ON EXPANSION IN EIGENFUNCTION FOR q-DIRAC SYSTEMS ON THE WHOLE LINE

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In this work, we establish a Parseval equality and expansion formula in eigenfunctions for the q-Dirac operator on the whole line.

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INTRODUCTION

The calculus without limits is known as the q-calculus (or quantum calculus). The concept of q-calculus was initiated by Jackson [11] in the first quarter of 19th century. Since then, the q-difference operators have been studied extensively. Hence, the q-calculus has a rich literature (see [7]). Moreover, the q-calculus has important applications in several mathematical areas such as the theory of relativity, orthogonal polynomials, combinatorics, number theory, quantum groups [5, 7].

On the other hand, spectral expansion theorems are important for solving various problems in mathematics. Specially, we lead to the problem of expanding an arbitrary function as a series of eigenfunctions when we seek a solution of a partial differential equation by separation of variables. The eigenfunction expansion is obtained by several methods. For instance, by the methods of integral equations, contour integration and the finite difference [3, 4, 8, 9, 14, 16].

In [2], the authors investigated the eigenfunction expansions for singular q-Dirac systems on $[0,\infty)$. In this work, we extend the results of [2] to obtain a Parseval equality and an expansion theorem for such operators on the whole line.

PRELIMINARIES

In this section, we recall some necessary fundamental concepts of q-analysis. Following the standard notations in [12] and [5], let q be a positive number with $0 < q < 1$, $A \subset \mathbb{R} := (-\infty, \infty)$ and $a \in A$. A *q-difference equation* is an equation that contains q-derivatives of a function defined on A. Let y be a complex-valued function on A. The *q-difference operator* D_q, the *Jackson q-derivative is* defined by

$$D_q y(x) = \frac{y(qx) - y(x)}{(q-1)x} \text{ for all } x \in A.$$

We know that there is a connection between the q-deformed Heisenberg uncertainty relation and the Jackson derivative on the q-basic numbers (see [15]). In the q-derivative, as $q \to 1$, the q-derivative is reduced to the classical derivative. The q-derivative at zero is defined by

$$D_q y(0) = \lim_{n \to \infty} \frac{y(q^n x) - y(0)}{q^n x} \quad (x \in A),$$

if the limit exists and does not depend on x. A *right-inverse* to D_q, the *Jackson q-integration* is given by

$$\int_0^x f(t)\, \mathrm{d}_q t = x(1-q) \sum_{n=0}^{\infty} q^n f(q^n x) \quad (x \in A),$$

provided that the series converges, and

$$\int_a^b f(t)\, \mathrm{d}_q t = \int_0^b f(t)\, \mathrm{d}_q t - \int_0^a f(t)\, \mathrm{d}_q t \quad (a, b \in A).$$

The q-integration for a function defined in [10] is given by the formulas

$$\int_0^{\infty} f(t)\, \mathrm{d}_q t = (1-q) \sum_{n=-\infty}^{\infty} q^n f(q^n),$$

$$\int_{-\infty}^{0} f(t)\, \mathrm{d}_q t = (1-q) \sum_{n=-\infty}^{\infty} q^n f(-q^n),$$

$$\int_{-\infty}^{\infty} f(t)\, \mathrm{d}_q t = (1-q) \sum_{n=-\infty}^{\infty} q^n \left[f(q^n) + f(-q^n) \right].$$

A function f which is defined on A, $0 \in A$, is said to be *q-regular at zero* if

$$\lim_{n \to \infty} f(xq^n) = f(0)$$

for every $x \in A$. Throughout the rest of the paper, we deal only with the functions which are q-regular at zero.

If f and g are q-regular at zero, then we have

$$\int_0^a g(t) D_q f(t)\, \mathrm{d}_q t + \int_0^a f(qt) D_q g(t)\, \mathrm{d}_q t = f(a) g(a) - f(0) g(0).$$

Let $L_q^2(-\infty,\infty)$ be the space of all complex-valued functions defined on $(-\infty,\infty)$ such that

$$\|f\| := \left(\int_{-\infty}^{\infty} |f(x)|^2 \, d_q x\right)^{1/2} < \infty.$$

The space $L_q^2(-\infty,\infty)$ is a separable Hilbert space with the inner product

$$(f,g) := \int_{-\infty}^{\infty} f(x)\overline{g(x)} d_q x, \quad f,g \in L_q^2(0,\infty)$$

(see [6]).

