# WARPED PRODUCTS, BIHARMONIC AND SEMI-CONFORMAL MAPS

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In this paper, biharmonicity of some special maps from or into a warped product manifold are studied. First, we obtain the biharmonicity conditions of semi-conformal maps from a Riemannian manifold into a warped product manifold. Next, we give some characterization for semi-conformal surjective maps from a warped product manifold onto a Riemannian manifold to be biharmonic. Finally, two new classes of non-harmonic biharmonic maps are constructed by using products of semi-conformal surjective maps $\Phi = \phi \times \psi : M \times N \longrightarrow P \times Q$ and warping the domain or target.

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## 1. INTRODUCTION

The notion of warped product manifolds was first studied by Bishop and O'Neill for constructing negative curvature manifolds, [8]. In view of Physics, warped product manifolds play a crucial role in plethora of physical application such as general relativity, string and super gravity theories, [7]. For instance, the best model of space-time that describes the out space near black holes or bodies with large gravitational force is given as a warped product manifold, [11].

In 1969, biharmonic maps, as an extension of harmonic maps, were first introduced by Eells and Sampson, [9]. A smooth map $\phi : (M, g) \longrightarrow (N, h)$ is called biharmonic if $\phi$ is a critical point of the bienergy functional defined as follows

$$E_2(\phi) := \frac{1}{2} \int_M |\, \tau(\phi) \,|^2 \, \mathrm{d}\nu_g,$$

where $\tau(\phi) := trace \nabla \mathrm{d}\phi$ is the tension field of $\phi$. The Euler-Lagrange equation associated to $E_2$ is obtained as follows

$$\tau_2(\phi) := -J^\phi(\tau(\phi)) = -\Delta^\phi \tau(\phi) - trace\, R^N(\mathrm{d}\phi, \tau(\phi))\mathrm{d}\phi = 0, \tag{1.1}$$

here $J^\phi$ is a Jacobi operator and $\Delta^\phi$ is the rough Laplacian on the sections of $\phi^{-1}(TN)$, [10]. The equation (1.1) is called biharmonic equation of $\phi$.

Since 2000, harmonic and biharmonic maps have been extensively studied by many scholars, see for instance, [1–6, 9, 10, 12, 13]. Two main research directions related to this topic are differential geometry and partial differential equation. From the differential geometric aspect, constructing the examples and classification results have become important. We shall try to follow this direction in this paper. In view of partial differential equation, biharmonic maps are solutions of a fourth order strongly elliptic semi-linear PDE.

In [6], the authors studied the biharmonicity of special maps between warped product manifolds such as projections, inclusions and product maps and gave examples. In [1], the authors constructed new examples of non-harmonic biharmonic maps by conformally deforming the domain metric of harmonic ones, while in [5], there were given new methods to construct non-harmonic biharmonic map by conformal change of metric on the target manifold of harmonic Riemannian submersions. Moreover, recently in [2], by exploiting biconformal transformations of the metric, the authors constructed biharmonic functions and mappings from Riemannian manifolds. Biharmonic semi-conformal maps between Riemannian manifolds were studied in [3, 12]. In [6], two new classes of non-harmonic biharmonic maps are given by using product of harmonic maps and warped product manifolds. In the present paper, this idea is taken by replacing harmonic by biharmonic semi-conformal map.

The present article is organized as follows.

In Section 2, the concept of warped product manifolds and semi-conformal maps are reviewed. In Section 3, the biharmonicity conditions of some special semi-conformal maps from a Riemannian manifold $P$ into a warped product manifold $M \times_f N$ are analyzed. In Section 4, some characterization are given for semi-conformal maps from a warped product manifold $M \times_f N$ onto a Riemannian manifold to be biharmonic maps. In Section 5, two new classes of non-harmonic biharmonic maps are constructed by using products of semi-conformal maps $\Phi = \phi \times \psi : M \times N \longrightarrow P \times Q$ and warping the domain or target.

Throughout this paper, we consider that $(M^m, g), (N^n, h), (P^p, \varrho), (Q^q, \rho)$ be Riemannian manifolds of dimensions $m, n, p$ and $q$, repectively. Let $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds. Denote the Levi-Civita connections on $M$ and $N$ by $\nabla^M$ and $\nabla^N$, respectively. Furthermore, the induced connection on $\phi^{-1}TN$ and $\otimes^2 T^*M \otimes \phi^{-1}TN$ are denoted by $\nabla^\phi$ and $\nabla$, respectively.

## 2. PRELIMINARIES

In this section, a few basic notions of warped product manifolds, distributions and semi-conformal maps are provided which will be used later. For more details see [3, 4, 9].

### 2.1. WARPED PRODUCT MANIFOLDS

Let $(M^m, g)$ and $(N^n, h)$ be Riemannian manifolds and let $f \in C^\infty(M)$ be a positive smooth function on $M$. The warped product $M \times_f N$ is the product manifold $M \times N$ equipped with the Riemannian metric

$$G_f = \pi_1^*(g) + (f \circ \pi_1)^2 \pi_2^*(h), \tag{2.1}$$

where $X, Y \in \Gamma(T(M \times N))$, $\pi_1 : M \times_f N \longrightarrow M$ and $\pi_2 : M \times_f N \longrightarrow N$ are canonical projection maps. The function $f$ is called warping function. A warped product manifold $M \times_f N$ with a constant warping function is said to be a direct product manifold.

THEOREM 2.1 ([6]). *Let $(M, g)$ and $(N, h)$ be Riemannian manifolds with the Levi-Civita connections $\nabla^M$ and $\nabla^N$, respectively. Let $\nabla$ and $\widetilde{\nabla}$ denote the Levi-Civita connections of direct product manifold $M \times N$ and warped product manifold $M \times_f N$, respectively. The Levi-Civita connection of warped product manifold is given by*

$$\widetilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2f^2} Y_1(f^2)(0, X_2) + \frac{1}{2f^2} X_1(f^2)(0, Y_2) - \frac{1}{2} h(X_2, Y_2)(grad\, f^2, 0), \tag{2.2}$$

*for any $(X_1, Y_1), (X_2, Y_2) \in \Gamma(T(M \times N))$, where $X_1, X_2 \in \Gamma(TM)$ and $Y_1, Y_2 \in \Gamma(TN)$. Here $(X_i, Y_i)$ , $i = 1, 2$, is identified with $(X_i, 0_2) + (0_1, Y_i)$.*

THEOREM 2.2 ([6]). *Let $(M, g)$ and $(N, h)$ be Riemannian manifolds with the Levi-Civita connections $\nabla^M$ and $\nabla^N$, respectively and let $R$ and $\tilde{R}$ denote the curvature tensors of $M \times N$ and $M \times_f N$, respectively. Then, the curvature tensor of warped product manifold $M \times_f N$ is given as follows:*

$$\begin{aligned}
\tilde{R}(X, Y) - R(X, Y) = &\frac{1}{2f^2} \{ (\nabla^M_{Y_1} grad\, f^2 - \frac{1}{2f^2} Y_1(f^2) grad\, f^2, 0) \wedge_{G_f} (0, X_2) \\
&- (\nabla^M_{X_1} grad\, f^2 - \frac{1}{2f^2} X_1(f^2) grad\, f^2, 0) \wedge_{G_f} (0, Y_2) \\
&- \frac{1}{2f^2} \mid grad\, f^2 \mid^2 (0, X_2) \wedge_{G_f} (0, Y_2) \}
\end{aligned} \tag{2.3}$$

*for any $X, Y \in \Gamma(T(M \times N))$, $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$, where $X_1, Y_1 \in \Gamma(TM)$ and $X_2, Y_2 \in \Gamma(TN)$.*

