A NEW CHARACTERIZATION OF $A_p(2)$ AND $A_{p-1}(2)$ WHERE $2^p - 1$ IS A PRIME

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Let G be a finite group, and $\mathrm{Irr}(G)$ be the set of complex irreducible characters of G. An element $g \in G$ is called a vanishing element if there exists an irreducible character $\chi \in \mathrm{Irr}(G)$ such that $\chi(g) = 0$. The set of orders of vanishing elements of G is denoted by $\mathrm{Vo}(G)$. A recent conjecture states that if G is a finite group and M is a finite nonabelian simple group such that $\mathrm{Vo}(G) = \mathrm{Vo}(M)$ and $|G| = |M|$, then $G \cong M$. In this paper, we give a positive answer to this conjecture for a family of classical simple groups, namely $A_p(2)$ and $A_{p-1}(2)$, where $p \neq 2, 3$ and $2^p - 1$ is a prime.

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1. INTRODUCTION

Let n be a positive integer. By $\pi(n)$ we mean the set of prime divisors of n. Let G be a finite group and $\pi(G)$ be the set of prime divisors of $|G|$. Denote by $\omega(G)$, the set of element orders of G. For a finite set of positive integers X, the prime graph $\Pi(X)$ is a graph whose vertices are the prime divisors of elements of X, and two distinct vertices p and q are adjacent if X has an element divisible by pq. We denote the graph $\Pi(\omega(G))$ by $GK(G)$ and we call it the prime graph or the Gruenberg-Kegel graph of G. The number of connected components of $GK(G)$ is denoted by $t(G)$, and the connected components of $GK(G)$ is denoted by $\pi_1(G), \cdots, \pi_{t(G)}(G)$. If there is no ambiguity, we use the notation π_i instead of $\pi_i(G)$. If $2 \in \pi(G)$, we assume that $2 \in \pi_1(G)$. It is easy to see that $|G|$ can be written as the product of coprime positive integers m_i such that $\pi(m_i) = \pi_i(G)$, for $i = 1, \cdots, t(G)$. These integers are called the order components of G.

We denote by $\mathrm{Irr}(G)$ the set of complex irreducible characters of G. We call an element $g \in G$, a vanishing element, if there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(g) = 0$. Put $\mathrm{Vo}(G)$, the set of orders of all vanishing elements of G. The prime graph $\Pi(\mathrm{Vo}(G))$ is denoted by $\Gamma(G)$ and is called the vanishing prime graph of G. Note that for every finite group G, $\Gamma(G)$ is a subgraph of $GK(G)$. There is a strong relation between the structure of a group G and the set $\mathrm{Vo}(G)$. For example, if a finite group G does not have any vanishing element whose order is divisible by p, where $p \in \pi(G)$, then G has a normal Sylow p-subgroup [2]. In [7], it is proved that if x is a non-vanishing element of a solvable group G, then x^2 is an element of the Fitting subgroup $F(G)$ and conjectured that $x \in F(G)$. In [13], this conjecture has been proved in a special case that if G is solvable and no Mersenne prime divides $|G|$, then every non-vanishing element of G is an element of $F(G)$. In [14], it is proved that the finite simple group A_5 is recognizable by its set of orders of vanishing elements. But not all finite simple groups are characterizable by their set of orders of vanishing elements. For example $\mathrm{Vo}(L_3(5)) = \mathrm{Vo}(Aut(L_3(5)))$, but $L_3(5) \ncong Aut(L_3(5))$. In [4], M. Foroudi Ghasemabadi et al. proposed the following conjecture that finite nonabelian simple groups are recognizable by their order and their set of orders of vanishing elements:

Conjecture. *Let G be a finite group and M be a finite nonabelian simple group such that* $\mathrm{Vo}(G) = \mathrm{Vo}(M)$ *and* $|G| = |M|$. *Then* $G \cong M$.

