ON GENERALIZED COMPLETION HOMOLOGY MODULES

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In this paper, the vanishing and non-vanishing of generalized completion homology modules $L_i\Lambda^I(N,M)$ has been studied. As a technical tool, several natural homomorphisms of $L_i\Lambda^I(N,M)$, generalized cohomology modules $H_I^i(N,M)$ and generalized homology modules $U_i^I(N,M)$ have been developed. Under some additional conditions, these natural homomorphisms were found isomorphisms. We will prove finitely generated modules M and N over a commutative Noetherian ring R $H_i(M) = 0$, for all $i \neq c = \operatorname{grade}(I,M)$.

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1. INTRODUCTION

Let I be an ideal of a commutative Noetherian ring R. The I-adic completion functor $L_i\Lambda^I(M)$ of an R-module M, was first studied by Matlis (see [17] and [18] for details). If I is generated by a regular sequence, then the following isomorphism was proved by him:

$$L_i\Lambda^I(M) \cong \operatorname{Ext}^i_R(\varinjlim R/I^s, M) \text{ for all } i \in \mathbb{Z}.$$

Later, Greenlees and May determined the criterion for computing $L_i\Lambda^I(M)$ in terms of certain local homology groups (see [9, Theorem 2.2]). Schenzel constructed the completion homology modules as dual to local cohomology modules $H_I^i(M)$ [30, Theorem 1.1]. The details of $H_I^i(M)$ are described in [8].

Afterward, Cuong and Nam introduced in [4] the local homology module $U_i^I(M)$, which is in fact a dualization of local cohomology modules. There it is proved that for an Artinian R-module M, the module $U_i^I(M)$ is isomorphic to $L_i\Lambda^I(M)$ for all $i \in \mathbb{Z}$. The duality between the local homology modules and local cohomology modules was also proved by them [4, Proposition 4.1].

Recently, in [11], Herzog define the generalization of local cohomology modules as follows:

$$H^i_I(N,M) \cong \varinjlim \operatorname{Ext}^i_R(N/I^sN, M) \text{ for all } i \in \mathbb{Z},$$

where N is an arbitrary R-module. Adequate study is being done on the vanishing, non-vanishing and Artinianess properties of $H^i_I(N,M)$ (see [5,6,12, 16,20]). Yassemi defined the functor $\Gamma_I(N,M)$ and proved its cohomology modules are isomorphic to Herzog's generalized local cohomology functors [34, Theorem 3.4].

The dualization of generalized local cohomology functors was introduced by Nam, as generalized local homology modules $U^I_i(N,M)$, which are defined as $\varprojlim \operatorname{Tor}^R_i(N/I^sN, M)$ (see [25]).

Bijan-Zadeh and Moslehi have proved finiteness and vanishing properties of $U^I_i(N,M)$ (see [3, Theorems 3.1 and 4.4]). The non-vanishing results are given in [21, Theorem 2.4].

The notion of generalized completion homology modules $L_i\Lambda^I(N,M)$, defined by Nam as $L_i\Lambda^I(N,M) = H_i(\varprojlim(N/I^sN \otimes_R F^R_\cdot))$, where $F^R_\cdot$ is a flat resolution of M (see [25]). The generalized completion homology module $L_i\Lambda^I(N,M)$ become ordinary I-adic completion functor of M, when $N = R$.

We obtain the vanishing and non-vanishing results of generalized completion homology modules. The Matlis dual functor can be defined for a local ring R as $D(\cdot) = \operatorname{Hom}_R(\cdot, E_R(k))$. We proved that:

$$H^i_I(N,M) = 0 \text{ for all } i \neq c,$$

$$\Leftrightarrow U^I_i(N, D(M)) = 0 \text{ for all } i \neq c,$$
$$\Leftrightarrow L_i\Lambda^I(N, D(M)) = 0 \text{ for all } i \neq c,$$

where $c = \operatorname{grade}(I,M)$ (see Corollary 3.2).

To aid our analysis, we established various natural homomorphisms of $L_i\Lambda^I(N,M)$, $H^i_I(N,M)$ and $U^I_i(N,M)$. These homomorphisms become isomorphisms, as stated in the following Theorem:

THEOREM 1.1. *Let I be an ideal and M a non-zero module over R. With $c = \operatorname{grade}(I,M)$, the following conditions are equivalent:*

(i) $H^i_I(M) = 0$ *for all $i \neq c$.*

(ii) *For any finitely generated R-module N, the natural homomorphism*

$$H^i_I(N, H^c_I(M)) \to H^{i+c}_I(N,M),$$

is an isomorphism for all $i \in \mathbb{Z}$.

In addition, if R is local, then the above conditions can be equivalently described as follows:

(iii) *For any finitely generated R-module N, the natural homomorphism*

$$H^i_I(N, D(M)) \to H^{i+c}_I(N, D(H^c_I(M))),$$

is an isomorphism for all $i \in \mathbb{Z}$.

(iv) *For any finitely generated R-module N, the natural homomorphism*

$$U^I_{i+c}(N, D(M)) \to U^I_i(N, D(H^c_I(M))),$$

is an isomorphism for all $i \in \mathbb{Z}$.

(v) *For any finitely generated R-module N, the natural homomorphism*

$$L_{i+c}\Lambda^I(N, D(M)) \to L_i\Lambda^I(N, D(H^c_I(M))),$$

is an isomorphism for all $i \in \mathbb{Z}$.

Further, if we assume that M is Artinian, then the following conditions are equivalent to the above conditions:

(vi) *For any finitely generated R-module N, the natural homomorphism*

$$U^I_{i+c}(N, D(D(H^c_I(M)))) \to U^I_i(N, M),$$

is an isomorphism for all $i \in \mathbb{Z}$.

(vii) *For any finitely generated R-module N, the natural homomorphism*

$$L_{i+c}\Lambda^I(N, D(D(H^c_I(M)))) \to L_i\Lambda^I(N, M),$$

is an isomorphism for all $i \in \mathbb{Z}$.

The natural homomorphisms of Theorems 1.1 are derived in Theorem 2.8 and Corollary 2.9.

If N is finitely generated and M is Artinian, we successfully proved $L_i\Lambda^I(N, M)$ isomorphic to $U^I_i(N, M)$ for all $i \in \mathbb{Z}$ (see Proposition 2.6). This result generalizes [4, Proposition 4.1].

