ROGERS-RAMANUJAN TYPE IDENTITIES FOR SPLIT $(n + t)$–COLOR PARTITIONS

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Using split $(n + t)$–color partitions as an elementary tool, three generalized q–series have been interpreted combinatorially. For some particular cases these series can be written as infinite products which produce three "Sum-Product" identities, which lead, in turn, to elegant three new Rogers-Ramanujan type identities for split $(n + t)$–color partitions. These new identities reveal the fact that our main results have the potential to yield Rogers-Ramanujan-MacMahon type partition identities linking split $(n+t)$–color partitions with ordinary partitions.

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1. INTRODUCTION

In the literature, we learn that analytical number theory and combinatorics are nicely connected with each other [9, 16, 18]. In combinatorics, partition theory [6] studies various enumeration problems related to partitions and is closely related to q–series.

For λ to be a natural number, the rising q–factorial of a with base q is defined by $(a; q)_0 = 1$ and $(a; q)_\lambda = (1-a)(1-aq)\cdots(1-aq^{\lambda-1})$, where $|q| < 1$. Any series involving this rising q–factorial is called a q–series (or basic series or Eulerian series). Let us recall the celebrated Rogers-Ramanujan identities:

$$\sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2}}{(q;q)_\lambda} = \prod_{\lambda=1}^{\infty} \frac{1}{(1-q^{5\lambda-1})(1-q^{5\lambda-4})},$$

$$\sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2+\lambda}}{(q;q)_\lambda} = \prod_{\lambda=1}^{\infty} \frac{1}{(1-q^{5\lambda-2})(1-q^{5\lambda-3})}.$$

The sum-product identities similar to Rogers-Ramanujan identities are referred as Rogers-Ramanujan type identities. MacMahon [17] interpreted the famous Rogers-Ramanujan identities combinatorially by using ordinary partitions, now popularly known as Rogers-Ramanujan-MacMahon partition identities. After this, several identities of Rogers-Ramanujan type have been interpreted combinatorially by several mathematicians [4, 8, 10–12, 19] using ordinary partitions. In 1985, Agarwal [1] introduced a new class of partitions which he named n–color partitions. In 1987, Agarwal and Andrews [3] generalized these n–color partitions to $(n+t)$–color partitions. These new partitions were used to interpret many more q–series identities combinatorially in [2, 3, 14, 15]. Recently, Agarwal and Sood [5] introduced split $(n+t)$–color partitions which further generalize Agarwal-Andrews $(n+t)$–color partitions. The authors [5] have shown that this new set of partitions is very helpful to interpret several q–series combinatorially which cannot be interpreted combinatorially using ordinary partitions or $(n+t)$–color partitions. Using split $(n+t)$–color partitions, Agarwal and Sood [5] interpreted two basic functions of Gordon-MacIntosh [13] combinatorially. The authors [5] also posed an open problem: "Is it possible to find Rogers-Ramanujan type identities for split $(n+t)$–color partitions?"

The purpose of this paper is to address this problem. In Section 2, we interpret combinatorially three generalized q–series by the aid of split $(n+t)$–color partitions which for some particular cases produce identities of Rogers-Ramanujan type for split $(n+t)$–color partitions. These results yield three new Rogers-Ramanujan-MacMahon type partition identities. Before we state our main results, we first recall some formal definitions:

Definition 1.1 ([6]). A *partition* of positive integer n is a finite non-increasing sequence of positive integers $\alpha_1, \alpha_2, \dots \alpha_r$ such that

$$\sum_{i=1}^{r} \alpha_i = n$$

where $\alpha_i's$ are called parts of the partition. The number of partitions of n is denoted by $p(n)$.

For example, $p(3) = 3$, where the relevant ordinary partitions of 3 are 3, 21, 111.

Definition 1.2 ([3]). A *partition with "$(n+t)$ copies of n"*, $t \geq 0$, is a partition in which a part of size n, $n \geq 0$, can come in $(n+t)$ different colors denoted by subscripts: $n_1, n_2, \dots, n_{n+t}$.

Note that zeros are permitted if and only if t is greater than or equal to one. Also, zeros are not permitted to repeat in any partition.

Remark 1.1. We note that if we take $t = 0$, then these are nothing but the n–color partitions.

Definition 1.3. The *weighted difference* of two parts a_p, b_q $(a \geq b)$ is defined by $a-b-p-q$ and is denoted by $((a_p - b_q))$.