They proved this conjecture for $M = A_1(q)$, where $q \in \{5, 7, 8, 9, 17\}$, $A_4(4)$, A_7, $Sz(8)$ and $Sz(32)$. Also in [5], the conjecture has been proved where M is a sporadic simple group, an alternating group, $A_1(p)$, for an odd prime p, and finite simple K_3-groups and K_4-groups. In this paper, we show that this conjecture is true for classical simple groups $M = A_p(2)$ and $A_{p-1}(2)$, where $2^p - 1$ is a prime and $GK(M)$ is disconnected. So if $M = A_p(2)$, we assume that $p \neq 2$, and if $M = A_{p-1}(2)$, we assume that $p \neq 2, 3$. In fact, we prove the following theorem:

Main Theorem. *Let G be a group and $M = A_p(2)$, where $p \neq 2$ and $2^p - 1$ is a prime; or $A_{p-1}(2)$, where $p \neq 2, 3$ and $2^p - 1$ is a prime. Then $G \cong M$ if and only if $Vo(G) = Vo(M)$ and $|G| = |M|$.*

Let k and n be coprime integers. We recall that if there exists an integer x such that $x^2 \equiv k \pmod{n}$, then k is called a quadratic residue mode n, otherwise k is called a quadratic nonresidue mode n. For a prime p, the Legendre symbol (a/p) is defined as follows: $(a/p) = 1$ if a is a quadratic residue mode p, $(a/p) = -1$ if a is a quadratic nonresidue mode p, and $(a/p) = 0$ if $p \mid a$. It is a well known result due to Euler that $(-1/p) = (-1)^{(p-1)/2}$.

Let n and m be positive integers and p be a prime. We write $p^m || n$, if $p^m \mid n$ but $p^{m+1} \nmid n$. We write $n_p = p^m$, if $p^m || n$. All further notation can be found in [1], for instance.

2. PRELIMINARY RESULTS

Definition 2.1. A finite group G is called a 2-Frobenius group if it has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K and G/H are Frobenius groups with kernels H and K/H, respectively.

The following lemma summarizes the basic structural properties of a Frobenius group and a 2−Frobenius group:

LEMMA 2.2 ([9]). (a) *Let G be a Frobenius group and let H, K be the Frobenius complement and the Frobenius kernel of G, respectively. Then $t(G) = 2$ and the prime graph components of G are $\pi(H)$ and $\pi(K)$. Moreover, K is nilpotent and hence $GK(K)$ is a complete graph.*

(b) *If G is a 2−Frobenius group then $t(G) = 2$. With the notations of Definition 2.1, we also have $\pi_1 = \pi(G/K) \cup \pi(H)$ and $\pi_2 = \pi(K/H)$.*

The next lemma is a consequence of Gruenberg–Kegel Theorem (see [12]):

LEMMA 2.3. *If G is a finite group with disconnected prime graph $GK(G)$, then one of the following holds:*
(1) *the finite group G is a Frobenius group and $t(G) = 2$;*
(2) *the finite group G is a 2-Frobenius group and $t(G) = 2$;*
(3) *the finite group G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups and K/H is a nonabelian simple group, where H is a nilpotent group and $|G/K| \mid |\mathrm{Out}\ (K/H)|$.*

LEMMA 2.4 ([2, 3]). *If G is a finite nonabelian simple group except A_7, then $GK(G) = \Gamma(G)$.*

As a consequense of [8, Corollary 22.26], we get the following lemma:

LEMMA 2.5. *If $\chi \in Irr(G)$ vanishes on a p-element for some prime p, then $p \mid \chi(1)$.*

Let p be a prime number. A character $\chi \in Irr(G)$ is said to be of p-defect zero if p is not a divisor of $|G|/\chi(1)$. It is a well-known result that if χ is of p-defect zero, then for every element $g \in G$ which order is divisible by p, we have $\chi(g) = 0$ (see for example [6, Theorem 8.17]).

LEMMA 2.6 ([9, Lemma 2.5]). *Let G be a finite group with $t(G) \geq 2$, and let N be a normal subgroup of G. If N is a π_i-group for some prime graph component of G and $m_1, m_2, \ldots, m_r$ are some order components of G but not π_i-numbers, then $m_1 m_2 \ldots m_r$ is a divisor of $|N| - 1$.*

LEMMA 2.7 ([10, Lemma 8]). *Assume $q > 1$ is a natural number, $s = \prod_{i=1}^{n}(q^i - 1)$, p is a prime, $p \mid s$. We denote the power of p in the standard* *factorization of s by s_p. $e = min\{d : p \mid q^d - 1\}$, $q^e = 1 + p^r k$, $p \nmid k$. If $p > 2$ or $r > 2$, then $s_p < q^{np/(p-1)}$.*

