GENERALIZED MATRICES, K-THEORY AND CYCLIC COHOMOLOGY

SEYED EBRAHIM AKRAMI* and REZA MOHAMMADI

Communicated by Henri Moscovici

We generalize the concept of a matrix over an ordinary algebra to the concept of a matrix over an algebra in a monoidal category. Based on this concept we extend K-theory and cyclic cohomology and Alain Connes' pairing between them.

AMS 2010 Subject Classification: 19D55, 46L80, 58B34, 20G42, 18D10.

Key words: generalized matrices, monoidal category, K-theory, cyclic cohomology.

1. INTRODUCTION

The idea of NCG (non-commutative geometry) is to extend the domain of classical concepts. The classical concept of a space is extended to the concept of an algebra or a C^*-algebra and the classical concept of a vector bundle over a space is extended to a finitely generated projective module over an algebra or idempotent matrices over this algebra. Then the classical topological K-theory is extended to algebraic and operator K-theory. Also, classical de Rham cohomology is extended to cyclic cohomology and classical index formula is extended as a pairing between operator K-theory and cyclic cohomology [2].

In this article, we extend the concept of a matrix over an algebra and therefore, we extend K-theory and cyclic cohomology. This work is based on the following simple idea. Let A be a complex associative algebra. Every matrix $X = (a_{ij})$ of size $m \times n$ over A can be regarded as a linear map $\mathbb{C}^m \xrightarrow{T} A \otimes \mathbb{C}^n$, $Te_i = \sum_j a_{ij} \otimes e_j$. The crucial point is that the later has meaning in any monoidal category $\mathcal{C}$ if we replace spaces $\mathbb{C}^m$ and $\mathbb{C}^n$ with objects V, W of the category and take A an algebra in $\mathcal{C}$. Thus, we may extend the concept of a matrix instead of being a rectangle table of elements of an algebra to be a morphism $V \to A \otimes W$ and call it a matrix over A of size $V \times W$. Notice that since $Hom_{A-Mod}(A \otimes \mathbb{C}^m, A \otimes \mathbb{C}^n) \cong Hom_{\mathbb{C}-Vec}(\mathbb{C}^m, A \otimes \mathbb{C}^n)$ thus, one can define similarly a matrix over an algebra $A \in \mathcal{C}$ as a morphism $A \otimes V \to A \otimes W$. This later viewpoint is used in [4] and [5].

* The corresponding author. This research was in part supported by a grant No.83810319 from IPM, Iran, Math. Department.

2. GENERALIZED MATRICES

Let $\mathcal{O}$ be the class of objects of a category $\mathcal{C}$. We denote the set of morphisms with initial object u and terminal object v, by $H_{u,v}$ and the set of *loops*, *i.e.* morphisms whose initial and terminal objects are the same object u, by E_u. We denote the composition of a morphism $f \in H_{u,v}$ with a morphism $g \in H_{v,w}$ by $f \circ g \in H_{u,w}$.

We define two maps s and r called *source* and *range* maps as follows. For any morphism $f \in H_{u,v}$, we set $s(f) := u$ and $r(f) := v$. We have $s(f \circ g) = s(f), r(f \circ g) = r(g)$.

We regard any object u as a loop by identifying it with the loop id_u. Thus with this agreement we have $s(u) = u, r(u) = u$ for all object u and $s(f) \circ f = f, f \circ r(f) = f$.

We suppose that $H_{u,u}$ is a complex vector space and the composition is bilinear.

We assume that $\mathcal{O}$ is equipped with an addition law $\oplus$ which makes it to an Abelian associative semigroup with a null object denoted by 0 and for any morphism $f_i \in H_{u_i,v_i}, i = 1, 2$, there exists an addition law

$$f_1 \oplus f_2 \in H_{u_1 \oplus u_2, v_1 \oplus v_2}$$

such that it is commutative and associative and there exists a null morphism $0 \in E_0$ such that $0 \oplus f = f, \forall f \in H_{u,v}$. Moreover, we assume that for any $u_1, u_2 \in \mathcal{O}$ there are morphisms $\pi_k = \pi_k(\{u_1, u_2\}) \in H_{u_1 \oplus u_2, u_k}$ and $\imath_k = \imath_k(\{u_1, u_2\}) \in H_{u_k, u_1 \oplus u_2}, k = 1, 2$ such that

$$\imath_k \pi_l = \delta_{kl} \mathrm{id}_{u_k}, \quad \sum_{k=1}^{2} \pi_k \imath_k = \mathrm{id}_{u_1 \oplus u_2}. \tag{2.1}$$

We recall [3], that $\mathcal{C}$ is called a monoidal category if there exists a tensor product in a category $\mathcal{C}$, *i.e.* a covariant functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which associates to each pair of objects $u, v \in \mathcal{O}$ an object $u \otimes v \in \mathcal{O}$ and to each pair of morphisms $f \in H_{u,v}, g \in H_{u',v'}$ a morphism $f \otimes g \in H_{u \otimes u', v \otimes v'}$ and there exists an object $\mathbf{1}$ such that the following conditions hold

$$u \otimes \mathbf{1} = \mathbf{1} \otimes u = u, \quad (u \otimes v) \otimes w = u \otimes (v \otimes w)$$

and

$$f \otimes \mathrm{id}_{\mathbf{1}} = \mathrm{id}_{\mathbf{1}} \otimes f = f, \quad (f \otimes g) \otimes h = f \otimes (g \otimes h).$$

To say that $\otimes$ is a covariant functor means that we have the following identities

$$(ff') \otimes (gg') = (f \otimes g)(f' \otimes g'), \quad \mathrm{id}_u \otimes \mathrm{id}_v = \mathrm{id}_{u \otimes v}.$$

Moreover, we assume that $\otimes$ of objects is distributive with respect to $\oplus$ of objects and $\otimes$ of morphisms is bilinear. We require that the following compatibility conditions hold

$$\imath_k^a = \mathrm{id}_a \otimes \imath_k,\ \pi_k^a = \mathrm{id}_a \otimes \pi_k, \tag{2.2}$$

for all $u_k, a \in \mathcal{O}, k = 1,2$, where $\imath_k = \imath_k(\{u_1,u_2\}), \pi_k = \pi_k(\{u_1,u_2\}), \imath_k^a = \imath_k(\{a\otimes u_1, a\otimes u_2\}), \pi_k^a = \pi_k(\{a\otimes u_1, a\otimes u_2\})$. Note that we immediately conclude

(2.3)

$$\pi_k^{a\otimes b} = \mathrm{id}_a \otimes \pi_k^b,\ \imath_k^{a\otimes b} = \mathrm{id}_a \otimes \imath_k^b,\ \imath_k^a(\mathrm{id}_a \otimes f) = \mathrm{id}_a \otimes \imath_k f,\ (\mathrm{id}_a \otimes h)\pi_k^a = \mathrm{id}_a \otimes h\pi_k$$

for all $u_1, u_2, a, b, u \in \mathcal{O}, f \in H_{u_1\oplus u_2, u}, h \in H_{u, u_1\oplus u_2}, k = 1,2$.

We define a product

$$H^a_{u,v} \times H^b_{v,w} \to H^{a\otimes b}_{u,w}, \quad f \odot g := f(\mathrm{id}_a \otimes g), \quad a,b,u,v \in \mathcal{O}. \tag{2.4}$$

This product is associative in the sense that we have

$$(f\odot g)\odot h = f\odot(g\odot h), \quad a,b,c,u,v,w,x \in \mathcal{O}, f \in H^a_{u,v}, g \in H^b_{v,w} H^c_{w,x}. \tag{2.5}$$

Note that we recover the composition law of the category *via* this product, *i.e.* we have

$$fg = f\odot g, \qquad f \in H_{u,v}, g \in H_{v,w}, \tag{2.6}$$

since we have $g = \mathrm{id}_{\mathbf{1}} \otimes g$, we conclude that $fg = f(\mathrm{id}_{\mathbf{1}} \otimes g) = f \odot g$.