In [5] the *split $(n+t)$–color partitions* are defined as:

Definition 1.4. Let a_p be a part in an $(n+t)$–color partition of a non-negative integer μ. We split the color 'p' into two parts-'the green part' and 'the red part' and denote them by 'g' and 'r' respectively, such that $1 \leq g \leq p$, $0 \leq r \leq p-1$ and $p = g+r$. An $(n+t)$–color partition in which each part is split in this manner is called a split $(n+t)$–color partition.

Example 1.1. In 5_{2+1}, the green part is 2 and the red part is 1.

Remark 1.2. We note that if $r=0$, then we will not write it. Thus for instance, we will write 5_3 for 5_{3+0}.

2. SPLIT $(n+t)$–COLOR PARTITIONS AS A COMBINATORIAL TOOL

In this section, we will use split $(n+t)$–color partitions as a combinatorial tool to interpret three generalized q–series combinatorially which lead to Rogers-Ramanujan type identities for split $(n+t)$–color partitions.

Definition 2.1. Let $T = \{-1, 1, 3, 5, 7, \cdots\}$. For $|q| < 1$, $i \;\varepsilon\; T$ and $1 \leq k \leq 3$, we define $f_k^i(q)$ by

$$f_1^i(q) = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda[1+(\lambda-1)(i+3)/2]}(-q;q^2)_\lambda}{(q^4;q^4)_\lambda (q;q^2)_\lambda} \tag{2.1}$$

$$f_2^i(q) = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda[3+(\lambda-1)(i+3)/2]}(-q;q^2)_\lambda}{(q^4;q^4)_\lambda (q;q^2)_\lambda} \tag{2.2}$$

$$f_3^i(q) = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda[1+(\lambda+1)(i+3)/2]}(-q;q^2)_\lambda}{(q^4;q^4)_\lambda (q;q^2)_{\lambda+1}} \tag{2.3}$$

For $i = -1$, (2.1)–(2.3) produces the following three q–series "Sum-Product" identities of Rogers-Ramanujan type, respectively.

$$\sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2}(-q;q^2)_\lambda}{(q^4;q^4)_\lambda (q;q^2)_\lambda} = \frac{(-q^2;q^{10})_\infty(-q^5;q^{10})_\infty(-q^8;q^{10})_\infty(-q;q^2)_\infty}{(q^{10};q^{10})_\infty^{-1}(q^3;q^{10})_\infty^{-1}(q^7;q^{10})_\infty^{-1}(q^2;q^2)_\infty}, \tag{2.4}$$

$$\sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2+2\lambda}(-q;q^2)_\lambda}{(q^4;q^4)_\lambda (q;q^2)_\lambda} = \frac{(-q^4;q^{10})_\infty(-q^5;q^{10})_\infty(-q^6;q^{10})_\infty(-q;q^2)_\infty}{(q^{10};q^{10})_\infty^{-1}(q;q^{10})_\infty^{-1}(q^9;q^{10})_\infty^{-1}(q^2;q^2)_\infty}, \tag{2.5}$$

$$\sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2+2\lambda}(-q;q^2)_\lambda}{(q^4;q^4)_\lambda(q;q^2)_{\lambda+1}} = \frac{(-q;q^2)_\infty}{(q^{20};q^{20})_\infty(q^5;q^{20})_\infty(q^{15};q^{20})_\infty(q^2;q^2)_\infty}. \tag{2.6}$$

The identities (2.4)–(2.6) are appearing in Chu and Zhang compendium [7]. We shall prove that the q–series (2.1)–(2.3) have their combinatorial counterparts in the form of the following theorems, respectively, which in conjunction with the identities (2.4)–(2.6) yield three new Rogers-Ramanujan type identities for split $(n+t)$–color partitions.