LEMMA 2.8 ([15, Zsigmondy Theorem]). *Let p be a prime and let n be a positive integer. Then one of the following holds:*

(i) *there is a primitive prime p' for $p^n - 1$, that is , $p' \mid (p^n - 1)$ but $p' \nmid (p^m - 1)$, for every $1 \leq m < n$,*

(ii) *$p = 2$, $n = 1$ or 6,*

(iii) *p is a Mersenne prime and $n = 2$.*

LEMMA 2.9 ([9, Lemma 2.9]). *The equation $p^m - q^n = 1$, where p and q are primes and $m, n > 1$ has only one solution, namely $3^2 - 2^3 = 1$.*

3. MAIN RESULTS

THEOREM 3.1. *Let G be a group and $M = A_p(2)$, where $p \neq 2$ and $2^p - 1$ is a prime number. Then $G \cong M$ if and only if $Vo(G) = Vo(M)$ and $|G| = |M|$.*

Proof. If $G \cong M$, the result is obvious. Let $Vo(G) = Vo(M)$ and $|G| = |M| = 2^{p(p+1)/2} \prod_{i=1}^{p}(2^{i+1} - 1)$. According to Lemma 2.4, we have $\Gamma(G) = \Gamma(M) = GK(M)$. Hence, $\Gamma(G)$ has 2 connected components and $l = 2^p - 1$ is an isolated vertex in $\Gamma(G)$. So G has an l-element g such that $\chi(g) = 0$ for some irreducible complex character χ of G. Now, Lemma 2.5, implies that l divides $\chi(1)$. Since $l \| |M|$ and $|G| = |M|$, χ is an irreducible character of l-defect zero of G. So, by the fact that l is an isolated vertex in $\Gamma(G)$, we conclude that l is an isolated vertex in $GK(G)$. Hence $t(G) \geq 2$.

Step 1. Let G be a Frobenius group and let H, K be the Frobenius complement and the Frobenius kernel of G, respectively. Consequently, $\Gamma(G)$ has two connected components, namely $\pi(H)$ and $\pi(K)$. Since $2^p - 1$ is an isolated vertex in $\Gamma(G)$, then $2^p - 1$ is a connected component. Since $|H| \mid (|K| - 1)$, we conclude that $|H| = 2^p - 1 = l$. Let $p \neq 3, 7$. There exists a primitive prime divisor x of $2^{p-1} - 1$. Set $S \in \mathrm{Syl}_x(K)$, so $S \trianglelefteq G$ and $|S| \mid (2^{p-1} - 1)$. On the other hand, H acts fixed point freely on S, and consequently $|S| \equiv 1 \pmod{l}$, which is a contradiction. If $p = 3$, take $S \in \mathrm{Syl}_5(K)$, and if $p = 7$, take $S \in \mathrm{Syl}_{31}(K)$. By a similar argument one can get a contradiction.

Step 2. Let G be a 2-Frobenius group, so G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that $\pi_2(G) = \pi(K/H)$ and $|G/K| \mid (|K/H| - 1)$. Therefore $|K/H| = 2^p - 1$ and $|G/K| \mid 2(2^{p-1} - 1)$. Then $(2^{p-2} - 1) \mid |H|$. Let $p \neq 3, 5$, and x be a primitive prime of $2^{p-2} - 1$ and $S \in \mathrm{Syl}_x(H)$. So similarly to Step 1 we get a contradiction. If $p = 3$, then Sylow 7-subgroup of G acts fixed point freely on Sylow 5-subgroup of H, which implies that $7 \mid 5 - 1$, a contradiction. If $p=5$, then Sylow 31-subgroup of G acts fixed point freely on Sylow 7-subgroup of H, and we get a contradiction similarly.

Step 3. Therefore, by Lemma 2.3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H and G/K are π_1-groups, K/H is a nonabelian simple group with disconnected prime graph, H is a nilpotent group and $|G/K| \mid |\mathrm{Out}\ (K/H)|$. Now by [11, Tables 1a-1c], we consider each possibility for K/H, separately:

Case 1. Let $K/H \cong A_{p'}$, where $p'-2$ is not odd prime.