Let $y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, $z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}$. Then, we define the q-Wronskian of $y(x)$ and $z(x)$ by

$$W_q(y,z)(x) = y_1(x)\, z_2\left(q^{-1}x\right) - z_1(x)\, y_2\left(q^{-1}x\right). \tag{1}$$

Now, we introduce the convenient Hilbert space $\mathcal{H} = L_q^2((-\infty,\infty);E)$ $(E := \mathbb{R}^2)$ of vector-valued functions by using the inner product

$$(f,g) := \int_{-\infty}^{\infty} (f(x), g(x))_E d_q x.$$

MAIN RESULTS

Let us consider the q-Dirac system

$$-\frac{1}{q} D_{q^{-1}} y_2 + p(x)\, y_1 = \lambda y_1, \tag{2}$$

$$D_q y_1 + r(x)\, y_2 = \lambda y_2, \tag{3}$$

where λ is a complex eigenvalue parameter, p and r are real-valued functions defined on $(-\infty,\infty)$ and continuous at zero, and $p,r \in L_{q,loc}^1(-\infty,\infty)$. This system, as $q \to 1$, is reduced to the classical one dimensional Dirac system

$$\begin{aligned} -y_2' + p(x)\, y_1 &= \lambda y_1, \\ y_1' + r(x)\, y_2 &= \lambda y_2. \end{aligned}$$

We will denote by $\phi_1(x,\lambda)=\begin{pmatrix} \phi_{11}(x,\lambda) \\ \phi_{12}(x,\lambda) \end{pmatrix}$ and $\phi_2(x,\lambda)=\begin{pmatrix} \phi_{21}(x,\lambda) \\ \phi_{22}(x,\lambda) \end{pmatrix}$, the solution of the system (2)-(3) which satisfy the initial conditions

$$\phi_{11}(0,\lambda) = 1, \ \phi_{12}(0,\lambda) = 0, \ \phi_{21}(0,\lambda) = 0, \ \phi_{22}(0,\lambda) = 1. \tag{4}$$

Let $[-q^{-m}, q^{-m}]$ be an arbitrary finite interval, where $m \in \mathbb{N} := \{1,2,...\}$.

Now we will consider the boundary value problem (2)-(3) with the boundary conditions

$$\begin{aligned} &y_2\left(-q^{-m}\right)\cos\alpha + y_1\left(-q^{-m}\right)\sin\alpha = 0, \\ &y_2\left(q^{-m}\right)\cos\beta + y_1\left(q^{-m}\right)\sin\beta = 0,\ \ \alpha,\beta\in\mathbb{R},\ \ m\in\mathbb{N}. \end{aligned} \tag{5}$$

In [1], the authors prove that the boundary value problem (2)–(3) with the boundary conditions (5) has a compact resolvent operator, thus it has a purely discrete spectrum.

Let $\lambda_0\,,\lambda_{\pm1}\,,\lambda_{\pm2}\,,...$ be the eigenvalues and $y_0\,,y_{\pm1}\,,y_{\pm2}\,,...$ be the corresponding eigenfunctions of the problem (2), (3), (5), where $y_{\pm n}\left(x\right)=\begin{pmatrix} y_{\pm n1}\left(x\right) \\ y_{\pm n2}\left(x\right)\end{pmatrix}$. Since the solutions of this problem are linearly independent, we get

$$y_n\left(x\right) = c_n\phi_1\left(x,\lambda_n\right) + d_n\phi_2\left(x,\lambda_n\right).$$

There is no loss of generality in assuming that $|c_n|\leq 1$ and $|d_n|\leq 1$. Now let us set

$$z_n^2 = \int_{-q^{-m}}^{q^{-m}} \left\|y_n\left(x\right)\right\|_E^2 \mathrm{d}_q x.$$

Let $f\left(.\right)=\begin{pmatrix} f_1\left(.\right) \\ f_2\left(.\right)\end{pmatrix}\in L_q^2\left(\left(-q^{-m},q^{-m}\right);E\right)$. If we apply the Parseval equality (see [2]) to $f\left(x\right)$, then we obtain

$$\begin{aligned} &\int_{-q^{-m}}^{q^{-m}} \left\|f\left(x\right)\right\|_E^2 \mathrm{d}_q x \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{z_n^2}\left\{\int_{-q^{-m}}^{q^{-m}} \left(f\left(x\right), y_n\left(x\right)\right)_E \mathrm{d}_q x\right\}^2 \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{z_n^2}\left\{\int_{-q^{-m}}^{q^{-m}} \left(f\left(x\right), c_n\phi_1\left(x,\lambda_n\right)+d_n\phi_2\left(x,\lambda_n\right)\right)_E \mathrm{d}_q x\right\}^2 \\ &= \sum_{n=-\infty}^{\infty} \frac{c_n^2}{z_n^2}\left\{\int_{-q^{-m}}^{q^{-m}} \left(f\left(x\right), \phi_1\left(x,\lambda_n\right)\right)_E \mathrm{d}_q x\right\}^2 \\ &+ 2\sum_{n=-\infty}^{\infty} \frac{c_nd_n}{z_n^2}\prod_{j=1}^{2}\left\{\int_{-q^{-m}}^{q^{-m}} \left(f\left(x\right), \phi_j\left(x,\lambda_n\right)\right)_E \mathrm{d}_q x\right\} \\ &+ \sum_{n=-\infty}^{\infty} \frac{d_n^2}{z_n^2}\left\{\int_{-q^{-m}}^{q^{-m}} \left(f\left(x\right), \phi_2\left(x,\lambda_n\right)\right)_E \mathrm{d}_q x\right\}^2. \end{aligned} \tag{6}$$