## 2.2. DISTRIBUTION

Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth submersion map between Riemannian manifolds. For each $x \in M$, the tangent space at $x$ splits $T_xM = T_x^{\mathcal{V}}M \oplus T_x^{\mathcal{H}}M$, where $T_x^{\mathcal{V}}M = ker\ \mathrm{d}\phi_x$ and $T_x^{\mathcal{H}}M$ is the orthogonal compliment of $T_x^{\mathcal{V}}M$. By this decomposition, there exists the horizontal distribution $\mathcal{H} = \cup_x T_x^{\mathcal{H}}M$ such that $TM = \mathcal{H} + \mathcal{V}$, where $\mathcal{V} = \cup_x T_x^{\mathcal{V}}M$ is the vertical distribution. This decomposition permits to write a vector field $X \in \chi(M)$ into the horizontal and vertical form $X = \mathcal{H}X + \mathcal{V}X$, uniquely. The fundamental tensor $A$ and $T$ associated to the horizontal and vertical distributions are given by

$$T(X,Y) = \mathcal{H}\nabla^M_{\mathcal{V}X}\mathcal{V}Y + \mathcal{V}\nabla^M_{\mathcal{V}X}\mathcal{H}Y, \tag{2.4}$$

and

$$A(X,Y) = \mathcal{H}\nabla^M_{\mathcal{H}X}\mathcal{V}Y + \mathcal{V}\nabla^M_{\mathcal{H}X}\mathcal{H}Y, \tag{2.5}$$

where $X, Y \in \Gamma(TM)$. Divergence and rough Laplacian of an arbitrary vector field $X$ on $M$ related to the horizontal distribution are defined as follows

$$div^{\mathcal{H}}X = \sum_{a=1}^{n} g(e_a, \nabla^M_{e_a}X), \tag{2.6}$$

and

$$\Delta^{\mathcal{H}}X = -\sum_{a=1}^{n}\Big\{\nabla^M_{e_a}\nabla^M_{e_a}X - \nabla^M_{\nabla^M_{e_a}e_a}X\Big\}, \tag{2.7}$$

where $\{e_a\}_{a=1}^n$ is a local orthonormal frame of $\mathcal{H}$. By reversing the roles of $\mathcal{V}$ and $\mathcal{H}$, the operators $div^{\mathcal{V}}$ and $\Delta^{\mathcal{V}}$ can be defined similarly. Furthermore, the mean curvator vector of the map $\phi$ is denoted by $\mu^{\mathcal{V}}$ and defined as follows

$$\mu^{\mathcal{V}} := \sum_{r=1}^{m-n} \mathcal{H}(\nabla^M_{e_r}e_r), \tag{2.8}$$

where $\{e_r\}_{r=1}^{m-n}$ is a local orthonormal frame of $\mathcal{V}$.

## 2.3. SEMI-CONFORMAL MAP

A smooth map $\phi : (M^m, g) \longrightarrow (N^n, h)$ is called semi-conformal map with dilation $\lambda$ if at each point $x \in M$, either $\mathrm{d}\phi_x \equiv 0$ or the linear map

$$\mathrm{d}\phi_x|_{\mathcal{H}_x} : \mathcal{H}_x \longrightarrow T_{\phi(x)}N, \tag{2.9}$$

is a surjective and conformal map with dilation $\lambda(x)$, [4]. For a semi conformal submersion map $\phi : (M^m, g) \longrightarrow (N^n, h)$ with dilation $\lambda$, the tension field of $\phi$ is given by $\tau(\phi) = \mathrm{d}\phi(\omega_\phi)$, where

$$\omega_\phi := (2-n)\mathcal{H}\, grad\, \ln\lambda - (m-n)\mu^{\mathcal{V}}, \tag{2.10}$$

here $\mu^{\mathcal{V}}$ denotes the mean curvature of the fibres, [4]. Now, we have

Lemma 2.3 ([3]). *Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a semi-conformal map with dilation $\lambda$ and $\upsilon$ be a vector field on $M$. Then*

$$\begin{aligned} trace_g(\nabla^\phi)^2(\mathrm{d}\phi(\upsilon)) &= \mathrm{d}\phi(trace\nabla^2\upsilon) + \nabla^\phi_\upsilon\tau(\phi) + 2 < \nabla\mathrm{d}\phi, \nabla\upsilon^\flat > \\ &\quad + \mathrm{d}\phi(Ricci^M(\upsilon)) - \lambda^2 Ricci^N(\mathrm{d}\varphi(\upsilon)), \end{aligned} \tag{2.11}$$

*where $\upsilon^\flat = g(\upsilon, \bullet)$ is the corresponding 1-form of $\upsilon$ determined by metric $g$ and $< \nabla\mathrm{d}\phi, \nabla\upsilon^\flat >= \sum_{i,j} \nabla^\phi\mathrm{d}\phi(e_i, e_j)\nabla\upsilon^\flat(e_i, e_j)$, here $\{e_i\}$ is a local orthonormal frame on M.*

By Lemma 2.3 and Equation (1.1), the biharmonic equation of $\phi$ can be written as follows

$$\mathrm{d}\phi(Tr\nabla^2\omega_\phi) + \nabla^\phi_{\omega_\phi}\mathrm{d}\phi(\omega_\phi) + 2 < \nabla\mathrm{d}\phi, \nabla\omega_\phi > + \mathrm{d}\phi(Ricci^M(\omega_\phi)) = 0, \tag{2.12}$$

where $\omega_\phi$ is defined by (2.10).

## 3. BIHARMONICITY OF $\phi : (P, \varrho) \longrightarrow (M \times_f N, G_f)$

In this section, we study the biharmonicity conditions of $\phi_{x_0}$, defined as follows

$$\begin{aligned} \phi_{x_0} : (P, \varrho) &\longrightarrow (M \times_f N, G_f) \\ y &\longrightarrow (x_0, \varphi(y)) \end{aligned} \tag{3.1}$$

where $\varphi : (P^p, \varrho) \longrightarrow (N^n, h)$ is a semi-conformal submersion map with dilation $\lambda$ and $x_0$ be an arbitrary point of $M$. Particularly, it is shown that the biharmonicity of the leaf $\{x_0\} \times N$ implies that $grad \mid grad\, f^2 \mid^2 = 0$, at $x_0$.

*Remark* 3.1. Let $\psi : (P^p, \varrho) \longrightarrow (M^m, g)$ be a biharmonic semi-conformal surjective map and $y_0$ be an arbitrary point of N. Then, $\phi_{y_0} : (P, \varrho) \longrightarrow (M \times_f N, G_f)$, defined by $\phi_{y_0}(y) = (\psi(y), y_0)$, is biharmonic.

Now, we use the methods of [1,3,4] to establish the following two lemmas.