In Corollary 3.7, as an application of Theorem 1.1, there are several characterizations of grade and co-grade. For the definition and basic results of co-grade, see [27].

2. GENERALIZED HOMOLOGIES AND COHOMOLOGIES

In the rest of paper, a commutative Noetherian ring will be denoted by R. Let $f : X \to Y$ be a morphism of R-complexes. If the map $H^i(X) \to H^i(Y)$ induced by f is an isomorphism for each $i \in \mathbb{Z}$, then f is called a quasi-isomorphism. In this case, it will be written as $f : X \xrightarrow{\sim} Y$. For well-known results on homological algebra, see [1] , [10] and [33]. In the rest of paper, cochain complexes will be used.

Let M and N be arbitrary R-modules. For an ideal I of R, $H^i_I(M)$ for all $i \in \mathbb{Z}$, denote the local cohomology modules of M with respect to I (see [8] for its definition). In [11], Herzog introduced the generalized local cohomology modules $H^i_I(N, M)$ as the direct limit of direct system $\{\mathrm{Ext}^i_R(N/I^sN, M) : i \in$

$\mathbb{Z}\}$. Later on, Nam defined the generalized local homology modules $U_i^I(N, M)$ as the inverse limit of inverse system $\{\mathrm{Tor}_i^R(N/I^sN, M) : i \in \mathbb{Z}\}$ (see [25]). For a flat resolution $F_\cdot^R$ of M, Nam introduced the notion of generalized completion homology modules as:

$$L_i\Lambda^I(N, M) := H_i(\varprojlim(N/I^sN \otimes_R F_\cdot^R)) \text{ for all } i \in \mathbb{Z},$$

(see [25]). Note that, $L_i\Lambda^I(N, M)$ are independent of the choice of $F_\cdot^R$. As the tensor product is not left exact while the inverse limit is not right exact, this implies that:

$$L_0\Lambda^I(N, M) \neq \varprojlim(N/I^sN \otimes_R M).$$

Clearly, if $N = R$, then $L_i\Lambda^I(R, M)$ is the usual left derived functors of the completion, $L_i\Lambda^I(M)$. For more details about $L_i\Lambda^I(M)$, one should see [9].

If R is a local ring with the unique maximal ideal $\mathfrak{m}$, then $E = E_R(k)$ is the injective hull of the residue field $k = R/\mathfrak{m}$, while $D(\cdot) = \mathrm{Hom}_R(\cdot, E)$ stands for the Matlis dual functor.

Definition 2.1. Let I be an ideal of R and M an R-module such that $IM \neq M$. Then, grade of M is defined as:

$$\mathrm{grade}(I, M) = \inf\{i \in \mathbb{Z} : H_I^i(M) \neq 0\}.$$

In the following, the definition of co-grade will be needed, which is defined in [27, Definition 3.10].

Definition 2.2. For an R-module N, an element $x \in R$ is co-regular, if $\mathrm{Ann}_N(xR) \neq 0$. A sequence $\underline{x} = x_1, \cdots, x_r \in R$ must satisfy the following conditions to be co-regular:

(i) $\mathrm{Ann}_N(\underline{x}R) \neq 0$.

(ii) Each x_i is an $\mathrm{Ann}_N((x_1, \cdots, x_{i-1})R)$-coregular element for all $i = 1, \cdots, r$.

Suppose that N is a finitely generated R-module and M is an Artinian R-module. Then, the length of any maximal M-coregular sequence contained in $\mathrm{Ann}_R(N)$ is called $\mathrm{Cograde}_M(N)$, see [27, Definition 3.10].

Definition 2.3. Let I be an ideal of any ring R(not necessarily local). If M and N are R-modules with $c = \mathrm{grade}(I, M)$, then M is cohomologically complete intersection with respect to the pair (N, I), if $H_I^i(N, M) = 0$ for all $i \neq c$. If $N = R$, M will be a cohomologically complete intersection with respect to I.

Since $U_i^I(N, M) = \varprojlim \mathrm{Tor}_i^R(N/I^sN, M)$, this naturally implies the following homomorphisms:

$$L_i\Lambda^I(N, M) \to U_i^I(N, M) \text{ for all } i \in \mathbb{Z}.$$

It can be proved that these natural homomorphisms are surjective. The following Lemma holds for specific module, $N = R$ [9, Proposition 1.1].

Lemma 2.4. *Let I be an ideal of any ring R. For arbitrary R-modules M and N, there exits an exact sequence:*

$$0 \to \overset{1}{\varprojlim} \operatorname{Tor}^R_{i+1}(N/I^sN, M) \to L_i\Lambda^I(N, M) \to U^I_i(N, M) \to 0,$$

for every $i \in \mathbb{Z}$.

Proof. The proof of this Lemma is along similar lines to that of [9, Proposition 1.1]. □

Lemma 2.5 (Hom-Tensor Duality). *Let M and N be any modules over a local ring R. For every $i \in \mathbb{Z}$, the following isomorphisms hold:*

(i) $\operatorname{Ext}^i_R(N, D(M)) \cong D(\operatorname{Tor}^R_i(N, M))$.

(ii) *For a finitely generated R-module N,*

$$D(\operatorname{Ext}^i_R(N, M)) \cong \operatorname{Tor}^R_i(N, D(M)).$$

Proof. For proof see [13, Example 3.6]. □

Under some assumptions, the natural homomorphisms $L_i\Lambda^I(N, M) \to U^I_i(N, M)$ can be proved to be isomorphisms, as shown in Proposition 2.6. A result of a similar kind is proved in [25, Theorems 3.2 and 3.6] and [4, Proposition 4.1].

Proposition 2.6. *Let I be an ideal over an arbitrary ring R. For a finitely generated R-module N, we have:*

(i) *If M is an Artinian R-module, then the natural homomorphism*

$$L_i\Lambda^I(N, M) \to U^I_i(N, M),$$

is an isomorphism for each $i \in \mathbb{Z}$. For a local ring R, we have the following isomorphism

$$D(H^i_I(N, D(M))) \cong U^I_i(N, M) \text{ for each } i \in \mathbb{Z}.$$

(ii) *If M is a module over a local ring R, then the natural homomorphism*

$$L_i\Lambda^I(N, D(M)) \to U^I_i(N, D(M)),$$

is an isomorphism for each $i \in \mathbb{Z}$. Also, $D(H^i_I(N, M)) \cong U^I_i(N, D(M))$ for each $i \in \mathbb{Z}$.