Now, we will define the nondecreasing step function $\mu_{ij,q^{-m}}$ $(i,j=1,2)$ on $(-q^{-m},q^{-m})$ by

$$\mu_{11,q^{-m}}(\lambda)=\begin{cases} -\sum_{\lambda<\lambda_n<0}\frac{c_n^2}{z_n^2}, & \text{for } \lambda\le 0\\ \sum_{0\le\lambda_n<\lambda}\frac{c_n^2}{z_n^2} & \text{for } \lambda>0,\end{cases}$$

$$\mu_{12,q^{-m}}(\lambda)=\begin{cases} -\sum_{\lambda<\lambda_n<0}\frac{c_nd_n}{z_n^2}, & \text{for } \lambda\le 0\\ \sum_{0\le\lambda_n<\lambda}\frac{c_nd_n}{z_n^2} & \text{for } \lambda>0,\end{cases}$$

$$\mu_{12,q^{-m}}(\lambda)=\mu_{21,q^{-m}}(\lambda),$$

$$\mu_{22,q^{-m}}(\lambda)=\begin{cases} -\sum_{\lambda<\lambda_n<0}\frac{d_n^2}{z_n^2}, & \text{for } \lambda\le 0\\ \sum_{0\le\lambda_n<\lambda}\frac{d_n^2}{z_n^2} & \text{for } \lambda>0.\end{cases}$$

From (6), we obtain

$$\int_{-q^{-m}}^{q^{-m}}\|f(x)\|_E^2\,\mathrm{d}_qx=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}F_i(\lambda)F_j(\lambda)\,\mathrm{d}\mu_{ij,q^{-m}}(\lambda), \tag{7}$$

where

$$F_i(\lambda)=\int_{-q^{-m}}^{q^{-m}}(f(x),\phi_i)_E\,\mathrm{d}_qx\ \ (i=1,2).$$

Now we will prove a lemma, but first we recall some definitions.

A function f defined on an interval $[a,b]$ is said to be of *bounded variation* if there is a constant $C>0$ such that

$$\sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|\le C$$

for every partition

$$a=x_0<x_1<...<x_n=b \tag{8}$$

of $[a,b]$ by points of subdivision $x_0,x_1,...,x_n$.

Let f be a function of bounded variation. Then, by the *total variation* of f on $[a,b]$, denoted by $\overset{b}{\underset{a}{V}}(f)$, we mean the quantity

$$\overset{b}{\underset{a}{V}}(f):=\sup\sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|,$$

where the least upper bound is taken over all (finite) partitions (8) of the interval $[a,b]$ (see [13]).

LEMMA 1. *There exists a positive constant $\Lambda = \Lambda(\xi)$, $\xi > 0$ such that*

$$\underset{-\xi}{\overset{\xi}{V}} \left\{ \mu_{ij,q^{-m}}(\lambda) \right\} < \Lambda \ \ (i,j = 1,2), \tag{9}$$

where Λ does not depend on q^{-m}.

Proof. From (4), we have

$$\phi_{ij}(0,\lambda) = \delta_{ij},$$

where δ_{ij} $(i,j = 1,2)$ is the Kronecker delta. So there exists a $k > 0$ such that

$$|\phi_{ij}(x,\lambda) - \delta_{ij}| < \varepsilon, \ \varepsilon > 0, \ |\lambda| < \xi, \ x \in [0,k]. \tag{10}$$

Let $f_k(x) = \begin{pmatrix} f_{k1}(x) \\ f_{k2}(x) \end{pmatrix}$ be a nonnegative vector-valued function such that $f_{k1}(x)$ vanishes outside the interval $[0,k]$ with

$$\int_0^k f_{k1}(x)\,\mathrm{d}_q x = 1, \tag{11}$$

and $f_{k2}(x) = 0$. Set

$$\begin{aligned} F_{ik}(\lambda) &= \int_0^k (f_k(x), \phi_i)_E \,\mathrm{d}_q x \\ &= \int_0^k f_{k1}(x)\,\phi_{i1}(x,\lambda)\,\mathrm{d}_q x \quad (i = 1,2). \end{aligned}$$

Using (10) and (11), we obtain

$$|F_{1k}(\lambda) - 1| < \varepsilon, \ |F_{2k}(\lambda)| < \varepsilon, \quad |\lambda| < \xi. \tag{12}$$