Therefore $p' = 2^p - 1$, and so $|K/H| = (2^p-1)!/2 \leq |G|$. The only possibility is $p=3$ and therefore $p'=7$, which is impossible since $p'-2$ is odd prime.

Similarly, K/H cannot be isomorphic to A_m, where $m \in \{p'+1, p'+2\}$ and m or $m-2$ is not odd prime and K/H cannot be isomorphic to $A_{p'}$, where p' and $p'-2$ are prime numbers.

Case 2. Let $K/H \cong {}^2A_{p'-1}(q)$, where $q = r^f$ and p' is an odd prime.

Therefore $(q^{p'}+1)/((q+1)(p',q+1)) = 2^p-1$. We know that

$$q^{p'-1} - 1 \geq \frac{q^{p'}+1}{(q+1)(p',q+1)} = 2^p - 1 \quad \Rightarrow \quad q^{p'-1} \geq 2^p.$$

On the other hand, we know that $q^{p'(p'-1)/2} \mid |S|$, where $S \in \mathrm{Syl}_r(G)$. Let $r \neq 2$, so by Lemma 2.7, $|S| < 2^{(p+1)r/(r-1)} \leq 2^{3(p+1)/2} \leq q^{3(p'-1)/2+3/2}$. Consequently, $p'(p'-1)/2 < 3p'/2$, which implies that $p'=3$.

First let $(p',q+1)=1$. Then

$$q(q-1) = 2(2^{p-1}-1) = 2(2^{(p-1)/2}-1)(2^{(p-1)/2}+1).$$

It is easy to see that either $q \mid 2^{(p-1)/2}-1$ or $q \mid 2^{(p-1)/2}+1$. If $q \mid 2^{(p-1)/2}-1$, then $2^{(p-1)/2}-1 = qB$ and $q-1 = 2(2^{(p-1)/2}+1)B$, for some positive integer B. Therefore, $2^{(p+1)/2}+3 \leq q \leq 2^{(p-1)/2}-1$, which is impossible. If $q \mid 2^{(p-1)/2}+1$, then $2^{(p-1)/2}+1 = qB$ and $q-1 = 2(2^{(p-1)/2}-1)B$, for some positive integer B. Therefore, $2^{(p+1)/2}-1 \leq q \leq 2^{(p-1)/2}+1$, which implies that $p=3$, $q=3$. But $|{}^2A_2(3)| \nmid |A_3(2)|$, a contradiction.

Let $(p',q+1)=3$, then

$$(q^2-q+1)/3 = (r^{3f}+1)/3(r^f+1) = 2^p-1 = l.$$

Obviously, l is a primitive prime divisor of $r^{6f}-1$. Since $|q(q-1)|_2 = 4$, it is obvious that f is odd. We claim that $\pi(f)=\{3\}$. Let $f = 3^i t$, for some non-negative integers i, t and $3 \nmid t$. So

$$(r^{3^{i+1}}+1)(1-r^{3^{i+1}}+\cdots+r^{3^{i+1}(t-1)})/3(r^{3^i t}+1) = l.$$

Therefore, $r^{3^{i+1}}+1 \mid 3(r^{3^i t}+1)$, so by Lemma 2.8 we get that $3^{i+1} \mid 3^i t$, a contradiction. So $\pi(f)=\{3\}$, and consequently $\pi(G/K) \subseteq \{3\}$ since

$|G/K| \mid |\mathrm{Out}\,(K/H)| = 3f$. Let $p \neq 3, 5$ and x be a primitive prime of $2^{p-2}-1$. Obviously $x \neq 2, 3, 7$. Therefore, x is a divisor of $2^p - 4 = (q^2 - q - 8)/3$. It is easy to get that $x \nmid |K/H| = q^3(q+1)(q^2-1)(q^2-q+1)/3$. So $x \in \pi(H)$. Let $T \in \mathrm{Syl}_x(H)$. So $T \trianglelefteq G$ and $|T| \mid 2^{p-2} - 1$. Now by Lemma 2.6 we have $|T| \equiv 1 \pmod{l}$, a contradiction. If $p = 3$, then $q = 5$, which is impossible since $|K/H| \nmid |G|$. If $p = 5$, then $q(q-1) = 92$, a contradiction.