Proof. To prove the statement in (i), we will follow the methodology of Cuong and Nam [4, Proposition 4.1]. Since, N is a finitely generated R-module and M an Artinian R-module, then $\operatorname{Tor}_{i+1}^R(N/I^sN, M)$ is also an Artinian R-module for all $i \in \mathbb{Z}$. This implies, $\varprojlim^1 \operatorname{Tor}_{i+1}^R(N/I^sN, M) = 0$ for all $i \in \mathbb{Z}$. Hence, in view of Lemma 2.4, the following natural homomorphism:

$$L_i\Lambda^I(N, M) \to U_i^I(N, M),$$

is an isomorphism. Note that, Hom functor transforms the direct systems into inverse systems in first variable. Hence, according to Lemma 2.5, we have:

$$D(H_I^i(N, D(M))) \cong \varprojlim \operatorname{Hom}_R(\operatorname{Ext}_R^i(N/I^sN, D(M)), E) \\ \cong \varprojlim \operatorname{Tor}_i^R(N/I^sN, M),$$

for all $i \in \mathbb{Z}$. It is important to note here that, $D(D(X)) \cong X$ for an Artinian R-module X.

To prove the statement in (ii), suppose that M is a module over a local ring R. According to the definition of the direct limit, there exists a short exact sequence:

$$0 \to \bigoplus_{s\in\mathbb{N}} \operatorname{Ext}_R^i(N/I^sN, M) \to \bigoplus_{s\in\mathbb{N}} \operatorname{Ext}_R^i(N/I^sN, M) \to H_I^i(N, M) \to 0,$$

for each $i \in \mathbb{Z}$. Since, N is finitely generated, application of the Matlis dual functor to the last sequence gives the following exact sequence:

$$0 \to D(H_I^i(N, M)) \to \prod_{s\in\mathbb{N}} \operatorname{Tor}_i^R(N/I^sN, D(M)) \\ \xrightarrow{\Psi_i} \prod_{s\in\mathbb{N}} \operatorname{Tor}_i^R(N/I^sN, D(M)) \to 0.$$

Note that $D(\operatorname{Ext}_R^i(N/I^sN, M)) \cong \operatorname{Tor}_i^R(N/I^sN, D(M))$ (see Lemma 2.5). It transforms the direct system $\{\operatorname{Ext}_R^i(N/I^sN, M) : i \in \mathbb{N}\}$ into the following inverse system:

$$\{\operatorname{Tor}_i^R(N/I^sN, D(M)) : i \in \mathbb{N}\}.$$

Now, by [33, Definition 3.5.1], it follows that:

$$\varprojlim \operatorname{Tor}_i^R(N/I^sN, D(M)) \cong D(H_I^i(N, M)) \text{ and } \varprojlim^1 \operatorname{Tor}_i^R(N/I^sN, D(M)) \\ \cong \operatorname{coker} \Psi_i = 0,$$

for all $i \in \mathbb{Z}$. On the other hand, by Lemma 2.4, we have the following exact sequence:

$$0 \to \varprojlim^1 \operatorname{Tor}_{i+1}^R(N/I^sN, D(M)) \to L_i\Lambda^I(N, D(M)) \to U_i^I(N, D(M)) \to 0.$$

Hence, the homomorphism $L_i\Lambda^I(N, D(M)) \to U_i^I(N, D(M))$ becomes an isomorphism for each $i \in \mathbb{Z}$. This completes the proof. $\square$

The next Corollary shows that the sequence of the functors $\{L_i\Lambda^I(N, M) : i \in \mathbb{Z}\}$ is positive strongly connected on the category of Artinian R-modules (see [28, p. 212]).

COROLLARY 2.7. *Suppose that I is an ideal of a local ring R and the following sequence of R-modules is exact:*

$$0 \to M_1 \to M_2 \to M_3 \to 0.$$

Then for a finitely generated R-module N, we have the following results:

(i) *There exists following long exact sequence of generalized completion homology modules:*

$$\cdots \to L_i\Lambda^I(N, D(M_3)) \to L_i\Lambda^I(N, D(M_2)) \to L_i\Lambda^I(N, D(M_1)) \to$$
$$L_{i-1}\Lambda^I(N, D(M_3)) \to \cdots \to L_0\Lambda^I(N, D(M_2)) \to L_0\Lambda^I(N, D(M_1)) \to 0.$$

(ii) *Suppose that M_i's is Artinian for each $i = 1, 2, 3$. There exists following long exact sequence of generalized completion homology modules:*

$$\cdots \to L_i\Lambda^I(N, M_1) \to L_i\Lambda^I(N, M_2) \to L_i\Lambda^I(N, M_3) \to$$
$$L_{i-1}\Lambda^I(N, M_1) \to \cdots \to L_0\Lambda^I(N, M_2) \to L_0\Lambda^I(N, M_3) \to 0.$$

Proof. These results are immediate consequence of the fact that $L_i\Lambda^I(N, M)$ is the ith left derived functor of the complex $\varprojlim(N/I^sN \otimes_R F_\cdot^R)$, where $F_\cdot^R$ is a flat resolution of M. $\square$

2.1. NATURAL HOMOMORPHISMS

In this section, some natural homomorphisms of the aforementioned modules, $H_I^i(N, M)$, $U_i^I(N, M)$ and $L_i\Lambda^I(N, M)$ will be derived. For a Gorenstein ring, the truncation complex was first constructed by Hellus and Schenzel (see [14, Definition 2.1]). Later, a more generalized form of truncation complex was presented in [24, Definition 2.6]. Let I be an ideal of a ring R. For an R-module M, the minimal injective resolution is denoted by $E_R^\cdot(M)$. Note that:

$$E_R^\cdot(M)^i \cong \bigoplus_{\mathfrak{p} \in \operatorname{Supp} M} E_R(R/\mathfrak{p})^{\mu_i(\mathfrak{p}, M)},$$

where $\mu_i(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})}(\operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}))$. Since $\Gamma_I(E_R(R/\mathfrak{p})) = 0$ for all $\mathfrak{p} \notin V(I)$ and $\Gamma_I(E_R(R/\mathfrak{p})) = E_R(R/\mathfrak{p})$ for all $\mathfrak{p} \in V(I)$, it implies that for all $i < c = \operatorname{grade}(I, M)$, we have:

$$\Gamma_I(E_R^\cdot(M))^i = 0.$$

This gives a natural embedding of the complexes $H^c_I(M)[-c] \to \Gamma_I(E^{\cdot}_R(M))$. The cokernel $C^{\cdot}_M(I)$ of this embedding is called the truncation complex. Therefore, there exists an exact sequence of complexes:

$$0 \to H^c_I(M)[-c] \to \Gamma_I(E^{\cdot}_R(M)) \to C^{\cdot}_M(I) \to 0. \tag{2.1}$$