Now let φ be a morphism from a to b. We define

$$\varphi^* : H^a_{u,v} \to H^b_{u,v}, \quad \varphi^*(f) := f(\varphi \otimes \mathrm{id}_v), \quad u,v \in \mathcal{O}. \tag{2.7}$$

The proof of the following lemma is straightforward.

LEMMA 2.1. (i) *For all* $f \in H^a_{u,v}, g \in H^b_{v,w}, \varphi \in H_{a,a'}, \psi \in H_{b,b'}, u,v,w,a, a', b, b' \in \mathcal{O}$, *we have*

$$(\varphi\otimes\psi)^*(f\odot g) = \varphi^*(f)\odot\psi^*(g), \quad \varphi^*(f\odot g) = \varphi^*(f)\odot g,$$

(ii) *For all* $f \in H^a_{u,v}, \varphi \in H_{a,b}, \psi \in H_{b,c}, u,v,a,b,c \in \mathcal{O}$, *we have*

$$(\varphi\psi)^*(f) = \psi^*(\varphi^*(f)),$$

(iii) $\varphi^*(fg) = f\varphi^*(g)$,

(iv)

$$\varphi^*(\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}) = \begin{bmatrix} \varphi^*(f_{11}) & \varphi^*(f_{12}) \\ \varphi^*(f_{21}) & \varphi^*(f_{22}) \end{bmatrix}.$$

(v)

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \odot \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} f_{11}\odot g_{11} + f_{12}\odot g_{21} & f_{11}\odot g_{12} + f_{12}\odot g_{22} \\ f_{21}\odot g_{11} + f_{22}\odot g_{21} & f_{21}\odot g_{12} + f_{22}\odot g_{22} \end{bmatrix}$$

A triple (a, m, μ) is called an **associative unital algebra** in the monoidal category $\mathcal{C}$, if $a \in \mathcal{O}$ and $m \in H_{a\otimes a,a}, \mu \in H_{\mathbf{1},a}$ and the following axioms hold

$$(m \otimes \mathrm{id}_a)m = (\mathrm{id}_a \otimes m)m, \quad (\mu \otimes \mathrm{id}_a)m = \mathrm{id}_a = (\mathrm{id}_a \otimes \mu)m. \tag{2.8}$$

Definition 2.2. Let (a, m, μ) be an associative unital algebra in the monoidal category $\mathcal{C}$. Any element of the space $H^a_{u,v} := H_{u,a\otimes v}$ is called a **matrix of size** $u \times v$ **over algebra** a**,** where $u, v \in \mathcal{O}$.

PROPOSITION 2.3. *For any* $u, v, w \in \mathcal{O}$ *the following defines a bilinear map*

$$\circ_a : H^a_{u,v} \times H^a_{v,w} \to H^a_{u,w}, \quad f \circ_a g := f(\mathrm{id}_a \otimes g)(m \otimes \mathrm{id}_w). \tag{2.9}$$

We have $(f \circ_a g) \circ_a h = f \circ_a (g \circ_a h)$ *for* $f \in H^a_{u,v}, g \in H^a_{v,w}, h \in H^a_{w,x}, u, v, w, x \in \mathcal{O}$. *In particular,* $E^a_u := H^a_{u,u}$ *is an associative unital complex algebra under the above product. The unit is* $\mu \otimes \mathrm{id}_u$. *Moreover for any elements* $u_1, u_2 \in \mathcal{O}$*, we have the following identification as algebras*

$$E^a_{u_1\oplus u_2} = \begin{bmatrix} H^a_{u_1,u_1} & H^a_{u_1,u_2} \\ H^a_{u_2,u_1} & H^a_{u_2,u_2} \end{bmatrix}, \quad f \mapsto (f_{ij})^2_{i,j=1} \tag{2.10}$$

where $f_{ij} = \imath_i f \pi^a_j$. *The inverse is given by* $(f_{ij})^2_{i,j=1} \mapsto f := \sum^2_{i,j=1} \pi_i f_{ij} \imath^a_j$.

Proof. Associativity:

$$\begin{aligned}
(f \circ_a g) \circ_a h &= [f(\mathrm{id}_a \otimes g)(m \otimes \mathrm{id}_w)] \circ_a h \\
&= f(\mathrm{id}_a \otimes g)(m \otimes \mathrm{id}_w)(\mathrm{id}_a \otimes h)(m \otimes \mathrm{id}_x) \\
&= f(\mathrm{id}_a \otimes g)(m\mathrm{id}_a \otimes \mathrm{id}_w h)(m \otimes \mathrm{id}_x) \\
&= f(\mathrm{id}_a \otimes g)(\mathrm{id}_{a\otimes a} m \otimes h\mathrm{id}_{a\otimes x})(m \otimes \mathrm{id}_x) \\
&= f(\mathrm{id}_a \otimes g)(\mathrm{id}_{a\otimes a} \otimes h)(m \otimes \mathrm{id}_{a\otimes x})(m \otimes \mathrm{id}_x) \\
&= f(\mathrm{id}_a \otimes g)[\mathrm{id}_a \otimes (\mathrm{id}_a \otimes h)][(m \otimes \mathrm{id}_a) \otimes \mathrm{id}_x](m \otimes \mathrm{id}_x) \\
&= f(\mathrm{id}_a \otimes g)[\mathrm{id}_a \otimes (\mathrm{id}_a \otimes h)][(m \otimes \mathrm{id}_a)m \otimes \mathrm{id}_x \mathrm{id}_x] \\
&= f(\mathrm{id}_a \otimes g)[\mathrm{id}_a \otimes (\mathrm{id}_a \otimes h)][(\mathrm{id}_a \otimes m)m \otimes \mathrm{id}_x] \\
&= f(\mathrm{id}_a \otimes g)[\mathrm{id}_a \otimes (\mathrm{id}_a \otimes h)][\mathrm{id}_a \otimes (m \otimes \mathrm{id}_x)](m \otimes \mathrm{id}_x) \\
&= f[\mathrm{id}_a \otimes (g(\mathrm{id}_a \otimes h)(m \otimes \mathrm{id}_x)](m \otimes \mathrm{id}_x) \\
&= f[\mathrm{id}_a \otimes (g \circ_a h)](m \otimes \mathrm{id}_x) \\
&= f \circ_a (g \circ_a h)[-1.25pt]
\end{aligned}$$

The unit:

$$\begin{aligned}
f \circ_a (\mu \otimes \mathrm{id}_u) &= f(\mathrm{id}_a \otimes \mu \otimes \mathrm{id}_u)(m \otimes \mathrm{id}_u)] \\
&= f[(\mathrm{id}_a \otimes \mu)m \otimes \mathrm{id}_u \mathrm{id}_u] \\
&= f(\mathrm{id}_a \otimes \mathrm{id}_u) \\
&= f,
\end{aligned}$$

and

$$\begin{aligned}(\mu\otimes \mathrm{id}_u)\circ_a f &= (\mu\otimes \mathrm{id}_u)(\mathrm{id}_a\otimes f)(m\otimes \mathrm{id}_u)\\ &= (\mu\otimes f)(m\otimes \mathrm{id}_u)\\ &= (\mathrm{id}_{\mathbf{1}}\mu\otimes f\mathrm{id}_{a\otimes u})(m\otimes \mathrm{id}_u)\\ &= (\mathrm{id}_{\mathbf{1}}\otimes f)(\mu\otimes \mathrm{id}_a\otimes \mathrm{id}_u)(m\otimes \mathrm{id}_u)\\ &= f[(\mu\otimes \mathrm{id}_a)m\otimes \mathrm{id}_u\mathrm{id}_u]\\ &= f(\mathrm{id}_a\otimes \mathrm{id}_u)\\ &= f.\end{aligned}$$