Therefore $r = 2$ and so $(2^{fp'}+1)/((p', 2^f+1)(2^f+1)) = 2^p - 1$. Let x be a primitive prime divisor of $2^{2fp'} - 1$. Then $x \mid (2^p - 1)$, and so $2fp' \mid p$, which is a contradiction.

Similarly, $K/H \ncong {}^2A_{p'}(q)$, where $(q+1) \mid (p'+1)$ and p' is an odd prime.

Case 3. Let $K/H \cong C_n(q)$, where $q = r^f$ and $n = 2^m \geq 2$.

Therefore $(q^n+1)/(2, q-1) = 2^p - 1 = l$, it follows that $q^n \equiv -1 \pmod{l}$. So $(-1/l) = 1$, which implies that $l \equiv 1 \pmod 4$, a contradiction.

Similarly, K/H cannot be isomorphic to $B_n(q)$, where $n = 2^m \geq 4$ and q is odd, ${}^2D_n(q)$, where $n = 2^m \geq 4$, ${}^2D_n(2)$, where $n = 2^m + 1 \geq 5$, and ${}^2D_n(3)$, where $n = 2^m + 1 \neq p'$ and $m \geq 2$.

Case 4. Let $K/H \cong {}^2D_{p'}(3)$, where $p' \neq 2^n + 1$ and $p' \geq 5$.

Therefore $(3^{p'} + 1)/4 = 2^p - 1$. So $2^p - 1 < 3^{p'} - 1$, and hence $2^p < 3^{p'}$. Also we know that $3^{p'(p'-1)} \mid |K/H|$, so if $S \in \mathrm{Syl}_3(G)$, then $3^{p'(p'-1)} \mid |S|$. By Lemma 2.7, $|S| < 2^{3(p+1)/2}$. Therefore $3^{p'(p'-1)} < 2^{3(p+1)/2} < 3^{3(p'+1)/2}$, which is a contradiction.

Similarly, K/H cannot be isomorphic to ${}^2D_{p'}(3)$, where $p' = 2^n+1$, $B_{p'}(3)$ and $D_{p'}(q)$, where $p' \geq 5$ is a prime and $q = 2, 3, 5$.

Case 5. Let $K/H \cong C_{p'}(q)$, where $q = 2, 3$.

Let $q = 3$, then $(3^{p'}-1)/2 = 2^p - 1$, which is a contradiction by Lemma 2.9. Therefore $q = 2$, it follows that $2^{p'} - 1 = 2^p - 1$, and so $p' = p$. Consequently, $2^{p^2} \mid |G|$, which is a contradiction.

Similarly, K/H cannot be isomorphic to $D_{p'+1}(q)$, where $q = 2, 3$.

Case 6. Let $K/H \cong F_4(q)$, where $q = 2^m > 2$.

If $q^4 - q^2 + 1 = 2^p - 1$, then $q^2(q^2-1) = 2(2^{p-1}-1)$, which is a contradiction, since $4 \mid q^2(q^2-1)$. Therefore $q^4 + 1 = 2^p - 1$, it follows that $q^4 = 2(2^{p-1} - 1)$, which is a contradiction.

Similarly, K/H cannot be isomorphic to $F_4(q)$, where q is odd, ${}^3D_4(q)$.

Case 7. Let $K/H \cong {}^{\varepsilon}E_6(q)$, where $q = r^f$ and $\varepsilon = \pm 1$.

Therefore $(q^6 + \varepsilon q^3 + 1)/(3, q - \varepsilon 1) = 2^p - 1$. Since $(q^6 + \varepsilon q^3 + 1) \mid (q^{18} - 1)$, then $2^p < q^{18}$. Also we know that $q^{36} \mid |K/H|$, so if $S \in \mathrm{Syl}_r(G)$, then $q^{36} \mid |S|$. Let $r \neq 2$, then $|S| < 2^{r(p+1)/(r-1)} \leq 2^{3(p+1)/2}$, by Lemma 2.7. Therefore $q^{36} < 2^{3(p+1)/2} < q^{27+3/2}$, which is a contradiction. Hence $r = 2$. If $3 \mid (q - \varepsilon 1)$, then $2^{6f} + \varepsilon 2^{3f} + 1 = 3(2^p - 1)$. Therefore $2^{3f}(2^{3f} + \varepsilon) = 3 \cdot 2^p - 4$, which is a contradiction. Therefore, $3 \nmid (q-\varepsilon 1)$, so $2^{6f}+\varepsilon 2^{3f}+1 = 2^p-1$, which is a contradiction.