Hence:

$$H^i(C^{\cdot}_M(I)) \cong \begin{cases} 0, & \text{if } i \leq c; \\ H^i_I(M), & \text{if } i > c. \end{cases}$$

Let $\underline{x} = x_1, \ldots, x_r \in I$ be a system of elements such that $\operatorname{Rad} I = \operatorname{Rad}(\underline{x})R$. Consider the Čech complex $\check{C}_{\underline{x}}$ with respect to $\underline{x}$. That is:

$$\check{C}_{\underline{x}} = \bigotimes_i^r \check{C}_{x_i},$$

where $\check{C}_{x_i}$ is the complex $0 \to R \to R_{x_i} \to 0$.

As a first application of this, we will prove the following result:

Theorem 2.8. *With the previous notion, the following conditions are satisfied for any finitely generated R-module N:*

(i) *For all $i \in \mathbb{Z}$, there exist the natural homomorphisms*

$$H^i_I(N, H^c_I(M)) \to H^{i+c}_I(N, M).$$

These are isomorphisms for all $i \in \mathbb{Z}$ if and only if $H^i_I(N, C^{\cdot}_M(I)) = 0$ for all $i \in \mathbb{Z}$.

(ii) *In addition, if R is local, then for all $i \in \mathbb{Z}$, there exist the natural homomorphisms*

$$H^i_I(N, D(M)) \to H^{i+c}_I(N, D(H^c_I(M))).$$

These are isomorphisms for all $i \in \mathbb{Z}$ if and only if $H^i_I(N, D(C^{\cdot}_M(I))) = 0$ for all $i \in \mathbb{Z}$.

Proof. Let $F_{\cdot}(N/I^sN)$ be a free resolution of N/I^sN, where $s \in \mathbb{N}$. Apply the functor $\operatorname{Hom}_R(F_{\cdot}(N/I^sN), .)$ to the exact sequence (2.1). Then, it induces the following short exact sequences of R-complexes:

$$0 \to \operatorname{Hom}_R(F_{\cdot}(N/I^sN), H^c_I(M))[-c] \to \operatorname{Hom}_R(F_{\cdot}(N/I^sN), \Gamma_I(E^{\cdot}_R(M))) \to \operatorname{Hom}_R(F_{\cdot}(N/I^sN), C^{\cdot}_M(I)) \to 0.$$

Let us investigate the cohomology of the complex in the middle of above exact sequence. Denote the complex $\operatorname{Hom}_R(F_{\cdot}(N/I^sN), \Gamma_I(E^{\cdot}_R(M)))$ by X. Since the functor Γ_I sends injective modules to injective modules, it follows that $H^i(X) \cong H^i(\operatorname{Hom}_R(N/I^sN, \Gamma_I(E^{\cdot}_R(M))))$ for all $i \in \mathbb{Z}$ and $s \in \mathbb{N}$.

Since $\mathrm{Supp}_R(N/I^sN) \subseteq V(I)$, then by [32, Lemma 2.2], there is an isomorphism of complexes:

$$\mathrm{Hom}_R(N/I^sN, \Gamma_I(E^{\cdot}_R(M))) \cong \mathrm{Hom}_R(N/I^sN, E^{\cdot}_R(M)).$$

It implies that:

$$H^i(X) \cong H^i(\mathrm{Hom}_R(F_{\cdot}(N/I^sN), E^{\cdot}_R(M))) \cong \mathrm{Ext}^i_R(N/I^sN, M),$$

for all $i \in \mathbb{Z}$ and $s \in \mathbb{N}$. Then the aforementioned sequence induces the following exact sequence of cohomology:

(2.2) $\mathrm{Ext}^{i-c}_R(N/I^sN, H^c_I(M)) \to \mathrm{Ext}^i_R(N/I^sN, M) \to \mathrm{Ext}^i_R(N/I^sN, C^{\cdot}_M(I))$,

for all $i \in \mathbb{Z}$ and $s \in \mathbb{N}$. Since the direct limit is an exact functor. On passing to the direct limit, we obtain the natural homomorphisms in (i). Clearly, these homomorphisms become isomorphisms if and only if $H^i_I(N, C^{\cdot}_M(I)) = 0$ for all $i \in \mathbb{Z}$.

In order to construct the natural homomorphisms in (ii), the Matlis dual functor will be applied to the short exact sequence (2.1). Then we obtain the exact sequence:

$$0 \to D(C^{\cdot}_M(I)) \to D(\Gamma_I(E^{\cdot}_R(M))) \to D(H^c_I(M))[c] \to 0.$$

The above sequence provides us with the following exact sequence:

$$0 \to \mathrm{Hom}_R(F_{\cdot}(N/I^sN), D(C^{\cdot}_M(I))) \to \mathrm{Hom}_R(F_{\cdot}(N/I^sN), D(\Gamma_I(E^{\cdot}_R(M)))) \to$$
$$\mathrm{Hom}_R(F_{\cdot}(N/I^sN), D(H^c_I(M)))[c] \to 0.$$

We are interested in the cohomology of the complex $\mathrm{Hom}_R(F_{\cdot}(N/I^sN), D(\Gamma_I(E^{\cdot}_R(M))))$, denoted by X. Note that there is an isomorphism of the following R-complexes:

$$X \cong D(F_{\cdot}(N/I^sN) \otimes_R \Gamma_I(E^{\cdot}_R(M))),$$

(see [10, Proposition 5.15]). Since the Matlis dual functor $D(\cdot)$ is exact and cohomology commutes with exact functor, the last isomorphism induces that:

$$H^i(X) \cong D(H^{-i}(F_{\cdot}(N/I^sN) \otimes_R \Gamma_I(E^{\cdot}_R(M)))),$$

for all $i \in \mathbb{Z}$. In order to compute the cohomology of X, we will calculate the cohomology of $Y := F_{\cdot}(N/I^sN) \otimes_R \Gamma_I(E^{\cdot}_R(M))$. Since $E^{\cdot}_R(M)$ is a complex of injective R-modules. Then according to [30, Theorem 3.2], we have:

$$Y \xrightarrow{\sim} F_{\cdot}(N/I^sN) \otimes_R \check{C}_{\underline{y}} \otimes_R E^{\cdot}_R(M).$$

Here $\check{C}_{\underline{y}}$ denotes the Čech complex with respect to $\underline{y} = y_1, \ldots, y_r \in I$ such that $\mathrm{Rad}(IR) = \mathrm{Rad}(\underline{y}R)$.