Lemma 3.2. *Let $\varphi : (M, g) \longrightarrow (N, h)$ be a semi-conformal submersion map with dilation $\lambda$ and $X, Y \in \mathcal{H}$. Then*

$$\mathcal{H}\nabla_X Y = L(X, Y) - Y(\ln\lambda)X - \omega_{X,Y}\, grad\, \ln\lambda, \tag{3.2}$$

*where* $L(X,Y) := \sum_{i,j} X^\flat(e_i)e_i(Y^\flat(e_j))e_j$ *and* $\omega_{X,Y} := \sum_{i,j} \delta_{ij}X^\flat(e_i)Y^\flat(e_j)$, *here* $\{e_i\}$ *is a local orthonormal frame on M.*

*Proof.* Take an arbitrary point of $x_0 \in M$ and fix it. We choose a local orthonormal frame field $\{f_a\}_{a=1}^n$ corresponding to the normal coordinates in a neighborhood of $\varphi(x_0)$. Then, we define $e_a$ as the normalize horizontal lift of $f_a$ for $a = 1, \cdots, n$, such that $\mathrm{d}\varphi(e_a) = \lambda f_a$. Finally, we complete the horizontal local frame field $\{e_a\}_{1\leq a\leq n}$ to an orthonormal frame $\{e_j\}_{1\leq j\leq m} = \{e_a, e_r\}$, where $e_r \in ker\mathrm{d}\varphi$, $r \in \{n+1, \cdots, m\}$, in a neighborhood of $x_0$. On calculating at $x_0$, we have

$$\begin{aligned}\lambda^2 g([e_a, e_b], e_c) &= h(\mathrm{d}\phi([e_a, e_b]), \mathrm{d}\phi(e_c))\\ &= h(([\mathrm{d}\phi(e_a), \mathrm{d}\phi(e_b)]), \mathrm{d}\phi(e_c))\\ &= h(e_a(\lambda)f_b - e_b(\lambda)f_a, \lambda f_c)\end{aligned} \tag{3.3}$$

Thus

$$g([e_a, e_b], e_c) = e_a(\ln\lambda)\delta_{bc} - e_b(\ln\lambda)\delta_{ac} \tag{3.4}$$

Due to the fact that $\nabla^M$ is the Levi-Civita connection on M determined by the metric g, we have

$$2g(\nabla^M_{e_a} e_b, e_c) = g([e_a, e_b], e_c) + g([e_c, e_a], e_b) - g([e_b, e_c], e_a), \tag{3.5}$$

by combining (3.5) and (3.4) we obtain

$$\mathcal{H}\nabla^M_{e_a} e_b = -e_b(\ln\lambda)e_a + \delta_{ab}\, grad\ \ln\lambda, \tag{3.6}$$

at the point $x_0$. □

LEMMA 3.3. *Let* $\varphi : (M^m, g) \longrightarrow (N^n, h)$ *be a semi-conformal submersion map with dilation* $\lambda$ *between Riemannian manifolds. Then*

$$\begin{aligned}\nabla^\varphi_Y \mathrm{d}\varphi(X) &= \mathrm{d}\varphi(\nabla^M_Y X) + X(\ln\ \lambda)\mathrm{d}\varphi(Y) + Y(\ln\lambda)\mathrm{d}\varphi(X)\\ &\quad - \omega_{\mathcal{H}X,\mathcal{H}Y}\mathrm{d}\varphi(grad\ \ln\lambda) - \mathrm{d}\varphi\Big(A(X,Y) + A(Y,X)\\ &\quad + T(X,Y)\Big),\end{aligned} \tag{3.7}$$

*where* $X, Y \in \Gamma(TM)$ *and* $\omega_{X,Y} := \sum_{i,j} \delta_{ij}X^\flat(e_i)Y^\flat(e_j)$, *here* $\{e_i\}$ *is a local orthonormal frame on M.*

*Proof.* Taking an arbitrary point of $x_0 \in M$ and fix it. Let $\{e_i\}_{1\leq i\leq m} = \{e_a, e_r\}$ be the local orthonormal frame field in a neighborhood of $x_0$, as in proof of Lemma 3.2. By calculating at the point $x_0$ and considering (3.2), we have

$$
\begin{aligned}
\nabla^{\varphi}_{Y}\mathrm{d}\varphi(X) &= Y^{\flat}(e_i)e_i(X^{\flat}(e_a))\mathrm{d}\varphi(e_a) + X^{\flat}(e_a)Y^{\flat}(e_i)e_i(\ln\lambda)\mathrm{d}\varphi(e_a) \\
&= Y^{\flat}(e_i)e_i(X^{\flat}(e_j))\mathrm{d}\varphi(e_j) + Y(\ln\lambda)\mathrm{d}\varphi(X) \qquad (3.8)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{H}\nabla^{M}_{Y}X &= Y^{\flat}(e_i)e_i(X^{\flat}(e_a))e_a + X^{\flat}(e_j)Y^{\flat}(e_i)\mathcal{H}\nabla^{M}_{e_i}e_j \\
&= Y^{\flat}(e_i)e_i(X^{\flat}(e_a))e_a + X^{\flat}(e_a)Y^{\flat}(e_b)\mathcal{H}\nabla^{M}_{e_b}e_a \\
&\quad + X^{\flat}(e_r)Y^{\flat}(e_b)\mathcal{H}\nabla^{M}_{e_b}e_r + X^{\flat}(e_a)Y^{\flat}(e_s)\mathcal{H}\nabla^{M}_{e_s}e_a \\
&\quad + X^{\flat}(e_r)Y^{\flat}(e_s)\mathcal{H}\nabla^{M}_{e_s}e_r \qquad (3.9)
\end{aligned}
$$

Due to the fact that $\mathrm{d}\varphi(A(\mathcal{H}X,\mathcal{V}Y)) = \mathrm{d}\varphi(A(X,Y))$ and $\mathrm{d}\varphi(T(\mathcal{V}X,\mathcal{V}Y)) = \mathrm{d}\varphi(T(X,Y))$, we obtain

$$
\begin{aligned}
Y^{\flat}(e_i)e_i(X^{\flat}(e_a)\mathrm{d}\varphi(e_a) &= \mathrm{d}\varphi(\nabla^{M}_{Y}X) - \omega_{\mathcal{H}X,\mathcal{H}Y}\mathrm{d}\varphi(grad\ \ln\lambda) + X(\ln\lambda)\mathrm{d}\varphi(Y) \\
&\quad - \mathrm{d}\varphi(A(X,Y) + A(Y,X) + T(X,Y)). \qquad (3.10)
\end{aligned}
$$

By substituting (3.10) in (3.8), the formula (3.7) is obtained and hence completes the proof. $\square$

THEOREM 3.4. *Let $\varphi : (P^p, \varrho) \longrightarrow (N^n, h)$ be a non-harmonic biharmonic semi-conformal submersion map with dilation $\lambda$ between Riemannian manifold and $x_0$ be an arbitrary point of $M$. Then, $\phi_{x_0}$, defined by (3.1), is a non-harmonic biharmonic map if and only if $\lambda$ and $f$ and the following equations are satisfied*

$$
\omega_{\varphi} - \frac{1}{1+2(n-1)\lambda^2 f^2}(\mathcal{H}\,grad\ \ln\lambda^{n(n-6)} + n\mu) = 0, \qquad (3.11)
$$

*and*

$$
\begin{aligned}
&- div\ \omega_{\varphi} - div^{\mathcal{H}}\omega_{\varphi} - (n+2)\omega_{\varphi}(\ln\lambda) - 4n \mid grad\ \ln\lambda \mid^2 \\
&+ 2n\,\Delta\ln\lambda\, grad\, f^2 - \frac{n^2}{4}\lambda^2 grad \mid grad\ f^2 \mid^2 = 0, \qquad (3.12)
\end{aligned}
$$

*where $\omega_{\phi}$ defined by (2.10) and $\mu^{\mathcal{V}}$ is the mean curvature vector field.*

*Proof.* Take an arbitrary point of $p_0 \in P$ and fix it. Let $\{e_i\}$ be the local frame field in a neighbourhood of $p_0$, as in Lemma (3.2). By definition of tension field, we have

$$
\tau(\phi_{x_0}) = \sum_{i=1}^{p}\left\{\nabla^{\phi_{x_0}}_{e_i}\mathrm{d}\phi_{x_0}(e_i) - \mathrm{d}\phi_{x_0}(\nabla^{p}_{e_i}e_i)\right\}
$$

$$\begin{aligned}
&= \sum_{i=1}^{p} \Big\{ \tilde{\nabla}_{(0,\mathrm{d}\varphi(e_i))}(0, \mathrm{d}\varphi(e_i)) - (0, \mathrm{d}\varphi(\nabla^{p}_{e_i} e_i)) \Big\} \\
&= (0, \tau(\varphi)) - \frac{n}{2}\lambda^2 (grad\, f^2, 0) \circ \phi_{x_0} \\
&= (0, \mathrm{d}\varphi(\omega_\varphi)) - \frac{n}{2}\lambda^2 (grad\, f^2, 0) \circ \phi_{x_0}. \qquad (3.13)
\end{aligned}$$