Similarly, K/H cannot be isomorphic to $G_2(q)$, where $q \equiv \varepsilon \pmod 3$, $\varepsilon = \pm 1$ and $q > 2$.

Case 8. Let $K/H \cong A_1(q)$, where $q = r^f$.

(8.1) Let $4 \mid (q-1)$. Obviously $q \neq 2^p-1$. Hence $(q+1)/2 = 2^p-1 = l$, and so by Lemma 2.8, we have l is a primitive prime divisor of $r^{2f}-1$, which implies that $2f$ is a divisor of $l-1 = 2(2^{p-1}-1)$. Therefore $|G/K| \mid 2(2^{p-1}-1)$. Note that

$$|K/H| = 4(2^p-1)(2^{p+1}-3)(2^{p-1}-1).$$

Let $p \neq 3, 5$ and x be a primitive prime divisor of $2^{p-2}-1$. Obviously $x \neq 2, 3, 5, 7$. It is easy to see that $x \nmid |K/H|$. If $x \mid |G/K|$, we have x is a divisor of $2^{p-1}-1$, a contradiction. So $x \mid |H|$. Let S be a Sylow x-subgroup of H. Obviously $S \trianglelefteq G$ and $|S| \mid 2^{p-2}-1$. On the other hand, by Lemma 2.6 we have $|S| \equiv 1 \pmod l$, which is impossible. If $p = 3$, then $q = 13$, which is a contradiction since $13 \nmid |G|$. If $p = 5$, then $q = 61$, which is a contradiction since $|K/H| \nmid |G|$.

(8.2) Let $4 \mid (q+1)$. If $q = 2^p-1$, then $f = 1$, by Lemma 2.9. Moreover, $|K/H| = 2^p(2^p-1)(2^{p-1}-1)$ and $|G/K| \mid |\mathrm{Out}\ (K/H)| = 2$. Therefore, $(2^{p-3}-1) \mid |H|$. Let $p \neq 3$ and $s \in \pi(2^{p-3}-1)$. Let $S \in \mathrm{Syl}_s(H)$, so $S \trianglelefteq K$. On the other hand S is cyclic, it follows that there exists a unique subgroup S_1 of S such that $|S_1| = s$, and so $S_1 \trianglelefteq K$. Let $L \in \mathrm{Syl}_l(K)$, so $L \ltimes S_1$ is a Frobenius group. Therefore, $l \mid (s-1)$, which is a contradiction. If $p = 3$, then $5 \in \pi(H)$ and Sylow 5-subgroup of H is normal in G. One can easily get a contradiction by Lemma 2.6. If $(q-1)/2 = 2^p-1$, then we get a contradiction by Lemma 2.9.

(8.3) Let $q = 2^f$. Obviously, $2^f+1 \neq 2^p-1$. Therefore $2^f-1 = 2^p-1$ and so $f = p$. Hence, $|K/H| = (2^{2p}-1)2^p$, which is a contradiction since $|K/H| \nmid |G|$.

Case 9. Let $K/H \cong {}^2B_2(q)$, where $q = 2^{2m+1} > 2$.

Let $2^{2m+1}-2^{m+1}+1 = 2^p-1$. So $2(2^{2m}-2^m+1) = 2^p$, which is a contradiction, since $p > 1$. Similarly, $2^{2m+1}+2^{m+1}+1 \neq 2^p-1$. Therefore $2^{2m+1}-1 = 2^p-1$ and so $2m+1 = p$. Hence, $|K/H| = 2^{2p}(2^p-1)(2^{2p}+1)$, which is a contradiction, since $|K/H| \nmid |G|$.

Similarly, K/H cannot be isomorphic to ${}^2F_4(q)$, where $q = 2^{2m+1} > 2$.

Case 10. Let $K/H \cong G_2(q)$, where $q = 3^f$.