2.1. COMBINATORIAL IDENTITIES

THEOREM 2.1. *Let $A_1^i(\mu)$ denote the number of split n–color partitions of μ such that (i) if a_p is the smallest or the only part in the partition, then $a \equiv p(mod4)$, (ii) the red part of the subscripts cannot exceed 1 and (iii) the weighted difference between any two consecutive parts is $> i$ and is $\equiv i+1(mod4)$. Then*

$$\sum_{\mu=0}^{\infty} A_1^i(\mu)q^\mu = f_1^i(q)$$

THEOREM 2.2. *Let $A_2^i(\mu)$ denote the number of split n–color partitions of μ such that (i) if a_p is the smallest or the only part in the partition, then $a \equiv p+2(mod4)$, (ii) the red part of the subscripts cannot exceed 1, (iii) all parts are greater than or equal to 3 and (iv) the weighted difference between any two consecutive parts is $> i$ and is $\equiv i+1(mod4)$. Then*

$$\sum_{\mu=0}^{\infty} A_2^i(\mu)q^\mu = f_2^i(q)$$

THEOREM 2.3. *Let $A_3^i(\mu)$ denote the number of split $(n+2)$–color partitions of μ such that (i) the smallest part or the only part is of the form a_{a+2}, (ii) the red part of the subscripts cannot exceed 1, (iii) the red part of the subscript of the smallest part is 0 and (iv) the weighted difference between any two consecutive parts is $> i$ and is $\equiv i+1(mod4)$. Then*

$$\sum_{\mu=0}^{\infty} A_3^i(\mu)q^\mu = f_3^i(q)$$

Remark 2.1. In the weighted difference condition of the above theorems we consider the whole subscript p and not its parts g and r, separately.

2.2. PROOFS OF THEOREMS 2.1–2.3

To prove these theorems, we study a more general partition function $A_k^i(m,\mu)$ $(1 \le k \le 3)$ which counts the partitions of μ of the kind as described in Theorem 2.j $(1 \le j \le 3)$ with the added restriction that there be exactly m parts. This technique is easily implementable on computer to obtain the tables of $A_k^i(m,\mu)$.

For $1 \le k \le 3$, $f_k^i(z;q)$ will denote the 2–variable generating function

$$f_k^i(z;q) = \sum_{\mu=0}^{\infty}\sum_{m=0}^{\infty} A_k^i(m,\mu) z^m q^\mu, \tag{2.7}$$

where $|q| < 1$ and $|z| < |q|^{-1}$.

2.3. PROOF OF THEOREM 2.1

Proof. We shall prove that

$$\sum_{\mu=0}^{\infty} A_1^i(\mu) q^\mu = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda[1+(\lambda-1)(i+3)/2]}(-q;q^2)_\lambda}{(q^4;q^4)_\lambda (q;q^2)_\lambda}.$$

Let $A_1^i(m,\mu)$ denote the number of partitions enumerated by $A_1^i(\mu)$ into exactly m parts. We shall first prove the identity,

$$\begin{aligned} A_1^i(m,\mu) =& A_1^i(m,\mu-4m) + A_1^i(m-1,\mu-m(i+3)+i+2) + A_1^i(m-1,\mu \\ & - m(i+5)+i+3) + A_1^i(m,\mu-2m+1) - A_1^i(m,\mu-6m+1). \end{aligned} \tag{2.8}$$

To prove Theorem 2.1, we split the partitions enumerated by $A_1^i(m,\mu)$ into four classes:

(i) those that do not contain a_a or $a_{(a-1)+1}$ as a part,
(ii) those that contain 1_1 as a part,
(iii) those that contain 2_{1+1} as a part and
(iv) those that contain $a_a, (a \ge 2)$ or $a_{(a-1)+1}, (a \ge 3)$ as a part.

We now transform the partitions in class (i) by subtracting 4 from each part ignoring the subscripts. Obviously, this transformation will not disturb the inequalities between the parts and so the transformed partition will be of the type enumerated by $A_1^i(m,\mu-4m)$.

Next, we transform the partitions in class (ii) by deleting the part 1_1 and then subtracting $i+3$ from all the remaining parts ignoring the subscripts. The transformed partition will be of the type enumerated by $A_1^i(m-1,\mu-m(i+3)+i+2)$.

Next, we transform the partitions in class (iii) by deleting the part 2_{1+1} and then subtracting $i+5$ from all the remaining parts ignoring the subscripts. The transformed partition will be of the type enumerated by $A_1^i(m-1, \mu - m(i+5)+i+3)$.

Finally, we transform the partitions in class (iv) by replacing a_a by $(a-1)_{a-1}$ or $a_{(a-1)+1}$ by $(a-1)_{(a-2)+1}$ as the case may be and then subtracting 2 from all the remaining parts. This will produce a partition of $\mu-2m+1$ into m parts. It is important to note here that by this transformation we get only those partitions of $\mu-2m+1$ into m parts which contain a part of the form a_a or $a_{(a-1)+1}$. Therefore, the actual number of partitions which belong to class (iv) is $A_1^i(m,\mu-2m+1) - A_1^i(m,\mu-6m+1)$, where $A_1^i(m,\mu-6m+1)$ is the number of partitions of $\mu-2m+1$ into m parts which are free from the parts like a_a or $a_{(a-1)+1}$.