Here it's obvious that $\phi_{x_0}$ is harmonic if and only if $\varphi$ is harmonic and $f$ is constant, but this is in contradiction with the assumption that $\varphi$ is a non-harmonic biharmonic map. Now, we compute the rough Laplacian of section $\tau(\phi_{x_0})$. By (2.2), we have

$$\begin{aligned}
\nabla^{\phi_{x_0}}_{e_i} \tau(\phi_{x_0}) &= \nabla^{\phi_{x_0}}_{e_i}(0, \mathrm{d}\varphi(\omega_\varphi)) - \frac{n}{2}\nabla^{\phi_{x_0}}_{e_i} \lambda^2 (grad\, f^2, 0) \circ \phi_{x_0} \\
&= \tilde{\nabla}_{(0,\mathrm{d}\varphi(e_i))}(0, \mathrm{d}\varphi(\omega_\varphi)) - \frac{n}{2}\lambda^2 \tilde{\nabla}_{(0,\mathrm{d}\varphi(e_i))}(grad\, f^2, 0) \circ \phi_{x_0} \\
&- n\lambda^2 e_i(\ln\lambda)(grad\, f^2, 0) \circ \phi_{x_0} \\
&= (0, \nabla^{\varphi}_{e_i} \mathrm{d}\varphi(\omega_\varphi)) - \frac{1}{2}\lambda^2(\omega^{\flat}_{\varphi}(e_i) + 2n e_i(\ln\lambda))(grad\ f^2, 0) \circ \phi_{x_0} \\
&- n\lambda^2 \mid grad\ f \mid^2 (0, \mathrm{d}\varphi(e_i)), \qquad (3.14)
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\nabla^{\phi_{x_0}}_{e_i}\nabla^{\phi_{x_0}}_{e_i} \tau(\phi_{x_0}) &= \nabla^{\phi_{x_0}}_{e_i}\nabla^{\phi_{x_0}}_{e_i}(0, \mathrm{d}\varphi(\omega_\varphi)) - \frac{n}{2}\nabla^{\phi_{x_0}}_{e_i}\nabla^{\phi_{x_0}}_{e_i} \lambda^2 (grad\, f^2, 0) \circ \phi_{x_0} \\
&= -\lambda^2 \mid grad\, f \mid^2 (5n\ e_i(\ln\lambda) + \omega^{\flat}_{\varphi}(e_i))(0, \mathrm{d}\varphi(e_i)) \\
&+ (0, \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i} \mathrm{d}\varphi(\omega_\varphi)) - \frac{1}{2}\Big\{ h(\mathrm{d}\varphi(e_i), \nabla^{\varphi}_{e_i}\mathrm{d}\varphi(\omega_\varphi)) \\
&+ \lambda^2(2\omega^{\flat}_{\varphi}(e_i)e_i(\ln\lambda) + e_i(\omega^{\flat}_{\varphi}(e_i)) + 4n(e_i(\ln\ \lambda))^2 \\
&+ 2n e_i(e_i(\ln\ \lambda)) - n\ h(\mathrm{d}\varphi(e_i), \mathrm{d}\varphi(e_i)) \mid grad\, f \mid^2) \Big\} \\
&(grad\, f^2, 0) \circ \phi_{x_0}. \qquad (3.15)
\end{aligned}$$

Moreover

$$\begin{aligned}
\nabla^{\phi_{x_0}}_{\nabla^{p}_{e_i} e_i} \tau(\phi_{x_0}) &= \nabla^{\phi_{x_0}}_{\nabla^{P}_{e_i} e_i}(0, \mathrm{d}\varphi(\omega_\varphi)) - \frac{n}{2}\nabla^{\phi_{x_0}}_{\nabla^{P}_{e_i} e_i} \lambda^2 (grad\, f^2, 0) \circ \phi_{x_0} \\
&= (0, \nabla^{\varphi}_{\nabla^{P}_{e_i} e_i} \mathrm{d}\varphi(\omega_\varphi)) - n\lambda^2 \mid grad\, f \mid^2 (0, \mathrm{d}\varphi(\nabla^{P}_{e_i} e_i)) \\
&- \frac{1}{2}\Big\{ h(\mathrm{d}\varphi(\omega_\varphi), \mathrm{d}\varphi(\nabla^{P}_{e_i} e_i)) + 2n\lambda^2 d\ln\lambda(\nabla^{P}_{e_i} e_i) \Big\} \\
&(grad\, f^2, 0) \circ \phi_{x_0}, \qquad (3.16)
\end{aligned}$$

Furthermore,

$$\sum_{i=1}^{p} h(\mathrm{d}\varphi(e_i), \nabla^{\varphi}_{e_i}\mathrm{d}\varphi(\omega_\varphi)) = \sum_{a=1}^{n} h(\mathrm{d}\varphi(e_a), \nabla^{\varphi}_{e_a}\mathrm{d}\varphi(\omega_\varphi))$$
$$= \lambda^2 \sum_{a=1}^{n} \Big\{ \varrho(e_a, \nabla^{P}_{e_a}\omega_\varphi) + \omega_\varphi(\ln\lambda)\delta_{aa} \Big\}$$
$$= \lambda^2 div^{\mathcal{H}}(\omega_\varphi) + n\lambda^2\omega_\varphi(\ln\lambda) \tag{3.17}$$

and

$$\sum_{i=1}^{p} h(\mathrm{d}\varphi(\omega_\varphi), \mathrm{d}\varphi(\nabla^{p}_{e_i}e_i)) = \lambda^2 \sum_{i=1}^{p} \varrho(\omega_\varphi, \nabla^{P}_{e_i}e_i)$$
$$= \lambda^2 \sum_{i=1}^{p} \Big\{ e_i(\omega^{\flat}_\varphi(e_i)) - \varrho(e_i, \nabla^{P}_{e_i}\omega_\varphi) \Big\}$$
$$= -\lambda^2 (div\ \omega_\varphi - \sum_{i=1}^{p} e_i(\omega^{\flat}_\varphi(e_i)) \tag{3.18}$$

By (3.15)–(3.18), we get

$$-\Delta^{\phi_{x_0}}\tau(\phi_{x_0}) = (0, -\Delta^{\varphi}\tau(\varphi_{x_0})) + \frac{\lambda^2}{2}\Big\{ (-div\ \omega_\varphi - div^{\mathcal{H}}\omega_\varphi$$
$$- (n+2)\omega_\varphi(\ln\ \lambda) - 4n \mid grad\ \ln\lambda \mid^2 + 2n\Delta\ln\lambda$$
$$+ n^2\lambda^2 \mid grad\, f \mid^2) \Big\} (grad\, f^2, 0) \circ \phi_{x_0}$$
$$+ \lambda^2 \mid grad\, f \mid^2 (0, \mathrm{d}\varphi(grad \ln\lambda^{n(n-6)} + nT - \omega_\varphi)) \tag{3.19}$$