Let $q^2+q+1 = 2^p-1$. We know that $(q^2+q+1) \mid (q^3-1)$, so $2^p < q^3$. Let $S \in \mathrm{Syl}_3(G)$, then $3^{6f} \mid |S|$, by the order of $|K/H|$. On the other hand, $|S| < 2^{3(p+1)/2} < q^6$, by Lemma 2.7, which is a contradiction. Therefore $q^2 - q + 1 = 2^p - 1$ and so $3^f(3^f - 1) = 2(2^{p-1} - 1)$. Since $4 \nmid (3^f - 1)$ it follows that f is odd. Let $f > 1$, then $9 \mid (2^{p-1} - 1)$ and so $6 \mid (p - 1)$. Therefore $7 \mid (2^{p-1} - 1)$, so $7 \mid (3^f - 1)$, which is a contradiction, since f is odd. Consequently, $f = 1$ and so $p = 3$, which is a contradiction since $13 \in \pi(K/H) \setminus \pi(G)$.

Similarly, K/H cannot be isomorphic to ${}^2G_2(q)$, where $q = 3^{2m+1}$.

Case 11. Let $K/H \cong E_8(q)$, where $q = r^f$.

Let $q^8 - q^6 + q^4 - q^2 + 1 = 2^p - 1$. Since $(q^8 - q^6 + q^4 - q^2 + 1) \mid (q^{20} - 1)$, then $2^p < q^{20}$. Let $S \in \mathrm{Syl}_r(G)$, so $q^{120} \mid |S|$. If $r \neq 2$, then by Lemma 2.7, $|S| < 2^{3(p+1)/2} < q^{30+3/2}$, which is a contradiction. Therefore $r = 2$ and so $2^{8f} - 2^{6f} + 2^{4f} - 2^{2f} + 1 = 2^p - 1$, which is a contradiction. Similarly if

$$2^p - 1 \in \{q^8 + q^7 - q^5 - q^4 - q^3 + q + 1, q^8 - q^7 + q^5 - q^4 + q^3 - q + 1, q^8 - q^4 + 1\},$$

then we get a contradiction.

Case 12. Let $K/H \cong M_{22}$.

It is clear that $2^p - 1$ is not equal to 5 or 11. So $2^p - 1 = 7$, hence $p = 3$. In this case $|K/H| \nmid |G|$, which is a contradiction.

Similarly, K/H cannot be isomorphic to other sporadic groups.

Case 13. Let $K/H \cong A_{p'-1}(q)$, where $q = r^f$, p' is an odd prime and $(p', q) \neq (3, 2), (3, 4)$.

Therefore $(q^{p'} - 1)/((q - 1)(p', q - 1)) = 2^p - 1$, and so $q^{p'} \geq 2^p$. Let $S \in \mathrm{Syl}_r(G)$, then $q^{p'(p'-1)/2} \mid |S|$.

If $r \neq 2$, then $|S| < 2^{3(p+1)/2} \leq q^{3(p'+1)/2}$, by Lemma 2.7. Consequently, $p'(p' - 1)/2 < 3(p' + 1)/2$, so $p' = 3$.

First let $(p', q - 1) = 1$. Then

$$q(q + 1) = 2(2^{p-1} - 1) = 2(2^{(p-1)/2} - 1)(2^{(p-1)/2} + 1).$$

So either $q \mid 2^{(p-1)/2} - 1$ or $q \mid 2^{(p-1)/2} + 1$. If $q \mid 2^{(p-1)/2} - 1$, then there exists a positive integer B such that $2^{(p-1)/2} - 1 = qB$ and $q + 1 = 2(2^{(p-1)/2} + 1)B$. Therefore, $2^{(p+1)/2} + 1 \leq q \leq 2^{(p-1)/2} - 1$, which is impossible. If $q \mid 2^{(p-1)/2} + 1$, then $2^{(p-1)/2} + 1 = qB$ and $q + 1 = 2(2^{(p-1)/2} - 1)B$, for a positive integer B. Therefore, $2^{(p+1)/2} - 3 \leq q \leq 2^{(p-1)/2} + 1$, which implies that $p = 3$, and so $q = 2$, a contradiction.