We denote the map (2.10) by Φ. It is clear that Φ is linear. Let $f, g\in E^a_{u_1\oplus u_2}$, we have

$$\begin{aligned}(\Phi(f\circ_a g))_{ij} &= \imath_i(f\circ_a g)\pi^a_j\\ &= \imath_i f(\mathrm{id}_a\otimes g)(m\otimes \mathrm{id}_{u_1\oplus u_2})\pi^a_j\\ &= \sum_k \imath_i f\pi^a_k\imath^a_k(\mathrm{id}_a\otimes g)(m\otimes \mathrm{id}_{u_1\oplus u_2})\pi^a_j\\ &= \sum_k f_{ik}(id_a\otimes \imath_k g)\pi^{a\otimes a}_j(m\otimes \mathrm{id}_{u_j})\\ &= \sum_k f_{ik}(id_a\otimes \imath_k g)(\mathrm{id}_a\otimes \pi^a_j)(m\otimes \mathrm{id}_{u_j})\\ &= \sum_k f_{ik}(\mathrm{id}_a\otimes \imath_k g\pi^a_j)(m\otimes \mathrm{id}_{u_j})\\ &= \sum_k f_{ik}\circ_a g_{kj}\\ &= (\Phi(f)\Phi(g))_{ij}.\end{aligned}$$

Invertibility of Φ:

$$\begin{aligned}\Phi^{-1}(\Phi(f)) &= \sum_{i,j=1}^{2}\pi_i(\Phi(f))_{ij}\imath^a_j\\ &= \sum_{i,j=1}^{2}\pi_i\imath_i f\pi^a_j\imath^a_j\\ &= \mathrm{id}_{u_1\oplus u_2} f\mathrm{id}_{a\otimes u_1\oplus a\otimes u_2}\\ &= \mathrm{id}_{u_1\oplus u_2} f\mathrm{id}_{a\otimes(u_1\oplus u_2)}\\ &= f,\end{aligned}$$

and

$$\begin{aligned}
(\Phi(\Phi^{-1}((f)_{ij})))_{kl} &= \sum_{i,j=1}^{2} (\Phi(\pi_i f_{ij} \imath_j^a))_{kl} \\
&= \sum_{i,j=1}^{2} \imath_k \pi_i f_{ij} \imath_j^a \pi_l^a \\
&= \mathrm{id}_{u_i} f_{ij} \mathrm{id}_{a\otimes u_j} \\
&= f. \quad \square
\end{aligned}$$

Example 2.4. Let Vec denote the category whose objects are complex vector spaces and morphisms are linear maps. The sum $\oplus$ is just the direct sum of vector spaces. The morphisms π_k and $\imath_k$ are the canonical projection and embedding, respectively. The product $\otimes$ is just the tensor product of vector spaces and $\mathbf{1} = \mathbb{C}$. An associative unital algebra in this category is just an ordinary associative unital algebra, where $m(x \otimes y) := xy, \mu(\lambda) := \lambda 1_a, x, y \in a, \lambda \in \mathbb{C}$. As we explained in Introduction, a matrix in Vec of size $\mathbb{C}^m \times \mathbb{C}^n$ over a, is just an ordinary $m \times n$ matrix over ordinary algebra a.

Example 2.5. Let (h, Δ, S, ϵ) be a complex Hopf algebra and h-Com be the category of left h-comodules, [6, 10]. That is, an object is a vector space u equipped with a linear map $\alpha : u \to h \otimes u$, called coaction, satisfying $\alpha(\mathrm{id}_h \otimes \alpha) = \alpha(\Delta \otimes \mathrm{id}_u)$, $\alpha(\epsilon \otimes \mathrm{id}_u) = \mathrm{id}_u$. Sometimes we denote this object by (u, α). In Sweedler's notation, we write $\alpha(x) = \sum x_{(1)} \otimes x_{(2)}, x \in u$. The morphisms are h-comodule intertwiners, *i.e.* linear maps $f : (u, \alpha) \to (v, \beta)$ satisfying $f\beta = \alpha(\mathrm{id}_h \otimes f)$. The direct sum of two left h-comodules $(u, \alpha), (v, \beta)$ is the ordinary direct sum of vector spaces $u \oplus v$ equipped with the coaction $\alpha \oplus \beta : u \oplus v \to h \otimes (u \oplus v) = (h \otimes u) \oplus (h \otimes v)$, $(\alpha \oplus \beta)(x, y) := (\alpha(x), \beta(y))$. In Sweedler's notation, it means $(\alpha \oplus \beta)(x, y) = \sum x_{(1)} \otimes (x_{(2)}, 0) + \sum y_{(1)} \otimes (0, y_{(2)})$. Thus $(\alpha \oplus \beta)(\mathrm{id}_h \otimes (\alpha \oplus \beta))(x, y) = \sum x_{(1)} \otimes x_{(2)} \otimes (x_{(3)}, 0) + \sum y_{(1)} \otimes y_{(2)} \otimes (0, y_{(3)}) = (\alpha \oplus \beta)(\Delta \otimes \mathrm{id}_{\alpha\oplus\beta})(x, y)$ and $(\alpha \oplus \beta)(\epsilon \otimes \mathrm{id}_{\alpha\oplus\beta})(x, y) = \sum \epsilon(x_{(1)})(x_{(2)}, 0) + \sum \epsilon(y_{(1)})(0, y_{(2)}) = \sum(\epsilon(x_{(1)})x_{(2)}, 0) + \sum(0, \epsilon(y_{(1)})y_{(2)}) = (x, 0) + (0, y) = (x, y)$. Thus $(u \oplus v, \alpha \oplus \beta)$ is a left h-comodule. It is easy to see that the canonical projection and embedding π_k and $\imath_k$ are morphisms in this category. It is well-known that this category is monoidal category where the tensor product of two left h-comodules $(u, \alpha), (v, \beta)$ is the ordinary tensor product of vector spaces $u \otimes v$ equipped with the coaction $\alpha \otimes \beta : u \otimes v \to h \otimes u \otimes v$, $(\alpha \otimes \beta)(x \otimes y) = \sum x_{(1)} y_{(1)} \otimes x_{(2)} \otimes y_{(2)}$. It is easy that we have $u \otimes (v \oplus w) = (u \otimes v) \oplus (u \otimes w)$ not only as vector spaces but also as left h-comodules. Thus $\otimes$ is distributive. An associative algebra a in this category is just a left h-comodule algebra, *i.e.* an ordinary algebra a which is also a left h-comodule with coaction α satisfying

$\alpha(xy) = x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)}, x, y \in a$. Now let us see how a matrix over a in this category can be presented by ordinary matrices.

PROPOSITION 2.6. *Let u and v be two left h-comodules of dimension m and n, respectively. We fix a basis for each of them; $\{e_i \mid 1 \leq i \leq m\}$ and $\{f_i \mid 1 \leq i \leq n\}$. To any matrix $T \in H^a_{u,v}$ we assign a triple (X, U, V) of ordinary matrices $X \in M_{m,n}(a), U \in M_m(h), V \in M_n(h)$ satisfying*

(2.11)

$$\Delta(U_{ij}) = \sum_k U_{ik} \otimes U_{kj}, \quad \Delta(V_{ij}) = \sum_k V_{ik} \otimes V_{kj}, \quad \epsilon(U_{ij}) = \delta_{ij}, \quad \epsilon(V_{ij}) = \delta_{ij}$$

and

$$\gamma(X_{ij}) = \sum_{k,l} U_{ik}SV_{lj} \otimes X_{kl}. \tag{2.12}$$

This assignment is given as follows: The matrices U, V and X are defined by the equations $\alpha(e_i) := \sum_j U_{ij} \otimes e_j$, $\beta(f_i) := \sum_j V_{ij} \otimes f_j$, $T(e_i) := \sum_j X_{ij} \otimes f_j$, where α, β and γ are h-comodule structures of u, v and a. Conversely let (X, U, V) be a triple of matrices $X \in M_{m,n}(a), U \in M_m(h), V \in M_n(h)$ satisfying the above relations and u and v be m and n-dimensional vector spaces with a fixed bases $\{e_i \mid 1 \leq i \leq m\}$ and $\{f_i \mid 1 \leq i \leq n\}$. Now we define a matrix $T \in H^a_{u,v}$ as follows: The h-comodule structure of u and v are given by $\alpha(e_i) := \sum_j U_{ij} \otimes e_j$, $\beta(f_i) := \sum_j V_{ij} \otimes f_j$, and T is given by $T(e_i) := \sum_j X_{ij} \otimes f_j$.