On the other hand, by making use of (2.3), it can be seen that

$$trace\tilde{R}(\tau(\phi_{x_0}), \mathrm{d}\phi_{x_0})\mathrm{d}\phi_{x_0} = trace\tilde{R}(\mathrm{d}\phi_{x_0}(\omega_\varphi), \mathrm{d}\phi_{x_0})\mathrm{d}\phi_{x_0}$$
$$- \frac{n}{2}\lambda^2 trace\tilde{R}((grad\, f^2, 0), \mathrm{d}\phi_{x_0})\mathrm{d}\phi_{x_0}$$
$$= \lambda^2(0, Ricci^N(\mathrm{d}\varphi(\omega_\varphi)))$$
$$- 2(n-1)f^2\lambda^4 \mid grad\, f \mid^2 (0, \mathrm{d}\varphi(\omega_\varphi))$$
$$+ \frac{n^2}{8}\lambda^4 \Big\{ (grad\ \mid grad\ f^2 \mid^2, 0)$$
$$- \frac{1}{f^2} \mid grad\ f^2 \mid^2 (grad\ f^2, 0) \Big\} \tag{3.20}$$

by (1.1), (3.19) and (3.20), we have

$$\tau_2(\phi_{x_0}) = \frac{\lambda^2}{2}(-div\ \omega_\varphi - div^{\mathcal{H}}\omega_\varphi - (n+2)\omega_\varphi(\ln\ \lambda) - 4n \mid grad\ \ln\lambda \mid^2$$
$$+ 2n\Delta\ln\lambda)(grad\, f^2, 0)\circ\phi_{x_0} + \frac{n^2}{8}\lambda^4(grad \mid grad\, f^2 \mid^2, 0)\circ\phi_{x_0}$$
$$+ \lambda^2 \mid f \mid^2 \Big(0, \mathrm{d}\varphi(grad\ \ln\lambda^{n(n-6)}) + nT$$
$$- (1+2(n-1)\lambda^2 f^2)\omega_\varphi\Big) \tag{3.21}$$

Thus, biharmonicity of $\phi_{x0}$ implies that

$$- div\ \omega_\varphi - div^{\mathcal{H}}\omega_\varphi - (n+2)\omega_\varphi(\ln\lambda) - 4n \mid grad\ \ln\lambda \mid^2$$
$$+ 2n\Delta\ln\lambda\, grad\, f^2 - \frac{n^2}{4}\lambda^2 grad \mid grad\, f^2 \mid^2 = 0, \tag{3.22}$$

and

$$\omega_\varphi - \frac{1}{1+2(n-1)\lambda^2 f^2}(\mathcal{H}\, grad\ \ln\lambda^{n(n-6)} + n\mu) = 0, \tag{3.23}$$

This completes the proof. □

Now, we consider Theorem 3.4, when the map $\varphi$ is conformal between equidimensional manifolds. Note that, any conformal map $\phi : (M^n, g) \longrightarrow (N^n, h)$ between Riemannian manifolds of the same dimension ($n > 2$) is a local conformal diffeomorphism, (cf. [3, Theorem 11.4.6]).

Corollary 3.5. *Let $\varphi : (P^n, \varrho) \longrightarrow (N^n, h)$ be a non-harmonic biharmonic conformal map between manifolds of the same dimension $(n > 2)$ and $x_0$ be an arbitrary point of $M$. Then, $\phi_{x_0}$ defined by (3.1) is never a non-harmonic biharmonic map.*

*Proof.* Due to the fact that $\varphi$ has no critical points and considering (2.10), we have

$$\omega_\varphi = grad\ \ln\lambda^{2-n} \tag{3.24}$$

By using (3.11) and (3.24), the biharmonicity of $\phi_{x_0}$ implies that $grad\ ln\lambda \equiv 0$, but this is in contradiction with that $\varphi$ is a non-harmonic biharmonic map, and hence completes the proof. □

By Corollary 3.5, we get

Corollary 3.6 ([6]). *Let $x_0$ be an arbitrary point of $M$. Then, the inclusion map $i_{x_0} : (N, h) \longrightarrow (M \times_f N, G_f)$, defined by $i_{x_0}(y) = (x_0, y)$, is a* *non-harmonic biharmonic map if and only if*

$$grad \mid grad\, f^2 \mid^2 = 0, \tag{3.25}$$

*at* $x_0$.

*Example* 3.7. Let $\mathbb{S}^m$ be the $m-$dimensional unit sphere, $\mathbb{R}^n$ be the $n$-dimensional Euclidean space and $x_0$ be an arbitrary point of $\mathbb{R}^n - \{0\}$. Also, let $f(x) = \sqrt{\mid x \mid}$, for $x \in \mathbb{R}^n - \{0\}$. By (3.25), it can be shown that the map

$$\begin{aligned} i_{x_o} : \mathbb{S}^m &\longrightarrow \mathbb{R}^n - \{0\} \times_f \mathbb{S}^m \\ y &\longrightarrow (x_0, y) \end{aligned}$$

is a non-harmonic biharmonic map.

## 4. BIHARMONICITY OF $\phi : (M \times_f N, G_f) \longrightarrow (P, \varrho)$

Let $\varphi : (M^m, g) \longrightarrow (P^p, \varrho)$ and $\psi : (N^n, h) \longrightarrow (Q^q, \rho)$ are biharmonic semi-conformal maps with dilation $\lambda$ and $\sigma$, respectively. In this section, we give the same characterization for

$$\begin{aligned} \phi_1 : (M^m \times_f N^n, G_f) &\longrightarrow (P^p, \varrho) \\ (x, y) &\longrightarrow \varphi(x) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \phi_2 : (M^m \times_f N^n, G_f) &\longrightarrow (Q^q, \rho) \\ (x, y) &\longrightarrow \psi(y) \end{aligned} \tag{4.2}$$

to be biharmonic. In particular case, biharmonicity of projection maps from the warped product manifold onto a first (or second) factor are studied.

By using the methods of [3] and [12], we have the following lemma

Lemma 4.1. *Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a semi-conformal map with dilation $\lambda$ and $f \in C^\infty(M)$ be a smooth positive function. Then*

$$\begin{aligned} -J^\phi(\mathrm{d}\phi(grad\ \ln f)) &= \mathrm{d}\phi(grad\,(\Delta\ \ln f) + 2Ricci^M(grad\ \ln f)) + (2\Delta^{\mathcal{H}}\ \ln f \\ &- 2grad\ \ln\lambda(\ln f) - \omega_\phi(\ln\lambda))\mathrm{d}\phi(grad\ \ln\lambda) \\ &+ (\omega_\phi(\ln\lambda) + 2 \mid \mathcal{H}\ grad\ \ln\lambda \mid^2)\mathrm{d}\phi(grad\ \ln f) \\ &+ grad\ \ln\lambda(\ln f)\mathrm{d}\phi(\omega_\phi) + 4\mathrm{d}\phi(\nabla_{\mathcal{H}\ grad\ \ln\lambda}\mathcal{H}\ grad\ \ln f) \\ &- \mathrm{d}\phi(A(\omega_\phi, grad\ \ln f)) - 4\nabla df(e_a, e_r)\mathrm{d}\phi(A(e_a, e_r)) \\ &- 2\nabla df(e_r, e_s)\mathrm{d}\phi(T(e_r, e_s)) \end{aligned} \tag{4.3}$$

*Proof.* By making use of (2.11) and considering the following relation

$$trace\nabla^2(grad\ ln f) = grad\,\Delta\ln f + Ricci^M(grad\ \ln f),$$

see ([3], Eq. 3, pp. 406), we have

$$
\begin{aligned}
-J^{\phi}(\mathrm{d}\phi(grad\ \ln f)) &= \nabla^{\phi}_{gard\ \ln f}\tau(\phi) + 2 < \nabla \mathrm{d}\phi, \nabla d \ln f > \\
&\quad + \mathrm{d}\phi(gard\ \Delta \ln f) + 2Ricci^{M}(gard \ln f).
\end{aligned} \tag{4.4}
$$

First, we calculate the first term of the right hand side of (4.4). By (3.7), we get

$$
\begin{aligned}
\nabla^{\phi}_{grad\, lnf}\tau(\phi) &= \nabla^{\phi}_{grad\, lnf}\mathrm{d}\phi(\omega_\phi) \\
&= \mathrm{d}\phi(\nabla^{M}_{grad\ \ln f}\omega_\phi + grad\ \ln f(\ln\lambda)\omega_\phi - \omega_\phi(\ln\lambda) grad\ \ln\lambda \\
&\quad + \omega_\phi(\ln\lambda) grad\ \ln f - A(\omega_\phi, grad \ln f)).
\end{aligned} \tag{4.5}
$$

