s-3} - \frac{p(p-1)}{2}T^{2p-3}_{2s-4} \\ 0 = \frac{s(s-1)}{2}T^{2p}_{2s-1} - (p-s+1)(2\lambda+p-s+1)T^{2p}_{2s-3} - \frac{(p-s+2)(p-s+1)}{2}T^{2p}_{2s-4} \\ -p(s-1)T^{2p-1}_{2s-2} + \frac{p(p-1)}{2}T^{2p-3}_{2s-4}. \end{cases}$$

$$\forall (s,p) \in \mathbb{N}^2, s \le p \le k - \frac{1}{2},$$
(6.80)
$$\begin{cases} 0 = sT^{2p+1}_{2s} - (p-s+1)T^{2p+1}_{2s-1} - (2\lambda+p)T^{2p}_{2s-1} \\ 0 = (2\mu - 2\lambda - 2p + s - 1)T^{2p+1}_{2s+1} - (2\lambda+p-s)T^{2p+1}_{2s} + (2\lambda+p)T^{2p}_{2s} \\ 0 = s(2\mu - 2\lambda - 2p + \frac{s-3}{2})T^{2p+1}_{2s+1} - (p-s+1)(2\lambda+\frac{p-s}{2})T^{2p+1}_{2s-2} \\ +s(2\lambda+p)T^{2p}_{2s} + p(2\lambda+p)T^{2p-1}_{2s-1} - p(2\lambda+\frac{p-1}{2})T^{2p-2}_{2s-2} \\ 0 = \frac{s(s+1)}{2}T^{2p+1}_{2s+1} - (p-s+1)(2\lambda+p-s)T^{2p+1}_{2s-1} \\ -(p-s+1)(2\lambda+\frac{p-s}{2})T^{2p+1}_{2s-2} - s(2\lambda+p)T^{2p}_{2s} + p(2\lambda+\frac{p-1}{2})T^{2p-2}_{2s-2} \end{cases}$$

$$\begin{cases} 0 = s(2\mu - 2\lambda - 2p + \frac{s-3}{2})T_{2s}^{2p+1} - (p-s+1)(2\lambda + p - s + 1)T_{2s-2}^{2p+1} \\ -\frac{(p-s+2)(p-s+1)}{2}T_{2s-3}^{2p+1} + s(2\lambda + p)T_{2s-1}^{2p} + p(2\lambda + p)T_{2s-2}^{2p-1} \\ -p(2\lambda + \frac{p-1}{2})T_{2s-3}^{2p-2} \\ 0 = s(s-1)T_{2s}^{2p+1} + (p-s+2)(p-s+1)T_{2s-3}^{2p+1} + 2p(s-1)T_{2s-1}^{2p} \\ +2p(2\lambda + \frac{p-1}{2})T_{2s-3}^{2p-2}. \end{cases}$$

Proof. It is well known that, if we identify S^1 with $\mathbb{RP}^1$ with homogeneous coordinates $(x_1 : x_2)$ and choose the affine coordinate $x = x_1/x_2$, the vector fields $\frac{\mathrm{d}}{\mathrm{d}x}, x\frac{\mathrm{d}}{\mathrm{d}x}, x^2\frac{\mathrm{d}}{\mathrm{d}x}$ are globally defined and correspond to the standard projective structure on $\mathbb{RP}^1$. In this adapted coordinate the action of the algebra $\mathfrak{sl}(2) = \mathrm{Span}\left(\frac{\mathrm{d}}{\mathrm{d}x}, x\frac{\mathrm{d}}{\mathrm{d}x}, x^2\frac{\mathrm{d}}{\mathrm{d}x}\right)$ is well defined. Thus, in the corresponding adapted coordinate (x,θ) of $S^{1|1}$, since (see 2.8)

$$X_{x^2\theta} = -\theta x^2\overline{D}^2 + \frac{1}{2}D(\theta x^2)\overline{D} = \frac{1}{2}x^2(\theta\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}}{d\theta}),$$

the vector field $X_{x^2\theta}$ is further globally defined.

Now, let $P = a\alpha^{-p}, p \in \mathbb{N}$. We have by (6.2)

$$\begin{aligned} T(P) &= \alpha^{\mu-\lambda}\sum_{s=0}^{2p} T_s^{2p} D^s(a)\alpha^{-p+\frac{s}{2}} \\ &= \alpha^{\mu-\lambda}\sum_{s=0}^{p} T_{2s}^{2p} D^{2s}(a)\alpha^{-p+s} + \alpha^{\mu-\lambda}\sum_{s=0}^{p-1} T_{2s+1}^{2p} D^{2s+1}(a)\alpha^{-p+s+\frac{1}{2}}. \end{aligned}$$