The above transformations are clearly reversible and so establish a bijection between the partitions enumerated by $A_1^i(m,\mu)$ and those enumerated by $A_1^i(m,\mu-4m) + A_1^i(m-1,\mu-m(i+3)+i+2) + A_1^i(m-1,\mu-m(i+5)+i+3) + A_1^i(m,\mu-2m+1) - A_1^i(m,\mu-6m+1)$. This leads to the identity (2.8)

Let

$$h^i(z;q) = \sum_{m,\mu=0}^{\infty} A_1^i(m,\mu) z^m q^\mu \tag{2.9}$$

Substituting for $A_1^i(m,\mu)$ from (2.8) into (2.9) and then simplifying, we get

$$h^i(z;q) = h^i(zq^4;q) + zqh^i(zq^{i+3};q) + zq^2h^i(zq^{i+5};q) + q^{-1}h^i(zq^2;q) - q^{-1}h^i(zq^6;q)$$

setting

$$h^i(z;q) = \sum_{\lambda=0}^{\infty} \alpha_{i,\lambda}(q) z^\lambda$$

and then comparing the coefficients of z^λ, we get

$$\alpha_{i,\lambda}(q) = \frac{q^{1+(\lambda-1)(i+3)}(1+q^{2\lambda-1})\alpha_{i,\lambda-1}(q)}{(1-q^{4\lambda})(1-q^{2\lambda-1})} \tag{2.10}$$

Iterating (2.10) λ–times and observing $\alpha_{i,0}(q)=1$, we may easily get

$$\alpha_{i,\lambda}(q) = \frac{q^{\lambda[1+(\lambda-1)(i+3)/2]}(-q;q^2)_\lambda}{(q^4;q^4)_\lambda(q;q^2)_\lambda} \tag{2.11}$$

Thus

$$h^i(z;q) = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda[1+(\lambda-1)(i+3)/2]}(-q;q^2)_\lambda}{(q^4;q^4)_\lambda(q;q^2)_\lambda} = f_1^i(z;q)$$

Now

$$\sum_{\mu=0}^{\infty} A_1^i(\mu)q^{\mu} = \sum_{\mu=0}^{\infty}\left(\sum_{m=0}^{\infty} A_1^i(m,\mu)\right)q^{\mu} = f_1^i(1;q) = f_1^i(q)$$

This completes the proof of Theorem 2.1. □

2.4. PROOF OF THEOREM 2.2

Proof. As the proof of this theorem is almost similar to the previous theorem so we will omit its detailed proof. We give the outline of the proof of this theorem.

To prove Theorem 2.2, we split the partitions enumerated by $A_2^i(m,\mu)$ into four classes:
(i) those that do not contain a_{a-2} or $a_{(a-3)+1}$ as a part,
(ii) those that contain 3_1 as a part,
(iii) those that contain 4_{1+1} as a part and
(iv) those that contain $a_{a-2},\ (a\geq 4)$ or $a_{(a-3)+1},\ (a\geq 5)$ as a part.

Now by performing some elementary reversible transformations in these classes which establish a bijection between the partitions enumerated by $A_2^i(m,\mu)$ and those enumerated by $A_2^i(m,\mu-4m)+A_2^i(m-1,\mu-m(i+3)+i)+A_2^i(m-1,\mu-m(i+5)+i+1)+A_2^i(m,\mu-2m+1)-A_2^i(m,\mu-6m+1)$. This leads to the following recurrence relation

$$\begin{aligned}A_2^i(m,\mu) =&A_2^i(m,\mu-4m)+A_2^i(m-1,\mu-m(i+3)+i)+A_2^i(m-1,\mu\\&-m(i+5)+i+1)+A_2^i(m,\mu-2m+1)-A_2^i(m,\mu-6m+1).\end{aligned}$$

Now using (2.7) for $k=2$ and after simplification, we get the following q–functional equation

$$\begin{aligned}f_2^i(z;q) =&f_2^i(zq^4;q)+zq^3f_2^i(zq^{i+3};q)+zq^4f_2^i(zq^{i+5};q)+q^{-1}f_2^i(zq^2;q)\\&-q^{-1}f_2^i(zq^6;q)\end{aligned}$$