Now, we calculate the second term of the right hand side of (4.4). Taking an arbitrary point of $x_0 \in M$ and fix it. Let $\{e_j\} = \{e_a, e_r\}$ be an orthonormal frame at $x_0$, as in proof of Lemma 3.2, we get

$$
\begin{aligned}
< \nabla \mathrm{d}\phi, \nabla \mathrm{d}f > &= \nabla^{\phi}\mathrm{d}\phi(e_a, e_b)\nabla \mathrm{d}f(e_a, e_b) + (\nabla \mathrm{d}f(e_a, e_r) \\
&\quad + \nabla \mathrm{d}f(e_r, e_a))\nabla^{\phi}\mathrm{d}\phi(e_a, e_r) + \nabla^{\phi}\mathrm{d}\phi(e_r, e_s)\nabla \mathrm{d}f(e_r, e_s),
\end{aligned} \tag{4.6}
$$

By making use of (3.6), it follows that

$$
\begin{aligned}
\nabla^{\phi}\mathrm{d}\phi(e_a, e_b)\nabla \mathrm{d}f(e_a, e_b) &= 2\mathcal{H}\, grad\ \ln\lambda(e_a(\ln f))\mathrm{d}\phi(e_a) - e_a(e_a(\ln f)) \\
&\quad \mathrm{d}\phi(grad\ \ln\lambda) + \mid \mathcal{H}\, grad\ \ln\lambda \mid^2 grad \ln f \\
&\quad + (n-2) grad\ \ln\lambda(\ln f)\mathrm{d}\phi(grad \ln\lambda).
\end{aligned} \tag{4.7}
$$

By (3.6), at $x_0$, we have

$$
\begin{aligned}
\Delta^{\mathcal{H}} \ln f &= -e_a(e_a(\ln f)) + d \ln f(\nabla_{e_a} e_a) \\
&= -e_a(e_a(\ln f)) + (n-1) grad\ \ln\lambda(\ln f),
\end{aligned} \tag{4.8}
$$

and

$$
\begin{aligned}
\nabla^{\phi}_{\mathcal{H}\ grad\ ln\lambda}\mathrm{d}\phi(\mathcal{H} grad\ lnf) &= e_b(\ln\ \lambda)e_b(e_a(\ln f))\mathrm{d}\phi(e_a) \\
&\quad + e_b(\ln\ \lambda)e_a(\ln f)\nabla^{\phi}_{e_b}\mathrm{d}\phi(e_a) \\
&= \mathcal{H} grad\ \ln\lambda(e_a(\ln f))\mathrm{d}\phi(e_a) \\
&\quad + \mid \mathcal{H} grad\ \ln\lambda \mid^2 \mathrm{d}\phi(grad\ \ln f)
\end{aligned} \tag{4.9}
$$

By (4.7),(4.8) and (4.9) we get

$$
\begin{aligned}
\nabla^{\phi}\mathrm{d}\phi(e_a, e_b)\nabla \mathrm{d}f(e_a, e_b) &= \Delta^{\mathcal{H}} \ln f \mathrm{d}\phi(grad\ \ln\lambda) \\
&\quad + \mid \mathcal{H}\ grad\ \ln\lambda \mid^2 \mathrm{d}\phi(grad\ \ln f) \\
&\quad - grad\ \ln\lambda(\ln f)\mathrm{d}\phi(grad\ \ln\lambda) \\
&\quad + 2\mathrm{d}\phi(\nabla_{\mathcal{H} grad \ln\lambda}\mathcal{H}\ grad\ \ln f)
\end{aligned} \tag{4.10}
$$

Since $\nabla \mathrm{d}\phi(e_i, e_r) = -\mathrm{d}\phi(\nabla_{e_i} e_r)$, we have

$$\begin{aligned} -\nabla^{\phi}\mathrm{d}\phi(e_a, e_r)\nabla \mathrm{d}f(e_r, e_a)) = &\nabla^{\phi}\mathrm{d}\phi(e_a, e_r)(\nabla \mathrm{d}f(e_a, e_r) \\ &+2\nabla \mathrm{d}f(e_a, e_r)\mathrm{d}\phi(A(e_a, e_r)), \end{aligned} \tag{4.11}$$

and

$$\nabla^{\phi}\mathrm{d}\phi(e_r, e_s)\nabla \mathrm{d}f(e_r, e_s) = -\nabla \mathrm{d}f(e_r, e_s)T(e_r, e_s). \tag{4.12}$$

Substitute (4.10), (4.11), (4.12) into (4.7), we have

$$\begin{aligned} < \nabla \mathrm{d}\phi, \nabla \mathrm{d}f > = &(\Delta^{\mathcal{H}} \ln f - grad \ln \lambda(\ln f))\mathrm{d}\phi(grad\ \ln \lambda) \\ &+ \mid \mathcal{H}\ grad\ \ln \lambda \mid^2 \mathrm{d}\phi(grad\ \ln f) \\ &+ 2\mathrm{d}\phi(\nabla_{\mathcal{H}\ grad\ \ln \lambda}\mathcal{H}\ grad\ \ln f) \\ &- 2\nabla \mathrm{d}f(e_a, e_r)\mathrm{d}\phi(A(e_a, e_r)) - \nabla \mathrm{d}f(e_r, e_s)T(e_r, e_s), \end{aligned} \tag{4.13}$$

By substituting (4.5) and (4.13) in (4.4), we obtain (4.3). This completes the proof. $\square$

THEOREM 4.2. *Let $\varphi : (M^m, g) \longrightarrow (P^p, \varrho)$ be a biharmonic semi-conformal submersion map with dilation $\lambda$.Then, $\phi_1$ defined by (4.1) is a non-harmonic biharmonic map if and only if*

$$\begin{aligned} 0 = &\mathcal{H}grad\, \Delta\ \ln f + 2\mathcal{H}Ricci^M(grad\ \ln f) + (n/2)\mathcal{H}grad \mid grad\ \ln f \mid^2 \\ &+ 4\mathcal{H}\nabla_{\mathcal{H}grad\ \ln \lambda}\mathcal{H}grad\ \ln \lambda + (2\Delta^{\mathcal{H}}\ \ln f - 2grad\ \ln \lambda(\ln f) - \omega_\phi(\ln f) \\ &- \omega_\phi(\ln \lambda) - n \mid grad\ \ln f \mid^2)\mathcal{H}grad\ \ln \lambda + \mathcal{H}\nabla_{grad\ \ln f}\omega_\phi \\ &+ (2\omega_\phi(\ln \lambda) + 2 \mid \mathcal{H}grad\ \ln \lambda \mid^2 + 2n\ grad\ \ln \lambda(\ln f))\mathcal{H}grad\ \ln f \\ &+ 2grad\ \ln \lambda(\ln f)\omega_\phi - 4\nabla \mathrm{d}f(e_a, e_r)A(e_a, e_r) - 2\nabla \mathrm{d}f(e_r, e_s)T(e_r, e_s) \\ &- 2nA(\mathcal{H}grad \ln f, \mathcal{V}grad\ \ln f) - nT(\mathcal{V}grad\ \ln f, \mathcal{V}grad\ \ln f) \end{aligned} \tag{4.14}$$