Now, by applying the Parseval equality (7) to $f_k(x)$, we get

$$\begin{aligned} \int_0^k f_{k1}^2(x)\,\mathrm{d}_q x &\geq \int_{-\xi}^{\xi} F_{1k}^2(\lambda)\,\mathrm{d}\mu_{11,q^{-m}}(\lambda) + 2\int_{-\xi}^{\xi} F_{1k}(\lambda)\,F_{2k}(\lambda)\,\mathrm{d}\mu_{12,q^{-m}}(\lambda) \\ &+ \int_{-\xi}^{\xi} F_{2k}^2(\lambda)\,\mathrm{d}\mu_{22,q^{-m}}(\lambda) \geq \int_{-\xi}^{\xi} F_{1k}^2(\lambda)\,\mathrm{d}\mu_{11,q^{-m}}(\lambda) \\ &- 2\int_{-\xi}^{\xi} |F_{1k}(\lambda)|\,|F_{2k}(\lambda)|\,\left|\mathrm{d}\mu_{12,q^{-m}}(\lambda)\right|. \end{aligned}$$

From (12), we have

$$\begin{aligned} \int_0^k f_{k1}^2(x)\,\mathrm{d}_q x &> \int_{-\xi}^{\xi} (1-\varepsilon)^2\,\mathrm{d}\mu_{11,q^{-m}}(\lambda) - 2\int_{-\xi}^{\xi} \varepsilon(1+\varepsilon)\left|\mathrm{d}\mu_{12,q^{-m}}(\lambda)\right| \\ &= (1-\varepsilon)^2 \left(\mu_{11,q^{-m}}(\xi) - \mu_{11,q^{-m}}(-\xi)\right) - 2\varepsilon(1+\varepsilon) \underset{-\xi}{\overset{\xi}{V}} \left\{ \mu_{12,q^{-m}}(\lambda) \right\}. \end{aligned}$$

Since

$$\overset{\xi}{\underset{-\xi}{V}}\left\{\mu_{12,q^{-m}}(\lambda)\right\} \le \frac{1}{2}\left[\mu_{11,q^{-m}}(\xi) - \mu_{11,q^{-m}}(-\xi) + \mu_{22,q^{-m}}(\xi) - \mu_{22,q^{-m}}(-\xi)\right], \tag{13}$$

we get

$$\int_0^k f_{k1}^2(x)\,\mathrm{d}_q x > (1-3\varepsilon)\left\{\mu_{11,q^{-m}}(\xi) - \mu_{11,q^{-m}}(-\xi)\right\}$$
$$- \varepsilon(1+\varepsilon)\left\{\mu_{22,q^{-m}}(\xi) - \mu_{22,q^{-m}}(-\xi)\right\}. \tag{14}$$

Let $g_k(x) = \begin{pmatrix} g_{k1}(x) \\ g_{k2}(x) \end{pmatrix}$ be a nonnegative vector-valued function such that $g_{k2}(x)$ vanishes outside the interval $[0,k]$ with $\int_0^k g_{k2}(x)\,\mathrm{d}_q x = 1$, and $g_{k1}(x) = 0$. Similar arguments apply to the function $g_k(x)$, and we obtain

$$\int_0^k g_{k2}^2(x)\,\mathrm{d}_q x > (1-3\varepsilon)\left\{\mu_{22,q^{-m}}(\xi) - \mu_{22,q^{-m}}(-\xi)\right\}$$
$$- \varepsilon(1+\varepsilon)\left\{\mu_{11,q^{-m}}(\xi) - \mu_{11,q^{-m}}(-\xi)\right\}. \tag{15}$$

If we add the inequalities (14) and (15), then we get

$$\int_0^k \left\{f_{k1}^2(x) + g_{k2}^2(x)\right\}\mathrm{d}_q x > \left(1-4\varepsilon-\varepsilon^2\right)\left\{\begin{array}{c} \mu_{11,q^{-m}}(\xi) - \mu_{11,q^{-m}}(-\xi) \\ +\mu_{22,q^{-m}}(\xi) - \mu_{22,q^{-m}}(-\xi) \end{array}\right\}.$$

If we choose $\varepsilon > 0$ such that $1-4\varepsilon-\varepsilon^2 > 0$, then we obtain the assertion of the lemma for the functions $\mu_{11,q^{-m}}(\lambda)$ and $\mu_{22,q^{-m}}(\lambda)$ relying on their monotonicity. From (13), we have the assertion of the lemma for the function $\mu_{12,q^{-m}}(\lambda)$. □

Now we recall Helly's theorems.