Since tensoring with the right bounded complexes of flat R-modules preserves quasi-isomorphisms. Moreover, the support of N/I^sN is contained in $V(I)$. So, we get the following quasi-isomorphisms:

$$F_{\cdot}(N/I^sN) \otimes_R \check{C}_{\underline{y}} \otimes_R M \xrightarrow{\sim} F_{\cdot}(N/I^sN) \otimes_R \check{C}_{\underline{y}} \otimes_R E^{\cdot}_R(M) \text{ and}$$

$$F_{\cdot}(N/I^sN) \otimes_R \check{C}_{\underline{y}} \xrightarrow{\sim} N/I^sN \otimes_R \check{C}_{\underline{y}} \cong N/I^sN.$$

Let $L^R_{\cdot}$ denote a free resolution of M. The following morphisms of complexes are homological isomorphisms:

$$F_{\cdot}(N/I^sN) \otimes_R \check{C}_{\underline{y}} \otimes_R L^R_{\cdot} \to F_{\cdot}(N/I^sN) \otimes_R \check{C}_{\underline{y}} \otimes_R M, \text{ and}$$

$$F_{\cdot}(N/I^sN) \otimes_R \check{C}_{\underline{y}} \otimes_R L^R_{\cdot} \to N/I^sN \otimes_R \check{C}_{\underline{y}} \otimes_R L^R_{\cdot} \cong N/I^sN \otimes_R L^R_{\cdot}$$

Hence, we conclude, $H^i(Y) \cong H^i(N/I^sN \otimes_R L^R_{\cdot}) \cong \operatorname{Tor}^R_{-i}(N/I^sN, M)$ for all $i \in \mathbb{Z}$ and $s \in \mathbb{N}$. By Hom-Tensor Duality (see Lemma 2.5), it implies that $H^i(X) \cong \operatorname{Ext}^i_R(N/I^sN, D(M))$ for all $i \in \mathbb{Z}$ and $s \in \mathbb{N}$. Then the long exact cohomology sequence provides the following exact sequence:

$$(2.3) \quad \operatorname{Ext}^i_R(N/I^sN, D(C^{\cdot}_M(I))) \to \operatorname{Ext}^i_R(N/I^sN, D(M)) \to \operatorname{Ext}^{i+c}_R(N/I^sN, D(H^c_I(M))),$$

for all $i \in \mathbb{Z}$ and $s \in \mathbb{N}$. By taking the direct limits of the above sequence, we get the natural homomorphisms of (ii) as follows:

$$H^i_I(N, D(M)) \to H^{i+c}_I(N, D(H^c_I(M))),$$

for all $i \in \mathbb{Z}$. Recall the following well-known isomorphism from [1]:

$$\varinjlim \operatorname{Ext}^i_R(N/I^sN, D(C^{\cdot}_M(I))) \cong H^i_I(N, D(C^{\cdot}_M(I))),$$

for all $i \in \mathbb{Z}$. Hence, the morphisms in (ii) become isomorphisms if and only if $H^i_I(N, D(C^{\cdot}_M(I))) = 0$ for all $i \in \mathbb{Z}$. □

Using Proposition 2.6 and Theorem 2.8, we can obtain the result stated in Corollary 2.9.

Corollary 2.9. *With the above notion, the following statements are true:*

(i) *For all $i \in \mathbb{Z}$, there are the natural homomorphisms*

$$U^I_{i+c}(N, D(M)) \to U^I_i(N, D(H^c_I(M))).$$

These are isomorphisms for all $i \in \mathbb{Z}$ if and only if $U^I_i(N, D(C^{\cdot}_M(I))) = 0$ for all $i \in \mathbb{Z}$.

(ii) *For all* $i \in \mathbb{Z}$, *there are the natural homomorphisms*

$$L_{i+c}\Lambda^I(N, D(M)) \to L_i\Lambda^I(N, D(H_I^c(M))).$$

These are isomorphisms for all $i \in \mathbb{Z}$ *if and only if* $L_i\Lambda^I(N, D(C_M^{\cdot}(I))) = 0$ *for all* $i \in \mathbb{Z}$.

(iii) *In addition, if* M *is Artinian, then for all* $i \in \mathbb{Z}$, *there are the natural homomorphisms*

$$U_{i+c}^I(N, D(D(H_I^c(M)))) \to U_i^I(N, M).$$

These are isomorphisms for all $i \in \mathbb{Z}$ *if and only if* $U_i^I(N, D(D(C_M^{\cdot}(I))))$ $= 0$ *for all* $i \in \mathbb{Z}$.

(iv) *For all* $i \in \mathbb{Z}$, *there are the natural homomorphisms*

$$L_{i+c}\Lambda^I(N, D(D(H_I^c(M)))) \to L_i\Lambda^I(N, M).$$

These are isomorphisms for all $i \in \mathbb{Z}$ *if and only if* $L_i\Lambda^I(N, D(D(C_M^{\cdot}(I))))$ $= 0$ *for all* $i \in \mathbb{Z}$.