*Proof.* Take an arbitrary point of $(x_0, y_0) \in M \times N$ and fix it. Let $\{U_i = (e_i, 0),\ U_{m+\alpha} = (0, \frac{1}{f}e_\alpha); 1 \leq i \leq m, 1 \leq \alpha \leq n\}$ be a local orthonormal frame around $(x_0, y_0)$, where $\{e_i\} = \{e_a, e_r\}$ is the local orthonormal frame in a neighbourhood of $x_0 \in M$, as in proof of Lemma (3.3) and $\{e_\alpha\}_{1\leq\alpha\leq n}$ be an arbitrary local orthonormal frame in a neighbourhood of $y_0 \in N$. By definition of tension field, we have

$$\tau(\phi_1) = \sum_{i=1}^{m} \left\{ \nabla^{\phi_1}_{U_i}\mathrm{d}\phi_1(U_i) - \mathrm{d}\phi_1(\tilde{\nabla}_{U_i}U_i) \right\}$$

$$
\begin{aligned}
&+\frac{1}{f^2}\sum_{\alpha=1}^{n}\left\{\nabla^{\phi_1}_{U_{m+\alpha}}\mathrm{d}\phi_1(U_{m+\alpha})-\mathrm{d}\phi_1(\tilde{\nabla}_{U_{m+\alpha}}U_{m+\alpha})\right\}\\
&=\tau(\phi)+n\,\mathrm{d}\phi(grad\ \ln f)\\
&=\mathrm{d}\phi(\omega_\phi)+n\mathrm{d}\phi(grad\ \ln f) \qquad (4.15)
\end{aligned}
$$

According to the biharmonic equation, we get

$$
\begin{aligned}
-J^{\phi_1}(\tau(\phi_1)) &= -\Delta^{\phi_1}\tau(\phi_1)-trace_{G_f}R^p(\mathrm{d}\phi_1,\tau(\phi_1))\mathrm{d}\phi_1\\
&=\tau_2(\phi)-nJ^{\phi}(\mathrm{d}\phi(grad\ \ln f))+n\nabla^{\phi}_{grad\ \ln f}\mathrm{d}\phi(\omega_\phi)\\
&+n^2\nabla^{\phi}_{grad\ \ln f}\mathrm{d}\phi(grad\ \ln f) \qquad (4.16)
\end{aligned}
$$

By Lemma (3.3), we have

$$
\begin{aligned}
n\nabla^{\phi}_{grad\ \ln f}\mathrm{d}\phi(\omega_\phi) &= n\mathrm{d}\phi(\nabla^{M}_{grad\ln f}\omega_\phi)+n\omega_\phi(\ln\lambda)\mathrm{d}\varphi(grad\ln f)\\
&-n\omega_\phi(\ln f)\mathrm{d}\varphi(grad\ln\lambda)+ngrad\ln\lambda(\ln f)\mathrm{d}\varphi(\omega_\phi)\\
&+nA(\omega_\phi,\mathcal{V}\,grad\ln f) \qquad (4.17)
\end{aligned}
$$

and

$$
\begin{aligned}
n^2\nabla^{\phi}_{grad\ \ln f}\mathrm{d}\phi(grad\ \ln f) &= \frac{n^2}{2}\mathrm{d}\phi(grad\ \mid grad\ \ln f\mid^2)\\
&+2n^2grad\ \ln f(\ln\lambda)\mathrm{d}\phi(grad\ \ln f)\\
&-n^2\mid grad\ \ln f\mid^2\mathrm{d}\phi(grad\ \ln\lambda)\\
&-n^2d\phi(T(grad\ \ln f,grad\ \ln f)\\
&+2A(grad\ \ln f,grad\ \ln f)) \qquad (4.18)
\end{aligned}
$$

Substituting (4.17) and (4.18) into (4.16), we obtain (4.14) and hence completes the proof. □

By Theorem 4.2, we have the following result

Corollary 4.3. *Let $\varphi:(M^n,g)\longrightarrow(P^n,\varrho)$be a biharmonic conformal map between Riemannian manifolds of the same dimension $n\geq 3$. Then, $\phi_1$ defined by (4.1) is biharmonic if and only if*

$$
\begin{aligned}
0 &= grad\,\Delta\ \ln f+2Ricci^M(grad\ \ln f)+(n/2)grad\mid grad\ \ln f\mid^2\\
&+4\nabla_{\mathcal{H}\,grad\ \ln\lambda}grad\ \ln\lambda+\nabla_{grad\ \ln f}grad\ln\lambda^{2-p}\\
&+(2\Delta\ln f-2grad\ln\lambda(\ln f)-(2-p)grad\ln\lambda(\ln f)\\
&-(2-p)grad\ln\lambda(\ln\lambda)-n\mid grad\ \ln f\mid^2)grad\ \ln\lambda\\
&+2((2-p)grad\ln\lambda(\ln\lambda)+\mid grad\ \ln\lambda\mid^2
\end{aligned}
$$

$$+ n\, grad\ \ln \lambda(\ln f))grad\ \ln f$$

$$+ (2-p) grad\ \ln \lambda(\ln f) grad \ln \lambda. \tag{4.19}$$

By Corollary 4.3, it can be easily seen that

COROLLARY 4.4. *The projection map of a warped product manifold onto its first factor is biharmonic if and only if*

$$gard\, \Delta \ln f + Ricci^M(gard\ \ln f) + \frac{n}{2} grad \mid grad\ \ln f \mid^2 = 0.$$

Similarly to the proof of Theorem 4.2, we get

THEOREM 4.5. *Let $\psi : (N^n, h) \longrightarrow (Q^q, \rho)$ be a biharmonic semi-conformal map with dilation $\lambda$ and let $M$ be a non-compact manifold. Then, the map $\phi_2 : (M^m \times_f N^n, G_f) \longrightarrow (Q^q, \rho)$, defined by (4.2) is biharmonic if and only if*

$$\Delta f^{-2} - \frac{q}{2} \mid gard\, f^{-2} \mid^2 = 0. \tag{4.20}$$

*Proof.* By calculating similar to (4.2), we have

$$\tau_2(\phi_2) = (\Delta\, f^{-2} - \frac{q}{2} \mid grad\, f^{-2} \mid^2) \mathrm{d}\varphi(\omega_\psi), \tag{4.21}$$

Since $\psi$ is a non-harmonic biharmonic map then $d\psi(\omega_\psi) \neq 0$. Therefore, the biharmonicity of $\phi_1$ implies that

$$\Delta f^{-2} - \frac{q}{2} \mid gard\, f^{-2} \mid^2 = 0. \tag{4.22}$$

This completes the proof. □

According to the proof of Theorem 4.5, we have

COROLLARY 4.6. *The projection map $\pi_2 : (M^m \times_f N^n, G_f) \longrightarrow (N^n, h)$, of a warped product manifold onto it's second factor is harmonic.*

*Remark* 4.7. The biharmonicity conditions of projection maps from a warped product manifolds are studied in [6].

## 5. PRODUCT MAPS

In this section, we study the biharmonicity conditions of the product maps between warped product manifolds. Let $\varphi : (M^m, g) \longrightarrow (P^p, \varrho)$ and $\psi : (N^n, h) \longrightarrow (Q^q, \rho)$ are biharmonic maps. It can be easily seen that the product map $\phi = \varphi \times \psi : (M \times N, g \oplus h) \longrightarrow (M \times N, g \oplus h)$ between direct product manifolds is biharmonic if and only if $\varphi$ and $\psi$ are biharmonic.

If the product metric (either as the domain or target) is modified, then the biharmonicity of product map may be failed. First, we consider the product map

$$\begin{aligned} \bar{\phi} : (M^m \times_f N^n, G_f) &\longrightarrow (P^p \times Q^q, \varrho \oplus \rho) \\ (x, y) &\longrightarrow (\varphi(x), \psi(y)) \end{aligned} \tag{5.1}$$

where $\varphi : (M, g) \longrightarrow (P, \varrho)$ and $\psi : (N, h) \longrightarrow (Q, \rho)$ are semi-conformal maps.