The vector space structure of $H^a_{u,v}$ corresponds to

$$z(X, U, V) + (X', U, V) = (zX + X', U, V), \tag{2.13}$$

$z \in \mathbb{C}, X, X' \in M_{m,n}(a), U \in M_m(h), V \in M_n(h)$, and the multiplication of elements of $H^a_{u,v}$ with elements $H^a_{v,w}$, where w is another h-comodule with a fixed basis, corresponds to

$$(X, U, V)(Y, V, W) = (XY, U, W), \tag{2.14}$$

where $X \in M_{m,n}(a), Y \in M_{n,p}(a), U \in M_m(h), V \in M_n(h), W \in M_p(h)$.

Proof. Relations (2.11) are the consequences of the assumption that u and v are comodules over h. As a consequence of these relations as well as the Hopf algebra axioms, we get $\sum_k V_{ik}SV_{kj} = \epsilon(V_{ij}) = \delta_{ij}$. Next, let us write $\gamma(X_{ij}) = \sum_l Y^l_{ij} \otimes Z^l_{ij}$. So since T is a h-comdule intertwiner, we get $\sum_{j,k,l} Y^l_{ij}V_{jk} \otimes Z^l_{ij} \otimes f_k = \sum_{j,k} U_{ij} \otimes X_{jk} \otimes f_k$. Thus $\sum_{j,l} Y^l_{ij}V_{jk} \otimes Z^l_{ij} = \sum_j U_{ij} \otimes X_{jk}$. Hence for each p we have $\sum_{j,k,l} Y^l_{ij}V_{jk}SV_{kp} \otimes Z^l_{ij} = \sum_{j,k} U_{ij}SV_{kp} \otimes X_{jk}$. So $\sum_{j,l} Y^l_{ij}(\sum_k V_{jk}SV_{kp}) \otimes Z^l_{ij} = \sum_{j,k} U_{ij}SV_{kp} \otimes X_{jk}$. Thus $\sum_{j,l} Y^l_{ij}\delta_{jp} \otimes Z^l_{ij} = \sum_{j,k} U_{ij}SV_{kp} \otimes X_{jk}$. Hence $\sum_l Y^l_{ip} \otimes Z^l_{ip} = \sum_{j,k} U_{ij}SV_{kp} \otimes X_{jk}$. So $\gamma(X_{ip}) =$

$\sum_{j,k} U_{ij}SV_{kp} \otimes X_{jk}$. Therefore, we proved the relation (2.12). To get the converse assertion, we need to prove that the linear map T given by $T(e_i) := \sum_j X_{ij} \otimes f_j$, is a h-comodules intertwiner. To show this we start from the relation $\gamma(X_{ip}) = \sum_{j,k} U_{ij}SV_{kp}\otimes X_{jk}$. Thus $\sum_l Y^l_{ip}\otimes Z^l_{ip} = \sum_{j,k} U_{ij}SV_{kp}\otimes X_{jk}$. Hence for each q we get $\sum_{l,p} Y^l_{ip}V_{pq} \otimes Z^l_{ip} = \sum_{j,k,p} U_{ij}SV_{kp}V_{pq} \otimes X_{jk}$. Thus $\sum_{l,p} Y^l_{ip}V_{pq}\otimes Z^l_{ip} = \sum_j U_{ij}\otimes X_{jq}$. So $\sum_{l,p,q} Y^l_{ip}V_{pq}\otimes Z^l_{ip}\otimes f_q = \sum_{j,q} U_{ij}\otimes X_{jq}\otimes f_q$. This means exactly that T is a h-comodules intertwiner.

Relation (2.13) is the consequence of the definition $(zT + T')(e_i) = \sum_j X_{ij} \otimes f_j + \sum_j X'_{ij} \otimes f_j = \sum_j(X_{ij} + X'_{ij}) \otimes f_j$, where $T, T' \in H^a_{u,v}$ and (X, U, V), (X', U, V) are their corresponding ordinary matrices. Similarly, the relation (2.14) is the consequence of the definition $(T \circ_a T')(e_i) = (m \otimes \mathrm{id}_w)((\mathrm{id}_a\otimes T')(\sum_j X_{ij}\otimes f_j)) = (m\otimes \mathrm{id}_w)(\sum_{j,k} X_{ij}\otimes X'_{jk}\otimes g_k) = \sum_{j,k} X_{ij}X'_{jk}\otimes g_k = \sum_k(XX')_{ik} \otimes g_k$, where $T \in H^a_{u,v}, T' \in H^a_{v,w}$ and $(X, U, V), (X', V, W)$ are their corresponding ordinary matrices. □

Example 2.7. We consider a special case of example 2 that is $h = \mathbb{C}[G]$ group Hopf algebra for a finite group G. It is well-known that the objects of this category are exactly the G-graded vector spaces. We choose $a = h = \mathbb{C}[G]$, with the coaction $\gamma = \Delta$, which is an algebra in the category $\mathbb{C}[G]$-Com, since an algebra in this category is nothing other than a G-graded algebra, see [8].

PROPOSITION 2.8. *Matrices over algebra $\mathbb{C}[G]$ in the category $\mathbb{C}[G]$-Com of size $u\times v$, where u and v are finite dimensional G-graded vector spaces, correspond to triples $(\{X^g\}_{g\in G}, \{U^g\}_{g\in G}, \{V^g\}_{g\in G})$ of matrices $X^g \in M_{m,n}(\mathbb{C})$, $U^g \in M_m(\mathbb{C}), V^g \in M_n(\mathbb{C})$ such that each family $\{U^g\}_g, \{V^g\}_g$ is a family of orthogonal idempotents and $\sum_g U^g = I_m, \sum_g V^g = I_n$ and*

$$\sum_{g_1g_2^{-1}=g} U^{g_1}X^{g'}V^{g_2} = \delta_{g,g'}X^g. \tag{2.15}$$

Proof. A matrix $X \in M_{m,n}(\mathbb{C}[G])$ is nothing other than a family of matrices $\{X^g \in M_{m,n}(\mathbb{C})\}_{g\in G}$. The relations mentioned in this proposition are just rewriting of the relations (2.11) and (2.12): We have $\Delta(U_{ij}) = \Delta(\sum_g U^g_{ij}g) = \sum_g U^g_{ij}g \otimes g$. On the other hand, $\sum_k U_{ik} \otimes U_{kj} = \sum_{k,g,g'} U^g_{ik}U^g_{kj}g \otimes g'$. Thus $\sum_k U^g_{ik}U^g_{kj} = \delta_{g,g'}U^g_{ij}$. That is $U^gU^{g'} = \delta_{g,g'}U^g$. Next $\epsilon(U_{ij}) = \epsilon(\sum_g U^g_{ij}g) = \sum_g U^g_{ij} = (\sum_g U^g)_{ij}$. On the other hand, $\delta_{ij} = (I_m)_{ij}$ where I_m is the identity matrix of size $m \times m$. Thus $\sum_g U^g = I_m$.