PROPOSITION 2.10. *With the same assumptions as in Theorem 2.8, we have the following results:*

(i) *For all* $i \in \mathbb{Z}$, *there are the natural homomorphisms*

$$U_{i+c}^I(N, H_I^c(M)) \to U_i^I(N, M).$$

(ii) *Let* M *be an Artinian* R*-module, then for all* $i \in \mathbb{Z}$, *there are the natural homomorphisms*

$$L_{i+c}\Lambda^I(N, H_I^c(M)) \to L_i\Lambda^I(N, M).$$

Proof. Let $F_{\cdot}^R(N/I^sN)$ be a free resolution of N/I^sN, where $s \in \mathbb{N}$. Then tensoring the sequence (2.1) with $F_{\cdot}(N/I^sN)$, we obtain the following short exact sequence of complexes:

$$0 \to (F_{\cdot}(N/I^sN) \otimes_R H_I^c(M))[-c] \to F_{\cdot}(N/I^sN) \otimes_R \Gamma_I(E_R^{\cdot}(M)) \to F_{\cdot}(N/I^sN) \otimes_R C_M^{\cdot}(I) \to 0.$$

The homology of $Y = F_{\cdot}(N/I^sN) \otimes_R \Gamma_I(E_R^{\cdot}(M))$, already calculated in Theorem 2.8(*ii*), are:

$$H^i(Y) \cong \operatorname{Tor}_{-i}^R(N/I^sN, M), \text{ for all } i \in \mathbb{Z}.$$

With these isomorphisms, the long exact cohomology sequence provides the following exact sequences:

(2.4)
$$\operatorname{Tor}_{c-i}^R(N/I^sN, H_I^c(M)) \to \operatorname{Tor}_{-i}^R(N/I^sN, M) \to \operatorname{Tor}_{-i}^R(N/I^sN, C_M^{\cdot}(I)),$$

for all $i \in \mathbb{Z}$. As $-i$ varies over $\mathbb{Z}$, we can replace it with i. Passing to the inverse limits, we obtain the natural homomorphisms in (i).

Now, suppose that M is Artinian. Using Lemma 2.4, we get the following natural homomorphisms:

$$L_{i+c}\Lambda^I(N, H_I^c(M)) \to U_{i+c}^I(N, H_I^c(M)),$$

for all $i \in \mathbb{Z}$. Also, from Proposition 2.6(ii), $U_i^I(N, M) \cong L_i\Lambda^I(N, M)$ for all $i \in \mathbb{Z}$. Now, using (i), we obtain the homomorphisms in (ii). □

In the next Corollary, we will relate the surjectivity and injectivity of natural homomorphisms obtained in Theorem 2.8 and Corollary 2.9.

Corollary 2.11. *Let I be an ideal and N a finitely generated module over a local ring R. Suppose that M is an R-module such that $c = \operatorname{grade}(I, M)$. Then for each $i \in \mathbb{Z}$, we obtain:*

(1) *The following conditions are equivalent:*
 (i) *The natural homomorphism*
$$H_I^i(N, H_I^c(M)) \to H_I^{i+c}(N, M),$$
 is injective (resp. surjective).
 (ii) *The natural homomorphism*
$$U_{i+c}^I(N, D(M)) \to U_i^I(N, D(H_I^c(M))),$$
 is surjective (resp. injective).
 (iii) *The natural homomorphism*
$$L_{i+c}\Lambda^I(N, D(M)) \to L_i\Lambda^I(N, D(H_I^c(M))),$$
 is surjective (resp. injective).

(2) *In addition, if M is Artinian, then the following conditions are equivalent:*
 (i) *The natural homomorphism*
$$H_I^i(N, D(M)) \to H_I^{i+c}(N, D(H_I^c(M))),$$
 is injective (resp. surjective).
 (ii) *The natural homomorphism*
$$U_{i+c}^I(N, D(D(H_I^c(M)))) \to U_i^I(N, M),$$
 is surjective (resp. injective).
 (iii) *The natural homomorphism*
$$L_{i+c}\Lambda^I(N, D(D(H_I^c(M)))) \to L_i\Lambda^I(N, M),$$
 is surjective (resp. injective).

Proof. Using Proposition 2.6, Theorem 2.8 and Corollary 2.9, the Matlis duality proves the results in (1) and (2). □

3. VANISHING AND NON-VANISHING PROPERTIES

In this last section, the vanishing and non-vanishing results of $L_i\Lambda^I(N, M)$ will be discussed. Also, with some additional conditions on M, the natural homomorphisms, described in the previous section are even isomorphisms. Further, the cohomologically complete intersection of M with respect to the pair (N, I) is studied from various homological points of view. As a last result, the characterization of grade and co-grade is presented.

Corollary 3.1. *Let I be an ideal and M a module over a local ring R. Then for each $i \in \mathbb{Z}$ and finitely generated R-module N, we have:*

(i) $H^i_I(N, M) = 0$, *if and only if* $U^I_i(N, D(M)) = 0$, *if and only if* $L_i\Lambda^I(N, D(M)) = 0$.

(ii) *If M is Artinian, then* $H^i_I(N, D(M)) = 0$, *if and only if* $U^I_i(N, M) = 0$, *if and only if* $L_i\Lambda^I(N, M) = 0$.

Proof. It is an immediate consequence of Proposition 2.6. □

Corollary 3.2. *With the same notion as in Corollary 3.1, the following conditions are equivalent:*

(i) *M is cohomologically complete intersection with respect to (N, I).*

(ii) $U^I_i(N, D(M)) = 0$ *for all* $i \neq c$.

(iii) $L_i\Lambda^I(N, D(M)) = 0$ *for all* $i \neq c$.

Proof. The proof can be deduced from Corollary 3.1(i). □

In the rest of paper, the theory of spectral sequences will be needed. For details, see [1, 28] and [33].

Proposition 3.3. *Let I be an ideal and N a finitely generated module over an arbitrary ring R. Suppose that M is an R-module with* $\operatorname{grade}(I, M) = c$. *Then the following conditions are satisfied:*

(i) *The natural homomorphism*

$$H^0_I(N, H^c_I(M)) \to H^c_I(N, M),$$

is an isomorphism and $H^i_I(N, M) = H^{i-c}_I(N, H^c_I(M)) = 0$ *for all* $i < c$.

(ii) *In addition, if R is local, then the natural homomorphism*

$$U^I_c(N, D(M)) \to U^I_0(N, D(H^c_I(M)),$$

is an isomorphism and $U^I_i(N, D(M)) = U^I_{i-c}(N, D(H^c_I(M))) = 0$ *for all* $i < c$.

(iii) *The natural homomorphism*

$$L_c\Lambda^I(N, D(M)) \to L_0\Lambda^I(N, D(H^c_I(M))),$$

is an isomorphism and $L_i\Lambda^I(N, D(M)) = L_{i-c}\Lambda^I(N, D(H^c_I(M))) = 0$ *for all* $i < c$.