Now proceeding in the same manner as the previous theorem we get our result. □

2.5. PROOF OF THEOREM 2.3

Proof. Let $C^i(\mu)$ denote the number of split n–color partitions of μ enumerated by $A_1^i(\mu)$ with the added restriction that the smallest part is of the form a_a and let $C^i(m,\mu)$ denote the number of split n–color partitions of μ enumerated by $C^i(\mu)$ into m–parts. Further let

$$g^i(q) = \sum_{\mu=0}^{\infty} C^i(\mu) q^{\mu},$$

$$g^i(z;q) = \sum_{m,\mu=0}^{\infty} C^i(m,\mu) z^m q^{\mu}.$$

Using the arguments of the proof of Theorem 2.1, we see that

$$\begin{aligned} C^i(m,\mu) =& A_1^i(m-1,\mu-m(i+3)+i+2) + \frac{1}{2}\Big[A_1^i(m-1,\mu-m(i+5) \\ &+i+3) + A_1^i(m,\mu-2m+1) - A_1^i(m,\mu-6m+1)\Big]. \end{aligned} \tag{2.12}$$

Translating (2.12) into a q–functional equation, we get

$$\begin{aligned} g^i(z;q) =& zqf_1^i(zq^{i+3};q) + \frac{1}{2}zq^2 f_1(zq^{i+5};q) + \frac{1}{2}q^{-1}f_1^i(zq^2;q) \\ &- \frac{1}{2}q^{-1}f_1^i(zq^6;q) \end{aligned} \tag{2.13}$$

setting

$$g^i(z;q) = \sum_{\lambda=0}^{\infty} \beta_{i,\lambda}(q) z^{\lambda}$$

and then comparing the coefficients of z^{λ} in (2.13), we get

$$\begin{aligned} 2\beta_{i,\lambda}(q) =& 2q^{(i+3)(\lambda-1)+1}\alpha_{i,\lambda-1}(q) + q^{(i+5)(\lambda-1)+2}\alpha_{i,\lambda-1}(q) + q^{2\lambda-1}\alpha_{i,\lambda}(q) \\ &- q^{6\lambda-1}\alpha_{i,\lambda}(q). \end{aligned}$$

Substituting the value of $\alpha_{i,\lambda}(q)$ from (2.11) and then simplifying, we get

$$\beta_{i,\lambda}(q) = \frac{q^{\lambda[1+(\lambda-1)(i+3)/2]}(-q;q^2)_{\lambda-1}}{(q^4;q^4)_{\lambda-1}(q;q^2)_{\lambda}}.$$

Thus

$$g^i(z;q) = \sum_{\lambda=0}^{\infty} \frac{q^{(\lambda+1)[1+\lambda(i+3)/2]}(-q;q^2)_{\lambda} z^{\lambda+1}}{(q^4;q^4)_{\lambda}(q;q^2)_{\lambda+1}} = zqf_3^i(z;q). \tag{2.14}$$

Define $P^i(m,\mu)$ by

$$f_3^i(z;q) = \sum_{\mu=0}^{\infty}\sum_{m=0}^{\infty} P^i(m,\mu) z^m q^{\mu}.$$

We see by coefficient comparison in (2.14) that

$$C^i(m+1,\mu+1) = P^i(m,\mu)$$

Now if we replace the part a_a in a partition enumerated by $C^i(m+1,\mu+1)$ by $(a-1)_{a+1}$, we see that the resulting partition is enumerated by $A_3^i(m+1,\mu)$. Thus we have

$$P^i(m,\mu) = A_3^i(m+1,\mu)$$

and so

$$\sum_{\mu=0}^{\infty}\sum_{m=0}^{\infty} A_3^i(m+1,\mu)z^m q^\mu = f_3^i(z;q)$$

Now

$$\sum_{\mu=0}^{\infty} A_3^i(\mu)q^\mu = \sum_{\mu=0}^{\infty}\left(\sum_{m=1}^{\infty} A_3^i(m,\mu)\right)q^\mu = \sum_{\mu=0}^{\infty}\left(\sum_{m=0}^{\infty} A_3^i(m+1,\mu)\right)q^\mu$$
$$= f_3^i(1;q) = f_3^i(q)$$

This completes the proof of Theorem 2.3. □

2.6. ROGERS-RAMANUJAN TYPE IDENTITIES FOR SPLIT $(n+t)$–COLOR PARTITIONS

For the case $i=-1$ of Theorems 2.1–2.3 in conjunction with the identities (2.4)–(2.6) produces the following Rogers-Ramanujan type identities for split $(n+t)$–color partitions which are similar to Rogers-Ramanujan-MacMahon type partition identities.