Finally, $\Delta(X_{ij}) = \Delta(\sum_g X^g_{ij}g) = \sum_g X^g_{ij}g \otimes g$. On the other hand,

$$\sum_{k,l} U_{ik}SV_{lj} \otimes X_{kl} = \sum_{k,l,g_1,g_2,g} U^{g_1}_{ik}V^{g_2}_{lj}X^g_{kl}g_1g_2^{-1} \otimes g.$$

Thus

$$\delta_{g,g'}X_{ij}^{g} = \sum_{k,l}\sum_{g_1g_2^{-1}=g} U_{ik}^{g_1}X_{kl}^{g'}V_{lj}^{g_2}.$$

That is $\delta_{g,g'}X^g = \sum_{g_1g_2^{-1}=g} U^{g_1}X^{g'}V^{g_2}$. $\square$

Example 2.9. We denote the space of all matrices of size $m \times n$ over an ordinary algebra A by $M_{m,n}(A)$ and the invertible matrices of size $m \times m$ over a field k by $GL_m(k)$. Let G be a group. The category $\mathcal{C} = Rep_k(G)$ of all representations of G over k is a monoidal category, with $\otimes$ being the tensor product of representations: if for a representation V one denotes by ρ_V the corresponding map $G \to GL(v)$, then

$$\rho_{V\otimes W}(g) := \rho_V(g) \otimes \rho_W(g).$$

The unit object in this category is the trivial representation $1 = k$, see [3]. The direct sum of two objects (V, ρ_V), (W, ρ_W) is given by $(V, \rho_V) \oplus (W, \rho_W) = (V \oplus W, \rho_{V\oplus W})$ where

$$\rho_{V\oplus W}(g) : V \oplus W \longrightarrow V \oplus W$$

$$(\rho_{V\oplus W}(g))(v, w) = (\rho_V(g)v, \rho_W(g)w).$$

We define a morphism between two objects (V, ρ_V), (W, ρ_W) to be a linear map $f : V \longrightarrow W$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "\rho_V(g)"] \arrow[d, "f"'] & V \arrow[d, "f"] \\ W \arrow[r, "\rho_W(g)"'] & W \end{tikzcd}$$

commutes, *i.e.*

$$\forall g \in G \quad f\rho_W(g) = \rho_V(g)f.$$

The class $\mathcal{O}$ of objects is an Abelian associative semigroup with a null object denoted by $(0, \rho_0)$. It is easy to see that the projection and injection π_k and ι_k are morphisms in this category and satisfy in conditions (2.1). On the other hand, clearly $\otimes$ of objects is distributive with respect to $\oplus$ of objects, *i.e.*

$$(V, \rho_V) \otimes \big((W, \rho_W) \oplus (U, \rho_U)\big) = \big((V, \rho_V) \otimes (W, \rho_W)\big) \oplus \big((V, \rho_V) \otimes (U, \rho_U)\big).$$

Since U, V and W are vector spaces we have $V \otimes (W \oplus U) = (V \otimes W) \oplus (V \otimes U)$ and thus

$$\rho_V \otimes (\rho_W \oplus \rho_U) = (\rho_V \otimes \rho_W) \oplus (\rho_V \otimes \rho_U),$$

$$f \otimes (g \oplus h) = (f \otimes g) \oplus (f \otimes h)$$

and therefore compatibility conditions hold.

Consider an algebra $\big((A, \rho_A), m, \mu\big)$ in this category. Thus $(A, \rho_A) \in \mathcal{C}$, $m \in H_{(A\otimes A, \rho_{A\otimes A}),(A,\rho_A)}$, $\mu \in H_{1,(A,\rho_A)}$ and the following diagram commutes

$$\begin{tikzcd}
A \otimes A \arrow[r, "\rho_{A\otimes A}(g)"] \arrow[d, "m"'] & A \otimes A \arrow[d, "m"] \\
A \arrow[r, "\rho_A(g)"'] & A.
\end{tikzcd}$$

Therefore an algebra A in this category, is a unital associative ordinary algebra A with a representation of group G on A

$$G \times A \longrightarrow A$$

$$(g, a) \longmapsto ga := \rho_A(g)(a)$$

which satisfies in the following properties

$$g(ab) = (ga)(gb), \qquad g(a + b) = ga + gb$$

$$g(ha) = (gh)a, \qquad\qquad g1 = 1.$$

We call such an algebra a G-algebra. In the following proposition, we study the structure of a matrix in this category.

Proposition 2.10. *Let U and V be two m and n-dimensional representations of G and A be a G-algebra. We fix a basis for each of U and V; $\{e_i \mid 1 \le i \le m\}$ and $\{f_i \mid 1 \le i \le n\}$. To any matrix $T \in H^A_{U,V}$ we assign a triple (X, λ, η), $X \in M_{n,m}(A), X = (a_{ij}), \lambda : G \to GL_m(k), \lambda(g) = \big(\lambda_{ij}(g)\big), \eta : G \to GL_n(k), \eta(g) = \big(\eta_{ij}(g)\big)$, by $Te_j = \sum_i a_{ij} \otimes f_i, ge_j = \sum_i \lambda_{ij}(g)e_i, gf_j = \sum_i \eta_{ij}(g)f_i$. The maps λ and η are group homomorphisms and*

$$X\lambda(g) = \eta(g)X, \quad \forall g \in G. \tag{2.16}$$

Conversely let (X, λ, η) be a triple, $X \in M_{n,m}(A)$, $\lambda : G \to GL_m(k)$, $\eta : G \to GL_n(k)$ group homomorphisms and satisfy the relation (2.16). Let U and V be m and n-dimensional vector spaces with some fixed bases $\{e_i \mid 1 \le i \le m\}$ and $\{f_i \mid 1 \le i \le n\}$. Now we define a matrix $T \in H^A_{U,V}$ as follows: the representation structures of U and V are given by $ge_j = \sum_i \lambda_{ij}(g)e_i, gf_j = \sum_i \eta_{ij}(g)f_i$, and T is given by $Te_j = \sum_i a_{ij} \otimes f_i$. Each triple (X, λ, η) is called a G-matrix over algebra A.

Proof. We have

$$U = \langle e_1, ..., e_m\rangle, \quad V = \langle f_1, ..., f_n\rangle.$$

We know that the following diagram is commutative:

$$\begin{CD} U @>{\rho_U(g)}>> U \\ @V{T}VV @VV{T}V \\ A\otimes V @>>{\rho_A(g)\otimes\rho_V(g)}> A\otimes V. \end{CD}$$

Therefore, since

$$e_j \xrightarrow{T} Te_j = \sum_{i=1}^{n} a_{ij}\otimes f_i \xrightarrow{\rho_A(g)\otimes\rho_V(g)} \sum_i ga_{ij}\otimes gf_i,$$

and

$$e_j \xrightarrow{\rho_U(g)} ge_j \xrightarrow{T} T(ge_j),$$

we get

$$\sum_i ga_{ij}\otimes gf_i = T(ge_j).$$

Now we have

$$T(ge_j) = \sum_l \lambda_{lj}(g)Te_l = \sum_i\sum_l \lambda_{lj}(g)a_{il}\otimes f_i,$$

and

$$\sum_i ga_{ij}\otimes gf_i = \sum_i\sum_l ga_{ij}\otimes \eta_{li}(g)f_l = \sum_{il}\eta_{il}(g)a_{lj}\otimes f_i,$$

therefore

$$\sum_l a_{il}\lambda_{lj}(g) = \sum_l \eta_{il}(g)a_{lj}.$$

Thus

$$X\lambda(g) = \eta(g)X, \quad \forall g.$$

Now we show that λ and η are group homomorphisms. We have

$$(g_2g_1)(e_i) = g_2(g_1e_i).$$

Therefore

$$\sum_r \lambda_{ri}(g_2g_1)e_r = \sum_{lr}\lambda_{li}(g_1)\lambda_{rl}(g_2)e_r,$$

hence

$$\lambda_{ri}(g_2g_1) = \sum_l \lambda_{rl}(g_2)\lambda_{li}(g_1).$$

So $\lambda(g_2g_1) = \lambda(g_2)\lambda(g_1)$, *i.e.* λ is a group homomorphism. Similarly, it can be shown that η is a group homomorphism. The converse is easily proved. $\square$

3. K-THEORY

In this section, we extend K-theory of an ordinary algebra for an algebra inside a monoidal category, see [9].