Proof. Let $E^R_{\cdot}$ be an injective resolution of the truncation complex $C^{\cdot}_M(I)$. Then by definition of the truncation complex, it follows that $H^i(E^R_{\cdot}) = 0$ for all $i \le c$. Also, note that:

$$\mathrm{Ext}^i_R(\cdot, C^{\cdot}_M(I)) \cong H^i(\mathrm{Hom}_R(\cdot, E^R_{\cdot})),$$

for all $i \in \mathbb{Z}$ (see [28, p. 331]). Now, for a fixed natural number $s \in \mathbb{N}$, consider the following spectral sequence

$$E^{p,q}_2 := \mathrm{Ext}^p_R(N/I^sN, H^q_I(E^R_{\cdot})) \Longrightarrow E^{p+q}_\infty = H^{p+q}(\mathrm{Hom}_R(N/I^sN, E^R_{\cdot})),$$

(see [28, Theorem 11.38]). Let $p+q \le c$. Then $p \ge 0$ implies $q \le c$. Therefore, it turns out that:

$$H^{p+q}(\mathrm{Hom}_R(N/I^sN, E^R_{\cdot})) = 0 \text{ for all } p+q \le c.$$

Hence, as a consequence of the spectral sequence, $H^i(\mathrm{Hom}_R(N/I^sN, E^R_{\cdot})) = 0$ for all $i \le c$ and $s \in \mathbb{N}$. This gives

$$H^i_I(N, C^{\cdot}_M(I)) = \varinjlim \mathrm{Ext}^i_R(N/I^sN, C^{\cdot}_M(I)) = 0,$$

for all $i \le c$. By [2, Proposition 5.5], it follows that $H^i_I(N, M) = 0$ for all $i < c$ and $H^c_I(N, M) \neq 0$. After taking the direct limit of exact sequence (2.2), we found that the following map

$$H^0_I(N, H^c_I(M)) \to H^c_I(N, M),$$

is an isomorphism and $H^{i-c}_I(N, H^c_I(M)) = 0$ for all $i < c$. This proves the claim in (*i*). By Corollaries 2.11(1) and 3.1(1), the statements in (*ii*) and (*iii*) are also true. epr

Corollary 3.4. *Let* I *be an ideal of an arbitrary ring* R. *Let* M *be a cohomologically complete intersection* R*-module with respect to* I. *Then for any finitely generated* R*-module* N, *the following are true:*

(i) *The natural homomorphism*

$$H^i_I(N, H^c_I(M)) \to H^{i+c}_I(N, M),$$

is an isomorphism for all $i \in \mathbb{Z}$.

(ii) *The natural homomorphism*

$$U^I_{i+c}(N, H^c_I(M)) \to U^I_i(N, M),$$

is injective for all $i \in \mathbb{Z}$.

(iii) *In addition, if R is local, then the natural homomorphism*

$$H_I^i(N, D(M)) \to H_I^{i+c}(N, D(H_I^c(M))),$$

is an isomorphism for all $i \in \mathbb{Z}$.

(iv) *The natural homomorphism*

$$U_{i+c}^I(N, D(M)) \to U_i^I(N, D(H_I^c(M))),$$

is an isomorphism for all $i \in \mathbb{Z}$.

(v) *The natural homomorphism*

$$L_{i+c}\Lambda^I(N, D(M)) \to L_i\Lambda^I(N, D(H_I^c(M))),$$

is an isomorphism for all $i \in \mathbb{Z}$.

(vi) *Assume further that M is Artinian, then the natural homomorphism*

$$U_{i+c}^I(N, D(D(H_I^c(M)))) \to U_i^I(N, M),$$

is an isomorphism for all $i \in \mathbb{Z}$.

(vii) *The natural homomorphism*

$$L_{i+c}\Lambda^I(N, D(D(H_I^c(M)))) \to L_i\Lambda^I(N, M),$$

is an isomorphism for all $i \in \mathbb{Z}$.

Proof. Since $H_I^i(M) = 0$ for al $i \neq c$. Then by definition of the truncation complex, $H^i(C_R^{\cdot}(I)) = 0$ for all $i \in \mathbb{Z}$. Hence, the complex $C_R^{\cdot}(I)$ is exact. Let $F_{\cdot}(N/I^sN)$ be a free resolution of N/I^sN, where $s \in \mathbb{N}$. The complex $\mathrm{Hom}_R(F_{\cdot}(N/I^sN), C_R^{\cdot}(I))$ is also exact for each $s \in \mathbb{N}$. This implies that

$$H_I^i(N, C_M^{\cdot}(I)) = \varinjlim \mathrm{Ext}_R^i(N/I^sN, C_M^{\cdot}(I)) = 0,$$

for all $i \in \mathbb{Z}$. Hence by Theorem 2.8(i), this completes the proof of (i). The statement in (iv) and (v) is straightforward from Corollary 2.11(1). With the similar arguments, Theorem 2.8(ii) and Corollary 2.11(2), we can prove the isomorphisms in (iii), (vi) and (vii).

From the above arguments, the complex $F_{\cdot}(N/I^sN) \otimes_R C_R^{\cdot}(I)$ is exact for each $s \in \mathbb{N}$. This proves the injectivity of the morphisms in (ii), by virtue of exact sequence (2.4). $\square$

Now, we are able to prove the following Theorems.

Theorem 3.5. *Let I be an ideal and M a non-zero module over R. With $c = \mathrm{grade}(I, M)$, the following conditions are equivalent:*

(i) *M is cohomologically complete intersection with respect to I.*

(ii) *For any finitely generated R-module N, the natural homomorphism*

$$H_I^i(N, H_I^c(M)) \to H_I^{i+c}(N, M),$$

is an isomorphism for all $i \in \mathbb{Z}$.

In addition, if R is local, then the above conditions can be equivalently described as follows:

(iii) *For any finitely generated R-module N, the natural homomorphism*

$$U_{i+c}^I(N, D(M)) \to U_i^I(N, D(H_I^c(M))),$$

is an isomorphism for all $i \in \mathbb{Z}$.

(iv) *For any finitely generated R-module N, the natural homomorphism*

$$L_{i+c}\Lambda^I(N, D(M)) \to L_i\Lambda^I(N, D(H_I^c(M))),$$

is an isomorphism for all $i \in \mathbb{Z}$.