THEOREM 2 ([13]). *Let $(w_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence of real nondecreasing functions on a finite interval $a \le \lambda \le b$. Then there exists a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ and a nondecreasing function w such that*

$$\lim_{k\to\infty} w_{n_k}(\lambda) = w(\lambda), \quad a \le \lambda \le b.$$

THEOREM 3 ([13]). *Assume that $(w_n)_{n\in\mathbb{N}}$ is a real, uniformly bounded sequence of nondecreasing functions on a finite interval $a \le \lambda \le b$, and suppose that*

$$\lim_{n\to\infty} w_n(\lambda) = w(\lambda),\ a \le \lambda \le b.$$

If f is any continuous function on $a \le \lambda \le b$, then

$$\lim_{n\to\infty} \int_a^b f(\lambda)\,\mathrm{d}w_n(\lambda) = \int_a^b f(\lambda)\,\mathrm{d}w(\lambda).$$

Now let ϱ be any nondecreasing function on $-\infty < \lambda < \infty$. Denote by $L_{\varrho}^{2}(-\infty,\infty)$ the Hilbert space of all functions $f:(-\infty,\infty)\to\mathbb{R}$ which are measurable with respect to the Lebesque-Stieltjes measure defined by ϱ and such that

$$\int_{-\infty}^{\infty} f^{2}(\lambda)\,\mathrm{d}\varrho(\lambda) < \infty,$$

with the inner product

$$(f,g)_{\varrho} := \int_{-\infty}^{\infty} f(\lambda)\,g(\lambda)\,\mathrm{d}\varrho(\lambda).$$

The main results of this paper are the following three theorems.

THEOREM 4. *Let* $f(.) = \begin{pmatrix} f_1(.) \\ f_2(.) \end{pmatrix} \in \mathcal{H}$. *Then, there exist monotonic functions* $\mu_{11}(\lambda)$ *and* $\mu_{22}(\lambda)$ *which are bounded over every finite interval, and a function* $\mu_{12}(\lambda)$ *which is of bounded variation over every finite interval with the property*

$$\int_{-\infty}^{\infty} \|f(x)\|_{E}^{2}\,\mathrm{d}_q x = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\lambda)\,F_j(\lambda)\,\mathrm{d}\mu_{ij}(\lambda), \tag{16}$$

where

$$F_i(\lambda) = \lim_{n\to\infty} \int_{-q^{-n}}^{q^{-n}} (f(x),\phi_i(x,\lambda))_{E}\,\mathrm{d}_q x.$$

We note that the matrix-valued function $\mu = (\mu_{ij})_{i,j=1}^{2}$ $(\mu_{12}=\mu_{21})$ *is called a spectral function for the system (2)-(3).*

Proof. Assume that the function $f_n(x) = \begin{pmatrix} f_{1n}(x) \\ f_{2n}(x) \end{pmatrix}$ satisfies the following conditions:

1) $f_n(x)$ vanishes outside the interval $[-q^{-n},q^{-n}]$, where $q^{-n} < q^{-m}$.

2) The functions $f_n(x)$ and $D_q f_n(x)$ are q-regular at zero.

If we apply the Parseval equality to $f_n(x)$, then we get

$$\int_{-q^{-n}}^{q^{-n}} \|f_n(x)\|_{E}^{2}\,\mathrm{d}_q x = \sum_{k=-\infty}^{\infty} \frac{1}{z_k^2} \left\{ \int_{-q^{-m}}^{q^{-m}} (f_n(x),y_k(x))_{E}\,\mathrm{d}_q x \right\}^{2}. \tag{17}$$

Then, via integrating by parts, we obtain

$$\int_{-q^{-m}}^{q^{-m}} (f_n(x),y_k(x))_{E}\,\mathrm{d}_q x$$
$$= \frac{1}{\lambda_k} \int_{-q^{-m}}^{q^{-m}} f_{1n}(x)\left[-\frac{1}{q}D_{q^{-1}}y_{k2}(x) + p(x)\,y_{k1}(x)\right]\mathrm{d}_q x$$

$$
\begin{aligned}
&+\frac{1}{\lambda_k}\int_{-q^{-m}}^{q^{-m}} f_{2n}(x)\left[D_q y_{k1}(x)+r(x)\,y_{k2}(x)\right]\mathrm{d}_q x \\
&=\frac{1}{\lambda_k}\int_{-q^{-m}}^{q^{-m}}\left[-\frac{1}{q}D_{q^{-1}}f_{2n}(x)+p(x)\,f_{1n}(x)\right]y_{k1}(x)\,\mathrm{d}_q x \\
&+\frac{1}{\lambda_k}\int_{-q^{-m}}^{q^{-m}}\left[D_q f_{1n}(x)+r(x)\,f_{2n}(x)\right]y_{k2}(x)\,\mathrm{d}_q x.
\end{aligned}
$$