Then using 6.78 one has

$$\begin{aligned} \mathcal{L}^{\lambda,\mu}_{X_{x^2\theta}}(T(P)) = \sum_{s=0}^{p} T_{2s}^{2p}\Big(&\theta x^2 D^{2s+2}(a) + \frac{1}{2}x^2\overline{D}(D^{2s}(a)) \\ &+ (2\mu - 2\lambda - 2p + 2s)\theta x D^{2s}(a)\Big)\alpha^{-p+s} \\ &+ \sum_{s=0}^{p-1} T_{2s}^{2p}\theta(p-s)(2\lambda + p - s - 1)D^{2s}(a)\alpha^{-p+s+1} \\ &+ \sum_{s=0}^{p-1} T_{2s}^{2p}(p-s)x(-1)^{|a|}D^{2s}(a)\alpha^{-p+s+\frac{1}{2}} \\ &- \sum_{s=0}^{p-2} T_{2s}^{2p}\frac{(p-s)(p-s-1)}{2}(-1)^{|a|}D^{2s}(a)\alpha^{-p+s+\frac{3}{2}} \end{aligned}$$

$$+\sum_{s=0}^{p-1}T_{2s+1}^{2p}\Big(\theta x^2D^{2s+3}(a)+\frac{1}{2}x^2\overline{D}(D^{2s+1}(a))$$
$$+(2\mu-2\lambda-2p+2s+1)\theta xD^{2s+1}(a)\Big)\alpha^{-p+s+\frac{1}{2}}$$
$$+\sum_{s=0}^{p-2}T_{2s+1}^{2p}\theta(p-s-1)(2\lambda+p-s-1)D^{2s+1}(a)\alpha^{-p+s+\frac{3}{2}}$$
$$+\sum_{s=0}^{p-2}T_{2s+1}^{2p}\theta(p-s-1)(2\lambda+\frac{p-s-2}{2})(-1)^{|a|}D^{2s+1}(a)\alpha^{-p+s+2}$$
$$-\sum_{s=0}^{p-1}T_{2s}^{2p+1}(2\lambda+p-s-1)x(-1)^{|a|}D^{2s+1}(a)\alpha^{-p+s+1}.$$

On the other hand,

$$T\Big(\mathcal{L}_{X_{x^2\theta}}^{\lambda,\mu}(P)\Big)=\sum_{s=0}^{p}T_{2s}^{2p}\Big[\theta x^2D^{2s+2}(a)+\frac{1}{2}x^2D^{2s}\Big(\overline{D}(a)\Big)$$
$$+(2\mu-2\lambda-2p+2s)\theta xD^{2s}(a)+sxD^{2s-2}\Big(\overline{D}(a)\Big)$$
$$+(2\mu-2\lambda-2p+s-1)s\theta D^{2s-2}(a)+\frac{s(s-1)}{2}D^{2s-4}\Big(\overline{D}(a)\Big)\Big]\alpha^{-p+s}$$
$$+\sum_{s=0}^{p-1}T_{2s}^{2p+1}\Big[x^2D^{2s+2}(a)-x^2\theta D^{2s+3}(a)+\frac{1}{2}x^2D^{2s+1}\Big(\overline{D}(a)\Big)$$
$$+(2\mu-2\lambda-2p+2s)xD^{2s}(a)-(2\mu-2\lambda-2p+2s-1)x\theta D^{2s+1}(a)$$
$$-(2\mu-2\lambda-2p+s-2)s\theta D^{2s-1}(a)+sxD^{2s-1}\Big(\overline{D}(a)\Big)$$
$$+(2\mu-2\lambda-2p+s-1)sD^{2s-2}(a)+\frac{s(s-1)}{2}D^{2s-3}\Big(\overline{D}(a)\Big)\Big]\alpha^{-p+s+\frac{1}{2}}$$
$$+p\sum_{s=0}^{p-1}T_{2s}^{2p-1}\Big[xD^{2s}\Big((-1)^{|a|}a\Big)+sD^{2s-2}\Big((-1)^{|a|}a\Big)\Big]\alpha^{-p+s+\frac{1}{2}}$$
$$+p\sum_{s=0}^{p-1}T_{2s+1}^{2p-1}\Big[\theta D^{2s}\Big((-1)^{|a|}a\Big)+xD^{2s+1}\Big((-1)^{|a|}a\Big)+sD^{2s-1}\Big((-1)^{|a|}a\Big)\Big]\alpha^{-p+s+1}$$
$$+p(2\lambda+p-1)\Big(\sum_{s=0}^{p-1}T_{2s}^{2p-2}\theta D^{2s}(a)\alpha^{-p+s+1}$$
$$+\sum_{s=0}^{p-2}T_{2s+1}^{2p-2}\Big[D^{2s}(a)-\theta D^{2s+1}(a)\Big]\alpha^{-p+s+\frac{3}{2}}\Big)$$

$$- \frac{p(p-1)}{2}\Big(\sum_{s=0}^{p-2} T_{2s}^{2p-3} D^{2s}\Big((-1)^{|a|}a\Big)\alpha^{-p+s+\frac{3}{2}} + \sum_{s=0}^{p-2} T_{2s+1}^{2p-3} D^{2s+1}\Big((-1)^{|a|}a\Big)\alpha^{-p+s+2}\Big).$$