Now let S be a subsemigroup of semigroup $(\mathcal{O}, \oplus)$. Using identification (2.10), we embed E^a_u in $E^a_{u\oplus v}$ *via* $f \mapsto \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}$ and let $M_\infty(a;S)$ be the union $\bigcup_{u\in S} E^a_u$ up to this identification. We define two equivalence relations on the set of idempotents living in $M_\infty(a;S)$, as follows: for any $u \in S$ and any two idempotents e, e' of the algebra $\in E^a_u$, we write $e \sim e'$ iff there exist $f, g \in E^a_u$ such that $e = fg$ and $e' = gf$ and we write $e \sim_s e'$ iff there exists an invertible $z \in E^a_u$ such that $e' = zez^{-1}$. As in algebraic K-theory for ordinary matrices, one can show that these are equivalence relations and the relation $e \sim_s e'$ implies $e \sim e'$. Also if $e \sim e'$ then $\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \sim_s \begin{bmatrix} e' & 0 \\ 0 & 0 \end{bmatrix}$, where the later matrices are regarded as idempotents of the algebra $E^a_{u\oplus u}$. So using the above mentioned embedding of E^a_u into E^a_{2u}, we conclude that the equivalence relations $\sim$ and $\sim_s$ over the idempotents of $M_\infty(a;S)$ are the same.

Now let $I(a;S)$ be the set of all equivalence classes of idempotents. There is a binary operation on $I(a;S)$: if $[e], [e'] \in I(a;S)$, where $e \in E^a_u, e' \in E^a_v, u, v \in S$, then $[e] + [e'] := [\mathrm{dia}(e, e')]$ where $\mathrm{dia}(e, e')$ is an idempotent in $E^a_{u\oplus v}$, regarding the identification (2.10).

Definition 3.1. $K_0(a;S)$ is the enveloping group of the semigroup $I(a;S)$.

Now we put some topological structures on the algebra E^a_u. Let for any $u \in S$, each algebra E^a_u is a local Banach algebra. For any two idempotents $e, e' \in E^a_u$ we write $e \sim_h e'$ iff there exists a norm-continuous path of idempotents in E^a_u from e to e'. Again like in algebraic K-theory for ordinary matrices, one can show that the relation $e \sim_h e'$ implies $e \sim_s e'$. Conversely, the relation $e \sim_s e'$ implies $\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \sim_h \begin{bmatrix} e' & 0 \\ 0 & 0 \end{bmatrix}$. So the three equivalence relations $\sim, \sim_s, \sim_h$ on the set of idempotents of $M_\infty(a;S)$ coincide. So no matter which equivalence relation we choose on the semigroup of idempotents, we get the same enveloping group.

Next we define K_1-theory. Let GL^a_u be the invertible elements of the algebra E^a_u. We embed GL^a_u in $GL^a_{u\oplus v}$ by $f \mapsto \mathrm{dia}(f, \mathrm{id}_v)$ and we set $K_1(a;S)$ to be the disjoint union of quotient groups $\bigsqcup_{u\in S} GL^a_u/[GL^a_u, GL^a_u]$ up to this identification.

4. CYCLIC COHOMOLOGY

In this section, we extend cyclic cohomology of an ordinary algebra for an algebra inside a monoidal category, see [7].

Let $\mathcal{C}$ be a monoidal category which admits braiding. That is, there exists a family of invertible morphisms $\psi_{u,v} \in H_{u\otimes v, v\otimes u}$ satisfying

$$\psi_{u\otimes v,w} = (\mathrm{id}_u \otimes \psi_{v,w})(\psi_{u,w} \otimes \mathrm{id}_v),\ \ \psi_{u,v\otimes w} = (\psi_{u,v} \otimes \mathrm{id}_w)(\mathrm{id}_v \otimes \psi_{u,w}), \tag{4.1}$$

and

$$(f \otimes g)\psi_{u',v'} = \psi_{u,v}(g \otimes f), \tag{4.2}$$

for all $u, v, w, u', v' \in \mathcal{O}$ and $f \in H_{u,u'}, g \in H_{v,v'}$. Let (a, m, μ) be an associative unital algebra in the category $\mathcal{C}$. We say that a is a **ribbon algebra** if there exists an invertible morphism $\sigma \in E_a$ satisfying

$$\psi^2_{a,a}(\sigma \otimes \sigma)m = m\sigma. \tag{4.3}$$

In the category Vec where the braiding is flip operator, the above condition just means σ is algebra automorphism. Also, in general, this condition is a combination of the algebra automorphism condition and the fundamental condition between braiding and twist in a ribbon category.

For arbitrary objects $b, c \in \mathcal{O}$ we define a linear operator

$$\lambda_{b,c} : H_{a\otimes b,c} \to H_{b\otimes a,c}, \quad \lambda_{b,c}(\varphi) := \psi_{b,a}(\sigma \otimes \mathrm{id}_b)\varphi. \tag{4.4}$$

We recall the notion of braided cyclic cohomology introduced in [1]. First of all, for simplicity we use the notations $\psi_{i,j} := \psi_{a^{\otimes i},a^{\otimes j}}$ and $\mathrm{id}_i := \mathrm{id}_{a^{\otimes i}}$. We set $\lambda_{(n)} := (-1)^n\lambda_{a^{\otimes n},\mathbf{1}}$. Explicitly $\lambda_{(n)}$ is the operator sending a morphism φ in the space $H_{a^{\otimes(n+1)},\mathbf{1}}$ to the following morphism in the same space

$$\lambda_{(n)}(\varphi) := (-1)^n\psi_{n,1}(\sigma \otimes \mathrm{id}_n)\varphi. \tag{4.5}$$

For simplicity, we will write λ instead of $\lambda_{(n)}$.

Let $C^n = C^n(a;\sigma) := \{\varphi \in H_{a^{\otimes(n+1)},\mathbf{1}} \mid \lambda^{n+1}(\varphi) = \varphi\}$. For $\varphi \in C^n$, we define

$$d_i^{(n)}(\varphi) = \begin{cases} (\mathrm{id}_i \otimes m \otimes \mathrm{id}_{n-i-1})\varphi, & 0 \le i \le n-1 \\ \psi_{n,1}(\sigma \otimes \mathrm{id}_n)(m \otimes \mathrm{id}_{n-1})\varphi, & i = n, \end{cases} \tag{4.6}$$

and

$$s_i^{(n)}(\varphi) = (\mathrm{id}_{i+1} \otimes \mu \otimes \mathrm{id}_{n-i})\varphi, \quad 0 \le i \le n. \tag{4.7}$$

Proposition 4.1. *We have $d_i^{(n)}(C^n) \subseteq C^{n+1}, s_i^{(n)}(C^n) \subseteq C^{n-1}$, $\lambda_{(n)}(C^n) \subseteq C^n$ and $\{C^n\}_{n\ge 0}$ with the linear maps $d_i^{(n)}, s_i^{(n)}$, and $\lambda_{(n)}$ as face, degeneracy and cyclic operators respectively, form a cocyclic module.*

Proof. The proof based on the very powerful graphical calculus is given in [1]. □

We denote the Hochschild cohomology of this cocyclic module by $HH(\mathcal{C}; a, \sigma)$ and the cohomology of the subcomplex

$$C^n_\lambda(\mathcal{C}; a, \sigma) = \{\varphi \in H_{a^{\otimes(n+1)},\mathbf{1}} \mid \lambda(\varphi) = \varphi\}$$

by $HC(\mathcal{C}; a, \sigma)$ and call them **Hochschild and cyclic cohomology** of ribbon algebra (a, m, μ, σ).