Let $(p', q - 1) = 3$, then

$$(q^2 + q + 1)/3 = (r^{3f} - 1)/3(r^f - 1) = 2^p - 1 = l.$$

Obviously, l is a primitive prime divisor of $r^{3f} - 1$. We claim that $\pi(f) =$

$\{3\}$. Let $f=3^it$, for some non-negative integers i, t and $3\nmid t$. So

$$(r^{3^{i+1}}-1)(1+r^{3^{i+1}}+\cdots+r^{3^{i+1}(t-1)})/3(r^{3^it}-1)=l.$$

Therefore, $r^{3^{i+1}}-1\mid 3(r^{3^it}-1)$, so by Lemma 2.8 we get that $3^{i+1}\mid 3^it$, a contradiction. Hence $\pi(f)=\{3\}$, and consequently $\pi(G/K)\subseteq\{2,3\}$ since $|G/K|\mid|\mathrm{Out}\ (K/H)|=6f$. Let $p\neq 3,5$ and x be a primitive prime divisor of $2^{p-2}-1$. Obviously $x\neq 2,3,7$. Therefore, x is a divisor of $2^p-4=(q^2+q-8)/3$. It is easy to get that

$$x\nmid|K/H|=q^3(q-1)^2(q+1)(q^2+q+1)/3.$$

So $x\in\pi(H)$. Let $T\in\mathrm{Syl}_x(H)$. So $T\trianglelefteq G$ and $|T|\mid 2^{p-2}-1$. Now by Lemma 2.6 we have $|T|\equiv 1 \pmod l$, a contradiction. If $p=3$, then $q=4$, which is impossible. If $p=5$, then $q(q+1)=92$, a contradiction.

Therefore $r=2$ and $(2^{fp'}-1)/((p',2^f-1)(2^f-1))=2^p-1$. Since $(p',q)\neq(3,4)$, $2^{fp'}-1$ has a primitive prime divisor, say x. Then $x\mid(2^p-1)$ and so $fp'\mid p$. Consequently, $f=1$, $p'=p$ and so $K/H\cong A_{p-1}(2)$. Obviously $2^{p+1}-1\mid|H|$. Let $p\neq 5$, and s be a primitive prime of $2^{p+1}-1$ and $S\in\mathrm{Syl}_s(H)$. Therefore $S\trianglelefteq G$ and $|S|\mid 2^{(p+1)/2}+1$. On the other hand, by Lemma 2.6 $|S|\equiv 1 \pmod l$, a contradiction. If $p=5$, one can easily get a contradiction.

Case 14. Let K/H be isomorphic to $A_{p'}(q)$, where $q=r^f$, $(q-1)\mid(p'+1)$ and p' is an odd prime.

Therefore $(q^{p'}-1)/(q-1)=2^p-1$, and so $q^{p'}\geq 2^p$. Let $S\in\mathrm{Syl}_r(G)$. So $q^{p'(p'+1)/2}\mid|S|$. If $r\neq 2$, then by Lemma 2.7,

$$q^{p'(p'+1)/2}\leq|S|<2^{3(p+1)/2}<q^{3(p'+1)/2},$$

which is a contradiction. So $r=2$ and $(2^{fp'}-1)/(2^f-1)=2^p-1$. If $2^{fp'}-1$ does not have a primitive prime, then $f=2$ and $p'=3$, which is impossible since $(q-1)\nmid(p'+1)$. So $2^{fp'}-1$ has a primitive prime, say x. Then $x\mid 2^p-1$, and so $fp'\mid p$, which implies that $f=1$ and $p=p'$. Therefore $K/H\cong A_p(2)$, $H=1$ and $G=K$. Therefore $G\cong A_p(2)$, as required.

Theorem 3.2. *Let G be a group and $M=A_{p-1}(2)$, where $p\neq 2,3$ and 2^p-1 is a prime number. Then $G\cong M$ if and only if $Vo(G)=Vo(M)$ and $|G|=|M|$.*

Proof. If $G\cong M$, the result follows obviously. Let $Vo(G)=Vo(M)$ and $|G|=|M|=2^{p(p-1)/2}\prod_{i=1}^{p-1}(2^{i+1}-1)$. Similarly to Theorem 3.1 we conclude that G has a normal series $1\trianglelefteq H\trianglelefteq K\trianglelefteq G$, such that $K/H\cong A_{p-1}(2)$, $H=1$ and $G=K$. Therefore $G\cong A_{p-1}(2)$, as required. □

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