Proof. By virtue of Corollaries 2.11(1) and 3.4(i), we only need to prove that (ii) implies (i). Suppose (ii) is true. Then for $N = R$, we get $H_I^i(C_M^{\cdot}(I)) = 0$ for all $i \in \mathbb{Z}$ (see Theorem 2.8(i)). By definition of the truncation complex and [23, Lemma 2.5], we obtain the following isomorphisms:

$$H_I^i(C_M^{\cdot}(I)) \cong H^i(C_M^{\cdot}(I)) \cong H_I^i(M) \text{ for } i > c.$$

Hence, M is cohomologically complete intersection with respect to I. □

THEOREM 3.6. *With the same assumptions as in Theorem 3.5, suppose that R is local. Then the following conditions are equivalent:*

(i) *M is cohomologically complete intersection with respect to I.*

(ii) *For any finitely generated R-module N, the natural homomorphism*

$$H_I^i(N, D(M)) \to H_I^{i+c}(N, D(H_I^c(M))),$$

is an isomorphism for all $i \in \mathbb{Z}$.

Further, assume that M is Artinian, then the following conditions are equivalent to the above conditions:

(iii) *For any finitely generated R-module N, the natural homomorphism*

$$U_{i+c}^I(N, D(D(H_I^c(M)))) \to U_i^I(N, M),$$

is an isomorphism for all $i \in \mathbb{Z}$.

(iv) *For any finitely generated R-module N, the natural homomorphism*

$$L_{i+c}\Lambda^I(N, D(D(H_I^c(M)))) \to L_i\Lambda^I(N, M),$$

is an isomorphism for all $i \in \mathbb{Z}$.

Proof. It suffices to prove (ii) and implies (i) (see Corollaries 2.11(2) and 3.4(iii)). To do this, assume (ii) is true. Then for $N=R$, we get $H_I^i(D(C_M^{\cdot}(I)))$ $=0$ for all $i \in \mathbb{Z}$ (see Theorem 2.8(ii)). We claim that $H_I^i(C_M^{\cdot}(I))=0$ for all $i \in \mathbb{Z}$. Note that by [15, Theorem 2.2], it is enough to prove $\operatorname{Ext}_R^i(R/I, C_M^{\cdot}(I))=0$ for all $i \in \mathbb{Z}$.

Let $\check{C}_{\underline{x}}$ be the Čech complex with respect to $\underline{x}=x_1,\ldots,x_s \in I$ such that $\operatorname{Rad} I=\operatorname{Rad}(\underline{x})R$. Then it implies that $\check{C}_{\underline{x}} \otimes_R D(C_M^{\cdot}(I))$ is an exact complex. This is due to the fact that $H_I^i(D(C_M^{\cdot}(I)))=0$ for all $i \in \mathbb{Z}$.

Suppose that $L_R^{\cdot}$ denotes a free resolution of $D(C_M^{\cdot}(I))$. Let $X := R/I \otimes_R L_R^{\cdot}$, then there is an isomorphism:

$$\operatorname{Tor}_{-i}^R(R/I, D(C_M^{\cdot}(I))) \cong H^i(X),$$

for all $i \in \mathbb{Z}$. Since the support of each module of X is contained in $V(I)$. It follows that there is an isomorphism of complexes:

$$\check{C}_{\underline{x}} \otimes_R X \cong X.$$

Let $F_{\cdot}^R$ be a free resolution of the finitely generated R-module R/I. Then by the above arguments, the following complex is exact:

$$Y := F_{\cdot}^R \otimes_R \check{C}_{\underline{x}} \otimes_R D(C_M^{\cdot}(I)).$$

As $F_{\cdot}^R$ is a right bounded complex of finitely generated free R-modules and $\check{C}_{\underline{x}}$ is a bounded complex of flat R-modules. So, Y is quasi-isomorphic to $\check{C}_{\underline{x}} \otimes_R F_{\cdot}^R \otimes_R L_R^{\cdot}$. Since Y is homologically trivial, so is $\check{C}_{\underline{x}} \otimes_R F_{\cdot}^R \otimes_R L_R^{\cdot}$.

Note that the morphism of complexes $\check{C}_{\underline{x}} \otimes_R F_{\cdot}^R \otimes_R L_R^{\cdot} \to \check{C}_{\underline{x}} \otimes_R X$, induces an isomorphism in cohomologies. It follows that the complex $\check{C}_{\underline{x}} \otimes_R X$ is homologically trivial. Therefore, $\operatorname{Tor}_i^R(R/I, D(C_M^{\cdot}(I)))=0$ for all $i \in \mathbb{Z}$. Now, by Lemma 2.5, we get the following vanishing result:

$$\operatorname{Ext}_R^i(R/I, C_M^{\cdot}(I))=0 \text{ for all } i \in \mathbb{Z}.$$

This proves our claim. Hence, following the same lines as in proof of Theorem 3.5, it can be proved that M is cohomologically complete intersection with respect to I. □

In the following, we will prove some results on grade and co-grade, as an application of our results.

Corollary 3.7. *Let I be an ideal of an arbitrary ring R. Let M and N be finitely generated modules over R with $c=\operatorname{grade}(I,M)$. Then the following identities are true:*

(i) $c=\inf\{i \in \mathbb{N} : H_I^i(N,M) \neq 0\} = \inf\{i \in \mathbb{N} : H_I^{i-c}(N, H_I^c(M)) \neq 0\}$.

(ii) *Assume further that R is local, then:*

$$c = \inf\{i \in \mathbb{N} : U_i^I(N, D(M)) \neq 0\} = \inf\{i \in \mathbb{N} : U_{i-c}^I(N, D(H_I^c(M)) \neq 0\}$$
$$= \inf\{i \in \mathbb{N} : L_i\Lambda^I(N, D(M)) \neq 0\} = \inf\{i \in \mathbb{N} : L_{i-c}\Lambda^I(N, D(H_I^c(M))) \neq 0\}.$$

(iii) *In addition, if M is Artinian and $t := \operatorname{Cograde}_M(N/IN)$, then:*

$$t = \inf\{i \in \mathbb{N} : L_i\Lambda^I(N, M) \neq 0\} = \inf\{i \in \mathbb{N} : H_I^i(N, D(M)) \neq 0\}.$$

Proof. Since it is already known that:

$$c = \inf\{i \in \mathbb{N} : H_I^i(N, M) \neq 0\} \text{ and } t = \inf\{i \in \mathbb{N} : U_i^I(N, M) \neq 0\},$$

(see [29, Theorem 2.3], [2, Proposition 5.5] and [3, Theorem 4.2]). In view of Corollary 3.1(*ii*) and Proposition 3.3, the results can be easily deduced. □

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