Thus, taking into account Remark 6.3, equations (6.79) are already available. A similar calculation should be approached to obtain (6.5). □

Proposition 6.6. *Let $n \in \frac{1}{2}\mathbb{N}^*, n \geq 2$ and $T : \mathfrak{D}^n_{\lambda,\mu} \to \mathfrak{D}^n_{\lambda,\mu}$ a differential linear operator $\mathcal{K}(1)$-invariant. Then T is completely determined by its restriction to the subspace of second-order differential operators $\mathfrak{D}^2_{\lambda,\mu}$.*

Proof. Since $[X_\theta, X_{\theta f}] = X_f, \forall f \in C^\infty(S^1)$, as a superalgebra, $\mathcal{K}(1)$ is generated by the set of odd vector fields $\{X_{\theta f}, f \in C^\infty(S^1)\}$. Moreover, we have $\frac{1}{2}[X_1, X_{x^2\theta}] = X_{x\theta}$, that is, for an Aff-invariant operator, the invariance with respect to $X_{x\theta}$ holds as soon as the invariance is with respect to $X_{x^2\theta}$. Thus, let T an Aff-invariant operator commuting with the action of $X_{x^2\theta}$, $\ell = 2p+1 \in \mathbb{N}, \ell \leq 2n$ and $p \geq 2$ (respectively $\ell = 2p$ and $p \geq 3$). Then

$$T(a\alpha^{-\frac{\ell}{2}}) = \alpha^{\mu-\lambda} \sum_{s=0}^{\ell} T_s^\ell D^s(a)\alpha^{-\frac{\ell-s}{2}},$$

moreover the scalars T_s^{2p+1} obey equation (6.5) (respectively (6.79)). Suppose that $T_s^q = 0, \forall q < \ell, \forall s < \ell$, that is we get the equations

(6.81)

$$\begin{cases} 0 = sT_{2s}^{2p+1} - (p-s+1)T_{2s-1}^{2p+1} \\ 0 = (2\mu - 2\lambda - 2k + s - 1)T_{2s+1}^{2p+1} - (2\lambda + p - s)T_{2s}^{2p+1} \\ 0 = s(2\mu - 2\lambda - 2p + \frac{s-3}{2})T_{2s+1}^{2p+1} - (p-s+1)(2\lambda + \frac{p-s}{2})T_{2s-2}^{2p+1} \\ 0 = \frac{s(s+1)}{2}T_{2s+1}^{2p+1} - (p-s+1)(2\lambda + p - s)T_{2s-1}^{2p+1} \\ 0 = s(2\mu - 2\lambda - 2p + \frac{s-3}{2})T_{2s}^{2p+1} - (p-s+1)(2\lambda + p - s + 1)T_{2s-2}^{2p+1} \\ \quad -\frac{(p-s+2)(p-s+1)}{2}T_{2s-3}^{2p+1} \\ 0 = s(s-1)T_{2s}^{2p+1} + (p-s+2)(p-s+1)T_{2s-3}^{2p+1} \end{cases}$$

We clearly observe from this system that if $T_s^{2p+1} = 0, \forall s \leq 5$, then, by induction, we get $T_s^{2p+1} = 0, \forall i \leq 2p+1$. We achieve the proof of the proposition by solving the system (6.81) for $s \leq 5$ and proving that, $\forall(\lambda,\mu) \in \mathbb{C}^2$, $T_s^{2p+1} = 0, \forall s \leq 5$. □

Now, we may end the proof of our main theorem.

Proof of Theorem 5.1. We solve the systems (6.79) and (6.5) explicitly for $k \leq 3$ and get the result in this case. From Proposition 6.6, it follows that the dimension of the algebras $\mathfrak{C}^k_{\lambda,\mu}$ of differential supersymmetries, with $k \geq 2$, can only decrease as k becomes $k + \frac{1}{2}$. On the other hand, for ≥ 3, one has

$$\mathrm{Span}\{\mathrm{Id}, P_0, P_0^*, P_1, C\} \subset \mathfrak{C}^k_{0,\frac{1}{2}},$$

$$\mathrm{Span}\{\mathrm{Id}, P_0\} \subset \mathfrak{C}^k_{0,\mu}, \mu \neq \frac{1}{2},$$

$$\mathrm{Span}\{\mathrm{Id}, P_0^*\} \subset \mathfrak{C}^k_{\lambda,\frac{1}{2}}, \lambda \neq 0,$$

and

$$\mathrm{Span}\{\mathrm{Id}, C\} \subset \mathfrak{C}^k_{\lambda,\frac{1}{2}-\lambda}, \lambda \neq 0,$$

which gives us a lower bound for the dimension of $\mathfrak{C}^k_{\lambda,\mu}$. □

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