Thus we have

$$
\sum_{|\lambda_k|\geq s}\frac{1}{z_k^2}\left\{\int_{-q^{-m}}^{q^{-m}}\left(f_n(x),y_k(x)\right)_E\mathrm{d}_q x\right\}^2
$$

$$
\begin{aligned}
&\leq\frac{1}{s^2}\sum_{|\lambda_k|\geq s}\frac{1}{z_k^2}\left\{\int_{-q^{-m}}^{q^{-m}}\left\{\begin{array}{c}\left[-\frac{1}{q}D_{q^{-1}}f_{2n}(x)+p(x)\,f_{1n}(x)\right]y_{k1}(x)\\ +\left[D_q f_{1n}(x)+r(x)\,f_{2n}(x)\right]y_{k2}(x)\end{array}\right\}\mathrm{d}_q x\right\}^2 \\
&\leq\frac{1}{s^2}\sum_{k=-\infty}^{\infty}\frac{1}{z_k^2}\left\{\int_{-q^{-m}}^{q^{-m}}\left\{\begin{array}{c}\left[-\frac{1}{q}D_{q^{-1}}f_{2n}(x)+p(x)\,f_{1n}(x)\right]y_{k1}(x)\\ +\left[D_q f_{1n}(x)+r(x)\,f_{2n}(x)\right]y_{k2}(x)\end{array}\right\}\mathrm{d}_q x\right\}^2 \\
&=\frac{1}{s^2}\int_{-q^{-n}}^{q^{-n}}\left\{\begin{array}{c}\left[-\frac{1}{q}D_{q^{-1}}f_{2n}(x)+p(x)\,f_{1n}(x)\right]^2\\ +\left[D_q f_{1n}(x)+r(x)\,f_{2n}(x)\right]^2\end{array}\right\}\mathrm{d}_q x.
\end{aligned}
$$

By using (17), we obtain

$$
\left|\int_{-q^{-n}}^{q^{-n}}\left(f_n(x),y_k(x)\right)_E\mathrm{d}_q x-\sum_{-s\leq\lambda_k\leq s}\frac{1}{z_k^2}\left\{\int_{-q^{-m}}^{q^{-m}}\left(f_n(x),y_k(x)\right)_E\mathrm{d}_q x\right\}^2\right|
$$

$$
\leq\frac{1}{s^2}\int_{-q^{-n}}^{q^{-n}}\left\{\begin{array}{c}\left[-\frac{1}{q}D_{q^{-1}}f_{2n}(x)+p(x)\,f_{1n}(x)\right]^2\\ +\left[D_q f_{1n}(x)+r(x)\,f_{2n}(x)\right]^2\end{array}\right\}\mathrm{d}_q x
$$

and

$$
\begin{aligned}
&\sum_{-s\leq\lambda_k\leq s}\frac{1}{z_k^2}\left\{\int_{-q^{-m}}^{q^{-m}}\left(f_n(x),y_k(x)\right)_E\mathrm{d}_q x\right\}^2 \\
&=\sum_{-s\leq\lambda_k\leq s}\frac{1}{z_k^2}\left\{\int_{-q^{-m}}^{q^{-m}}\left(f_n(x),c_k\phi_1(x,\lambda_k)+d_k\phi_2(x,\lambda_k)\right)_E\mathrm{d}_q x\right\}^2 \\
&=\int_{-s}^{s}\sum_{i,j=1}^{2}F_{in}(\lambda)\,F_{jn}(\lambda)\,\mathrm{d}\mu_{ij,q^{-m}}(\lambda),
\end{aligned}
$$

where

$$
F_{in}(\lambda)=\int_{-q^{-m}}^{q^{-m}}\left(f_n(x),\phi_i(x,\lambda)\right)_E\mathrm{d}_q x\ \ (i=1,2).
$$

Consequently, we get

$$\begin{aligned}
&\left|\int_{-q^{-n}}^{q^{-n}} (f_n(x), f_n(x))_E \,\mathrm{d}_q x - \int_{-s}^{s} \sum_{i,j=1}^{2} F_{in}(\lambda) F_{jn}(\lambda) \,\mathrm{d}\mu_{ij,q^{-m}}(\lambda)\right| \\
&\le \frac{1}{s^2} \int_{-q^{-n}}^{q^{-n}} \left[-\frac{1}{q} D_{q^{-1}} f_{2n}(x) + p(x) f_{1n}(x)\right]^2 \mathrm{d}_q x \qquad (18)\\
&+ \frac{1}{s^2} \int_{-q^{-n}}^{q^{-n}} \left[D_q f_{1n}(x) + r(x) f_{2n}(x)\right]^2 \mathrm{d}_q x.
\end{aligned}$$