5. PAIRING K-THEORY WITH CYCLIC COHOMOLOGY

In this section, we extend the pairing between K-theory and cyclic cohomology of an ordinary algebra for an algebra inside a monoidal category, see [2].

In this section, we assume that S is a subsemigroup of the semigroup $(\mathcal{O}, \oplus)$ and there exists an additive family $T_u : E_u \to \mathbb{C}, u \in S$ of linear maps satisfying a trace property which we now explain. By additivity we mean

$$T_{u\oplus v}(\begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix}) = T_u(f) + T_v(g), \quad u, v \in S, f \in E_u, g \in E_v. \tag{5.1}$$

Next we extend the family T_u to a family $T_{u,\varphi} : E^a_u \to \mathbb{C}, u \in S, \varphi \in H_{a,\mathbf{1}}$ by

$$T_{u,\varphi}(f) := T_u(\varphi^*(f)), \quad u \in S, a \in \mathcal{O}, \varphi \in H_{a,\mathbf{1}}, f \in E^a_u. \tag{5.2}$$

Now we express the promised axiom of trace property. We assume that

$$T_{u,\varphi}(f \odot g) = T_{v,\varphi}(g \odot f), \quad u, v \in S, b \in \mathcal{O}, f \in H_{u,v}, g \in H^b_{v,u}, \varphi \in H_{b,\mathbf{1}}, \tag{5.3}$$

and

$$T_{u,\varphi}(f \odot g) = T_{v,\lambda(\varphi)}(g \odot f), \quad u, v \in S, a, b \in \mathcal{O}, f \in H^a_{u,v}, g \in H^b_{v,u}, \varphi \in H_{a\otimes b,\mathbf{1}}, \tag{5.4}$$

where $\lambda = \lambda_{b,\mathbf{1}}$ was defined by (4.4). Using Lemma 2.1 one can easily see that this family is also additive.

For the trivial algebra $a = \mathbf{1}$ with $m = \mathrm{id}_{\mathbf{1}}, \mu = \mathrm{id}_{\mathbf{1}}, \sigma = \mathrm{id}_{\mathbf{1}}$ and for $b = \mathbf{1}$, the product $\odot$ is just the composition law of the category and the axioms (5.3) and (5.4) become the ordinary trace property.

In particular, we can use the following traces: if $E_{\mathbf{1}} = \mathbb{C}$ and S is also closed under tensor product and admits twist, *i.e.* there exists a natural family of invertible morphisms $\theta_u \in E_u, u \in S$ satisfying (5.1), and admits duality, *i.e.* there is an operation on S, $u \mapsto u^*$ and there are morphisms $b_u \in H_{\mathbf{1},u\otimes u^*}, d_u \in H_{u^*\otimes u,\mathbf{1}}$ satisfying (5.2), then we get the following family of traces.

PROPOSITION 5.1. *Under the conditions mentioned in the above last paragraph, the following linear maps*

$$T_u : E_u \to \mathbb{C}, \quad T_u(f) := b_u((f\theta_u) \otimes \mathrm{id}_{u^*})\psi_{u,u^*} d_u \tag{5.5}$$

satisfy the axiom (5.4) for any ribbon algebra a and any object b.

Remark. We can use also any ribbon knot to produce a nontrivial family T_u.

Now we come back to the general situation at the beginning of this section where we had just braiding morphisms and the family $T_u, u \in S$.

PROPOSITION 5.2. *For any $u \in S$, the map $C^*(\mathcal{C}; a, \sigma) \to C^*(E_u^a), \varphi \mapsto \varphi_u$ defined by*

$$\varphi_u(x_0, \ldots, x_n) := T_{u,\varphi}(x_0 \odot \cdots \odot x_n) \tag{5.6}$$

is a map of cocyclic modules.

Proof. For $\varphi \in C^{n-1}(\mathcal{C}; a, \sigma)$ and $u \in S$ we must show $(d_i(\varphi))_u = d_i(\varphi_u)$, $(s_i(\varphi))_u = s_i(\varphi_u), (\lambda(\varphi))_u = \lambda(\varphi_u)$, for all $0 \le i \le n$. We have

$$\begin{array}{rcl} (\lambda(\varphi))_u(x_0, \ldots, x_n) & = & T_{u,\lambda(\varphi)}(x_0 \odot \cdots \odot x_n) \\ & = & (-1)^n T_{u,\varphi}(x_n \odot x_0 \odot \cdots \odot x_{n-1}) \\ & = & (-1)^n \varphi_u(x_n, x_0, \ldots, x_{n-1}) \\ & = & (\lambda(\varphi_u))(x_0, \ldots, x_n). \end{array}$$

Let $0 \le i < n$. We have

$$\begin{array}{rcl} (d_i(\varphi))_u(x_0, \ldots, x_n) & = & T_u\big((d_i(\varphi))^*(x_0 \odot \cdots \odot x_n)\big) \\ & = & T_u\big(\varphi^*((\mathrm{id}_i \otimes m \otimes \mathrm{id}_{n-i-1})^*(x_0 \odot \cdots \odot x_n))\big) \\ & = & T_u\big(\varphi^*(\mathrm{id}_i^*(x_0 \odot \cdots \odot x_{i-1}) \odot m^*(x_i \odot x_{i+1}) \\ & \odot & \mathrm{id}_{n-i-1}^*(x_{i+2} \odot \cdots \odot x_n))\big) \\ & = & T_u\big(\varphi^*(x_0 \odot \cdots \odot x_{i-1} \odot (x_i \circ_a x_{i+1}) \\ & \odot & x_{i+2} \odot \cdots \odot x_n)\big) \\ & = & \varphi_u(x_0, \ldots, x_{i-1}, (x_i \circ_a x_{i+1}), x_{i+2}, \ldots, x_n) \\ & = & d_i(\varphi_u)(x_0, \ldots, x_n). \end{array}$$

The case $i = n$, now is the consequence of the relation $d_n(\varphi) = (-1)^n \lambda(d_0\varphi))$, which holds for any cocyclic module, and the relations $(d_0(\varphi))_u = d_0(\varphi_u)$,$(\lambda(\varphi))_u = \lambda(\varphi_u)$.

Next, by setting $S_i = (s_i(\varphi))_u(x_0, \ldots, x_n)$ we have

$$\begin{array}{rcl} S_i & = & T_u\big((s_i(\varphi))^*(x_0 \odot \cdots \odot x_n)\big) \\ & = & T_u\big(\varphi^*(\mathrm{id}_{i+1} \otimes \mu \otimes \mathrm{id}_{n-i})^*(x_0 \odot \cdots \odot x_n)\big) \end{array}$$

$$
\begin{aligned}
&= T_u\big(\varphi^*(\mathrm{id}_{i+1}\otimes\mu\otimes\mathrm{id}_{n-i})^*(x_0\odot\cdots\odot x_i\odot \mathrm{id}_u\odot x_{i+1}\odot\cdots\odot x_n)\big)\\
&= T_u\big(\varphi^*(\mathrm{id}_{i+1})^*(x_0\odot\cdots\odot x_i)\odot\mu^*(\mathrm{id}_u)\odot \mathrm{id}^*_{n-i}(x_{i+1}\odot\cdots\odot x_n)\big)\\
&= T_u\big(\varphi^*(x_0\odot\cdots\odot x_i\odot 1^a_u\odot x_{i+1}\odot\cdots\odot x_n)\big)\\
&= \varphi_u(x_0,\ldots,x_i,1^a_u,x_{i+1},\ldots,x_n)\\
&= s_i(\varphi_u)(x_0,\ldots,x_n). \quad\Box
\end{aligned}
$$