Theorem 2.4. *Let $A_1^{-1}(\mu)$ denote the number of split n–color partitions of μ such that (i) if a_p is the smallest or the only part in the partition, then $a \equiv p(mod 4)$, (ii) the red part of the subscripts cannot exceed 1 and (iii) the weighted difference between any two consecutive parts is nonnegative and is $\equiv 0(mod 4)$. Let*

$$B_1(\mu) = \sum_{l=0}^{\mu} C_1(\mu-l)D_1(l),$$

where $C_1(\mu)$ is the number of partitions of μ into parts $\equiv \pm 2, \pm 4, \pm 8(mod 20)$ and $D_1(\mu)$ denotes the number of partitions of μ into distinct parts $\equiv \pm 1, \pm 2$, $5(mod 10)$, where parts $\equiv 5(mod 10)$ are counted twice. Then $A_1^{-1}(\mu) = B_1(\mu)$, for all μ.

Example 2.1. $A_1^{-1}(5) = 5$, since the relevant partitions are: 5_5, 5_{4+1}, 5_1, 4_2+1_1, $4_{1+1}+1_1$.

Also, $B_1(5) = \sum_{l=0}^{5} C_1(5-l)D_1(l)$
$= C_1(5)D_1(0) + C_1(4)D_1(1) + \cdots + C_1(0)D_1(5)$
$= 0(1) + 2(1) + 0(1) + 1(1) + 0(0) + 1(2)$
$= 5.$

Theorem 2.5. *Let $A_2^{-1}(\mu)$ denote the number of split n–color partitions of μ such that (i) if a_p is the smallest or the only part in the partition, then $a \equiv p + 2(mod4)$, (ii) the red part of the subscripts cannot exceed 1, (iii) all parts are greater than or equal to 3 and (iv) the weighted difference between any two consecutive parts is nonnegative and is $\equiv 0(mod4)$. Let*

$$B_2(\mu) = \sum_{l=0}^{\mu} C_2(\mu - l)D_2(l),$$

where $C_2(\mu)$ is the number of partitions of μ into parts $\equiv \pm 4, \pm 6, \pm 8(mod20)$ and $D_2(\mu)$ denotes the number of partitions of μ into distinct parts $\equiv \pm 3, \pm 4,$ $5(mod10)$, where parts $\equiv 5(mod10)$ are counted twice. Then $A_2^{-1}(\mu) = B_2(\mu)$, for all μ.

Example 2.2. $A_2^{-1}(5) = 2$, since the relevant partitions are: 5_3, 5_{2+1}.
Also, $B_2(5) = \sum\limits_{l=0}^{5} C_2(5 - l)D_2(l)$
$= C_2(5)D_2(0) + C_2(4)D_2(1) + \cdots + C_2(0)D_2(5)$
$= 0(1) + 1(0) + 0(0) + 0(1) + 0(1) + 1(2)$
$= 2.$

Theorem 2.6. *Let $A_3^{-1}(\mu)$ denote the number of split $(n+2)$–color partitions of μ such that (i) the smallest part or the only part is of the form a_{a+2}, (ii) the red part of the subscripts cannot exceed 1, (iii) the red part of the subscript of the smallest part is 0 and (iv) the weighted difference between any two consecutive parts is nonnegative and is $\equiv 0(mod4)$. Let $B_3(\mu)$ denote the number of partitions of μ into parts $\equiv \pm 1, \pm 3, \pm 4 \pm 7, \pm 8 \pm 9(mod20)$. Then $A_3^{-1}(\mu) = B_3(\mu)$, for all μ.*

Example 2.3. $A_3^{-1}(5) = 3$, since the relevant partitions are: 5_7, $5_3 + 0_2$, $5_{2+1} + 0_2$. Also, $B_3(5) = 3$, since the relevant partitions are: $4 + 1$, $3 + 1 + 1$, $1 + 1 + 1 + 1 + 1$.

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