By Lemma 1 and Theorems 2 and 3, we can find sequences $\{-q^{-m_k}\}$ and $\{q^{-m_k}\}$ such that the functions $\mu_{ij,q^{-m_k}}(\lambda)$ $(m_k \to \infty)$ converge to a monotone function $\mu_{ij}(\lambda)$. Passing to the limit with respect to $\{-q^{-m_k}\}$ and $\{q^{-m_k}\}$ in (18), we have

$$\begin{aligned}
&\left|\int_{-q^{-n}}^{q^{-n}} (f_n(x), f_n(x))_E \,\mathrm{d}_q x - \int_{-s}^{s} \sum_{i,j=1}^{2} F_{in}(\lambda) F_{jn}(\lambda) \,\mathrm{d}\mu_{ij}(\lambda)\right| \\
&\le \frac{1}{s^2} \int_{-q^{-n}}^{q^{-n}} \left[-\frac{1}{q} D_{q^{-1}} f_{2n}(x) + p(x) f_{1n}(x)\right]^2 \mathrm{d}_q x \\
&+ \frac{1}{s^2} \int_{-q^{-n}}^{q^{-n}} \left[D_q f_{1n}(x) + r(x) f_{2n}(x)\right]^2 \mathrm{d}_q x.
\end{aligned}$$

As $s \to \infty$, we get

$$\int_{-q^{-n}}^{q^{-n}} (f_n(x), f_n(x))_E \,\mathrm{d}_q x = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_{in}(\lambda) F_{jn}(\lambda) \,\mathrm{d}\mu_{ij}(\lambda).$$

Now let $f(.) \in \mathcal{H}$. Choose functions $\{f_\eta(x)\}$ satisfying the conditions 1-2 and such that

$$\lim_{\eta\to\infty} \int_{-\infty}^{\infty} \|f(x) - f_\eta(x)\|_E^2 \,\mathrm{d}_q x = 0.$$

Let

$$F_{i\eta}(\lambda) = \int_{-\infty}^{\infty} (f_\eta(x), \phi_i(x,\lambda))_E \,\mathrm{d}_q x \ \ (i = 1, 2).$$

Then, we have

$$\int_{-\infty}^{\infty} \|f_\eta(x)\|_E^2 \,\mathrm{d}_q x = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_{i\eta}(\lambda) F_{j\eta}(\lambda) \,\mathrm{d}\mu_{ij}(\lambda).$$

Since

$$\int_{-\infty}^{\infty} \|f_{\eta_1}(x) - f_{\eta_2}(x)\|_E^2 \,\mathrm{d}_q x \to 0 \text{ as } \eta_1, \eta_2 \to \infty,$$

we have

$$\int_{-\infty}^{\infty} \sum_{i=1}^{2} \left(F_{i\eta_1}(\lambda) F_{j\eta_1}(\lambda) - F_{i\eta_2}(\lambda) F_{j\eta_2}(\lambda)\right) \mathrm{d}\mu_{ij}(\lambda)$$

$$= \int_{-\infty}^{\infty} \| f_{\eta_1}(x) - f_{\eta_2}(x) \|_E^2 \, \mathrm{d}_q x \to 0$$

as $\eta_1, \eta_2 \to \infty$. Therefore, there is a limit function F which satisfies

$$\int_{-\infty}^{\infty} \| f(x) \|_E^2 \, \mathrm{d}_q x = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\lambda) F_j(\lambda) \, \mathrm{d}\mu_{ij}(\lambda),$$

by the completeness of the space $L_\mu^2(-\infty, \infty)$.

Now, we will show that the sequence (K_η) given by

$$K_\eta(\lambda) = \int_{-q^{-\eta}}^{q^{-\eta}} f_1(x) \phi_1(x, \lambda) + f_2(x) \phi_2(x, \lambda) \, \mathrm{d}_q x$$

converges as $\eta \to \infty$ to F in the metric of the space $L_\mu^2(-\infty, \infty)$. Let g be another function in $\mathcal{H}$. By similar arguments, $G(\lambda)$ can be defined by g.

It is obvious that

$$\int_{0}^{\infty} \| f(x) - g(x) \|_E^2 \, \mathrm{d}_q x$$

$$= \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \{ (F_i(\lambda) - G_i(\lambda)) (F_j(\lambda) - G_j(\lambda)) \} \, \mathrm{d}\mu_{ij}(\lambda).$$

Let

$$g(x) = \begin{cases} f(x), & x \in [-q^{-\eta}, q^{-\eta}] \\ 0, & \text{otherwise.} \end{cases}$$

Then, we have

$$\int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \{ (F_i(\lambda) - K_{\eta i}(\lambda)) (F_j(\lambda) - K_{\eta j}(\lambda)) \} \, \mathrm{d}\mu_{ij}(\lambda)$$

$$= \int_{-\infty}^{-q^{-\eta}} \| f(x) \|_E^2 \, \mathrm{d}_q x + \int_{q^{-\eta}}^{\infty} \| f(x) \|_E^2 \, \mathrm{d}_q x \to 0 \ \ (\eta \to \infty),$$

which proves that (K_η) converges to F in $L_\mu^2(-\infty, \infty)$ as $\eta \to \infty$. $\square$

THEOREM 5. *Suppose that the functions* $f(.) = \begin{pmatrix} f_1(.) \\ f_2(.) \end{pmatrix}$, $g(.) = \begin{pmatrix} g_1(.) \\ g_2(.) \end{pmatrix}$ $\in \mathcal{H}$, *and* $F(\lambda)$, $G(\lambda)$ *are their Fourier transforms. Then, we have*

$$\int_{-\infty}^{\infty} (f(x), g(x))_E \, \mathrm{d}_q x = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\lambda) G_j(\lambda) \, \mathrm{d}\mu_{ij}(\lambda),$$

which is called the generalized Parseval equality.