PROPOSITION 5.3. *The following is a bilinear pairing between* $K_0(a;S)$ *and* $HC^{even}(\mathcal{C},a;\sigma)$,

(5.7) $$< [e],[\varphi] >:= \varphi_u(e,\ldots,e),\quad e\in E^a_u, u\in S.$$

Proof. We first show that if we replace $e\in E^a_u$ with $\begin{bmatrix} e & 0\\ 0 & 0\end{bmatrix} = \pi_1 e\imath^a_1 \in E^a_{u\oplus u}$, where $\pi_1=\pi_1(\{u,u\})$ and $\imath^a_1=\imath_1(\{a\otimes u, a\otimes u\})$, the result of the pairing does not change. For morphisms $x_i\in E^a_u$, $0\le i\le n$ and for any morphism $\varphi\in H_{a^{\otimes(n+1)},1}$ and by setting $\Phi_{u\oplus u}=\varphi_{u\oplus u}(\pi_1x_0\imath^a_1,\ldots,\pi_1x_n\imath^a_1)$ we have

$$
\begin{aligned}
\Phi_{u\oplus u} &= T_{u\oplus u,\varphi}(\pi_1x_0\imath^a_1\odot\cdots\odot\pi_1x_n\imath^a_1)\\
&= T_{u\oplus u,\varphi}(\pi_1x_0(\mathrm{id}_a\otimes\imath_1)\odot\cdots\odot\pi_1x_n(\mathrm{id}_a\otimes\imath_1))\\
&= T_{u\oplus u,\varphi}(\pi_1\odot x_0\odot\imath_1\odot\cdots\odot\pi_1\odot x_n\odot\imath_1)\\
&= T_{u\oplus u,\varphi}(\pi_1\odot x_0\odot\imath_1\odot\pi_1\odot x_1\odot\imath_1\cdots\odot\pi_1\odot x_n\odot\imath_1)\\
&= T_{u\oplus u,\varphi}(\pi_1\odot x_0\odot\imath_1\pi_1\odot x_1\odot\imath_1\cdots\odot\pi_1\odot x_n\odot\imath_1)\\
&= T_{u\oplus u,\varphi}(\pi_1\odot x_0\odot \mathrm{id}_u\odot x_1\odot\imath_1\cdots\odot\pi_1\odot x_n\odot\imath_1)\\
&= T_{u\oplus u,\varphi}(\pi_1\odot x_0\odot\cdots\odot x_n\odot\imath_1)\\
&= T_{u,\varphi}(\imath_1\odot\pi_1\odot x_0\odot\cdots\odot x_n)\\
&= T_{u,\varphi}(\imath_1\pi_1\odot x_0\odot\cdots\odot x_n)\\
&= T_{u,\varphi}(x_0\odot\cdots\odot x_n)\\
&= \varphi_u(x_0,\ldots,x_n).
\end{aligned}
$$

Next we must show that if φ is a coboundary then the value of the pairing vanishes. So let $d=\sum_{i=0}^{2m}(-1)^i d_i$ be the Hochschild coboundary then by Proposition 5.2 we have

$$
\begin{aligned}
(d(\varphi))_u(e,\ldots,e) &= d(\varphi_u)(e,\ldots,e)\\
&= \sum_{i=0}^{2m}(-1)^i\varphi_u(e,\ldots,e)\\
&= \varphi_u(e,\ldots,e).
\end{aligned}
$$

But from $\lambda(\varphi_u)=\varphi_u$ we conclude that $\varphi_u(e,\ldots,e)=0$. Therefore $(d(\varphi))_u(e,\ldots,e)=0$.

Now let e and f be two idempotents in the algebra E^a_u such that $e \sim f$. Since we proved that the value of the pairing does not change if we replace $e \in E^a_u$ with $\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \in E^a_{u\oplus u}$ we conclude that we may assume $e \sim_s f$ within E^a_u, that there exists an invertible element $z \in E^a_u$ such that $f = z^{-1}ez$. But using Proposition 1.8 in chapter 3 of [2], and the above fact that the pairing for coboundaries vanishes, we conclude that the value of the pairing will not change if one replaces e with f. Thus the pairing only depends on the equivalence class of e and the cohomology class of φ. To show that this pairing over the semigroup $I(a;S)$ of the equivalence classes of idempotents extend to the group $K_0(a;S)$, we need to show that this pairing is additive:

$$\begin{aligned}
\varphi_{u\oplus v}(\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}, \dots, \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}) &= T_{u\oplus v,\varphi}(\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} \odot \cdots \odot \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}) \\
&= T_{u,\varphi}(\begin{bmatrix} e^{\odot(n+1)} & 0 \\ 0 & f^{\odot(n+1)} \end{bmatrix}) \\
&= T_{u,\varphi}(e^{\odot(n+1)}) + T_{v,\varphi}(f^{\odot(n+1)}) \\
&= \varphi_u(e, \dots, e) + \varphi_v(f \dots, f). \quad \square
\end{aligned}$$

PROPOSITION 5.4. *The following is a bilinear pairing between $K_1(a;S)$ and $HC^{odd}(\mathcal{C};a,S)$,*

$$< [g], [\varphi] >:= \varphi_u(g^{-1} - 1, g - 1, \dots, g^{-1} - 1, g - 1), \quad g \in GL^a_u, u \in S. \tag{5.8}$$

Proof. Since for fixed u, this pairing is nothing other than the pairing of the cyclic cohomology of the algebra E^a_u with the quotient of the group of invertible elements of this algebra by the commutator subgroup, it is enough to apply the proof of Proposition 3.3, chapter 3 of [2] for the algebra $\mathcal{A} = E^a_u$ and for the cyclic cocycle φ_u for the case $k = 1$.

Also we need to show that the pairing is compatible with the inclusion $GL^a_u \subseteq GL^a_{u\oplus v}$, $f \mapsto \mathrm{dia}(f, \mathrm{id}_v)$ which is easy to show. $\square$

REFERENCES

[1] S.E. Akrami and S. Majid, *Braided cyclic cocycles and non-associative geometry.* J. Math. Phys. **45** (2004), 3883–3911.

[2] A. Connes, *Noncommutative Geometry.* Academic Press, 1994.

[3] P. Etingof, S. Gelaki, D. Nikshych and V. Ostrik, *Tensor Categories.* Math. Surveys Monogr. **205**, American Mathematical Society, Providence, RI, 2015.

[4] C. Jones and D. Penneys, *Operator algebras in rigid C*-tensor categories.* Comm. Math. Phys. **355** (2017), 1121–1188.

[5] J.R. Kirillov and V. Ostrik, *On a q-Analog of the Mckay Correspondence and the ADE Classification of $\hat{sl}_2$ Conformal Field Theories.* INSPIRE **math/0101219** (2001), 27 pages. Preprint.

[6] A. Klimik and K. Schmuedgen, *Quantum Groups and Their Representations.* Texts and Monographs in Physics, Springer-Verlag, 2012.

[7] J.L. Loday, *Cyclic Homology.* Springer-Verlag, 2013.

[8] S. Majid, *Foundations of Quantum Group Theory.* Cambridge Univ. Press, Cambridge, 2000.

[9] N.E. Wegge-Olsen, *K-theory and C^*-algebras: a Friendly Approach.* Oxford Univ. Press, Oxford, 1993.

[10] D.N. Yetter, *Quantum groups and representations of monoidal categories.* Math. Proc. Cambridge Philos. Soc. **108** (1990), 261–290.

Received 17 January 2017

Semnan University,
Faculty of Mathematics,
Statistics and Computer Science,
Iran, P.O. Box 35131-19111
akramisa@semnan.ac.ir
r.mohammadi@semnan.ac.ir