Proof. It is clear that $F \mp G$ are transforms of $f \mp g$. Therefore, we have

$$\int_{-\infty}^{\infty} \|f(x) + g(x)\|_E^2 \, \mathrm{d}_q x$$

$$= \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \left(F_i(\lambda) + G_i(\lambda)\right)\left(F_j(\lambda) + G_j(\lambda)\right) \mathrm{d}\mu_{ij}(\lambda)$$

and

$$\int_{-\infty}^{\infty} \|f(x) - g(x)\|_E^2 \, \mathrm{d}_q x$$

$$= \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \left(F_i(\lambda) - G_i(\lambda)\right)\left(F_j(\lambda) - G_j(\lambda)\right) \mathrm{d}\mu_{ij}(\lambda).$$

By subtracting one of these equalities from the other one, we get the desired result. □

THEOREM 6. *Let* $f(.) = \begin{pmatrix} f_1(.) \\ f_2(.) \end{pmatrix} \in \mathcal{H}$. *Then, the integrals*

$$\int_{-\infty}^{\infty} F_i(\lambda)\, \phi_j(x,\lambda)\, \mathrm{d}\mu_{ij}(\lambda) \quad (i,j = 1,2)$$

converge in $L_\mu^2(-\infty,\infty)$. *Consequently, we have*

$$f(x) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\lambda)\, \phi_j(x,\lambda)\, \mathrm{d}\mu_{ij}(\lambda),$$

which is called the expansion theorem.

Proof. Take any function $f_s \in \mathcal{H}$ and any positive number s, and set

$$f_s(x) = \int_{-s}^{s} \sum_{i,j=1}^{2} F_i(\lambda)\, \phi_j(x,\lambda)\, \mathrm{d}\mu_{ij}(\lambda).$$

Let $g(.) = \begin{pmatrix} g_1(.) \\ g_2(.) \end{pmatrix} \in \mathcal{H}$ be a vector function which is equal to zero outside the finite interval $[-q^{-\tau}, q^{-\tau}]$, where $q^{-\tau} < q^{-m}$. Thus we obtain

$$\int_{-q^{-\tau}}^{q^{-\tau}} \left(f_s(x), g(x)\right)_E \mathrm{d}_q x$$

$$= \int_{-q^{-\tau}}^{q^{-\tau}} \left(\int_{-s}^{s} \sum_{i,j=1}^{2} F_i(\lambda)\, \phi_j(x,\lambda)\, \mathrm{d}\mu_{ij}(\lambda), g_1(x) \right)_E \mathrm{d}_q x$$

$$= \int_{-s}^{s} \sum_{i,j=1}^{2} F_i(\lambda) \left\{ \int_{-q^{-\tau}}^{q^{-\tau}} \left(g(x), \phi_j(x,\lambda)\right)_E \mathrm{d}_q x \right\} \mathrm{d}\mu_{ij}(\lambda)$$

$$= \int_{-s}^{s} \sum_{i,j=1}^{2} F_i(\lambda) G_j(\lambda) \mathrm{d}\mu_{ij}(\lambda). \tag{19}$$

From Theorem 5, we get

$$\int_{-\infty}^{\infty} \left(f(x), g(x)\right)_E \mathrm{d}_q x = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} F_i(\lambda) G_j(\lambda) \mathrm{d}\mu_{ij}(\lambda). \tag{20}$$

By (19) and (20), we have

$$(f - f_s, g)_{\mathcal{H}} = \int_{|\lambda|>s} \sum_{i,j=1}^{2} F_i(\lambda) G_j(\lambda) \mathrm{d}\mu_{ij}(\lambda).$$

Apply this equality to the function

$$g(x) = \begin{cases} f(x) - f_s(x), & x \in [-q^{-s}, q^{-s}] \\ 0, & \text{otherwise}, \end{cases}$$

we get

$$\|f - f_s\|_{\mathcal{H}}^2 \leq \sum_{i,j=1}^{2} \int_{|\lambda|>s} F_i(\lambda) F_j(\lambda) \mathrm{d}\mu_{ij}(\lambda).$$

Letting $s \to \infty$ yields the expansion result. □

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