THEOREM 5.1. *Let $\varphi : (M^m, g) \longrightarrow (P^p, \varrho)$ and $\psi : (N^n, h) \longrightarrow (Q^q, \rho)$ are semi-conformal maps with dilation $\lambda$ and $\sigma$, respectively. Then, $\bar{\phi}$, defined by* (5.1)*, is biharmonic if and only if $\phi_1$ and $\phi_2$ defined by* (4.1)–(4.2) *are biharmonic.*

*Proof.* Take an arbitrary point of $(x_0, y_0) \in M \times N$ and fix it. Let $\{U_i, U_{m+\alpha}; i = 1, \cdots, m, \alpha = 1, \cdots, n\}$ be the local orthonormal frame, as in proof of Theorem (4.2). By definition of tension field, we have

$$\begin{aligned} \tau(\bar{\phi}) &= \sum_{i=1}^{m} \Big\{ \nabla_{U_i} \mathrm{d}\bar{\phi}(U_i) - \mathrm{d}\bar{\phi}(\tilde{\nabla}_{U_i} U_i) \Big\} \\ &+ \frac{1}{f^2} \sum_{\alpha=1}^{n} \Big\{ \nabla_{U_\alpha} \mathrm{d}\bar{\phi}(U_\alpha) - \mathrm{d}\bar{\phi}(\tilde{\nabla}_{U_\alpha} U_\alpha) \Big\} \\ &= (\tau(\varphi), 0) + \frac{1}{f^2}(0, \tau(\psi)) + n(\mathrm{d}\varphi(grad\ \ln f), 0) \end{aligned} \tag{5.2}$$

by (1.1) and (5.2), we have

$$\begin{aligned} -J^{\bar{\phi}}(\tau(\bar{\phi})) &= -J^{\bar{\phi}}(\mathrm{d}\bar{\phi}(\omega_1)) - J^{\bar{\phi}}(\frac{1}{f^2}\mathrm{d}\bar{\phi}(\omega_2)) - nJ^{\bar{\phi}}(\mathrm{d}\bar{\phi}(grad\ \ln f)) \\ &= (\tau_2(\varphi), 0) + \frac{1}{f^2}(0, \tau_2(\psi)) - n\Big( \Delta^{\varphi}(\mathrm{d}\varphi(grad\ \ln f) \\ &+ Ricci^P(\mathrm{d}\varphi(grad\ \ln f)), 0\Big) + \Big( \Delta f^{-2} - \frac{n}{2} f^2 \mid grad\, f^{-2} \mid^2) \\ &(0, d\psi(\omega_2)) + n(\nabla^{\varphi}_{grad\ \ln f} \mathrm{d}\varphi(\omega_1), 0) + n^2(\nabla^{\varphi}_{grad\ \ln f} \mathrm{d}\varphi(grad\ \ln f), 0\Big) \\ &= (\tau(\phi_1), 0) + (0, \tau(\phi_2)) \end{aligned} \tag{5.3}$$

Thus, biharmonicity of $\bar{\phi}$ implies $\phi_1$ and $\phi_2$ are biharmonic and hence completes the proof . □

Finally, we study the biharmonicity conditions of $\hat{\phi}$ defined as follows

$$\hat{\phi} = \widehat{id_M \times \varphi} : (M \times P, g \oplus h) \longrightarrow (M \times_f N, G_f)$$

$$(x,y)\longrightarrow(x,\varphi(y)) \tag{5.4}$$

where $id_M:(M,g)\longrightarrow(M,g)$ is an identity map and $\varphi:(P^p,\varrho)\longrightarrow(N^n,h)$ is a surjective semi-conformal map with dilation $\lambda$.

THEOREM 5.2. *Let $\varphi:(P^p,\varrho)\longrightarrow(Q^q,\rho)$ be a biharmonic semi-conformal map with dilation $\lambda$. Then, $\hat\phi$ defined by (5.4), is biharmonic if and only if*

$$\begin{aligned}0=&(-div\ \omega_\varphi-div^{\mathcal{H}}\omega_\varphi-(n+2)\omega_\varphi(\ln\lambda)-4n\mid grad\ \ln\lambda\mid^2\\ &+2n\Delta\ln\lambda)grad\, f^2-\frac{n^2}{4}\lambda^2 grad\mid grad\, f^2\mid^2-n\,Ricci^M grad\ f^2\end{aligned} \tag{5.5}$$

*and*

$$\begin{aligned}0=&(\lambda^2\mid grad\, f\mid^2(1+2(n-1)\lambda^2f^2)+\Delta\ \ln f+grad\ \ln f)\omega_\varphi\\ &-\lambda^2\mid grad\, f\mid^2(\mathcal{H}grad\ \ln\lambda^{n(n-6)}+nT),\end{aligned} \tag{5.6}$$

*Proof.* Taking an arbitrary point of $(x_0,y_0)\in M\times P$. Let $\{U_i=(e_i,0), U_{m+j}=(0,f_j),1\le i\le m,1\le j\le p\}$ is a local orthonormal frame in a neighborhood of $(x_0,y_0)$, where $\{e_i\}$ and $\{f_j\}$ be local orthonormal frames around $x_0$ and $y_0$, respectively. By definition of tension field, we have

$$\begin{aligned}\tau(\hat\phi)&=\sum_{i=1}^m\left\{\nabla^{\hat\phi}_{U_i}\mathrm{d}\hat\phi(U_i)-\mathrm{d}\hat\phi(\nabla_{U_i}U_i)\right\}+\sum_{j=1}^P\left\{\nabla^{\hat\phi}_{U_j}\mathrm{d}\hat\phi(U_j)-\mathrm{d}\hat\phi(\nabla_{U_j}U_j)\right\}\\ &=-\frac{n}{2}\lambda^2(grad\, f^2,0)+(0,\tau(\varphi))\\ &=-n\lambda^2f^2(grad\ \ln f,0)+\mathrm{d}\hat\phi(\omega_\varphi)\end{aligned} \tag{5.7}$$

By calculating bitension field, we get

$$\begin{aligned}\tau_2(\hat\phi)=&\sum_{i=1}^m\left\{\nabla^{\hat\phi}_{U_i}\nabla^{\hat\phi}_{U_i}\tau(\hat\phi)-\nabla^{\hat\phi}_{\nabla_{U_i}U_i}\tau(\hat\phi)+\tilde R(\tau(\hat\phi),\mathrm{d}\hat\phi(U_i))\mathrm{d}\hat\phi(U_i)\right\}\\ &+\sum_{j=1}^p\left\{\nabla^{\hat\phi}_{U_j}\nabla^{\hat\phi}_{U_j}\tau(\hat\phi)-\nabla^{\hat\phi}_{\nabla_{U_j}U_j}\tau(\hat\phi)+\tilde R(\tau(\hat\phi),\mathrm{d}\hat\phi(U_j))\mathrm{d}\hat\phi(U_j)\right\}\\ =&(\Delta\ln f+grad\ \ln f)(0,\mathrm{d}\varphi(\omega_\varphi))-\frac{n}{2}\lambda^2(trace(\nabla^M)^2grad\ f^2,0)\\ &-\frac{n}{2}\lambda^2(Ricci^M grad\ f^2,0)\\ &+\sum_{j=1}^p\left\{\nabla^{\hat\phi}_{U_j}\nabla^{\hat\phi}_{U_j}\tau(\hat\phi)-\nabla^{\hat\phi}_{\nabla_{U_j}U_j}\tau(\hat\phi)+\tilde R(\tau(\hat\phi),\mathrm{d}\hat\phi(U_j))\mathrm{d}\hat\phi(U_j)\right\}\\ =&\frac{\lambda^2}{2}(div\ \omega_\varphi-div^{\mathcal{H}}\omega_\varphi-(n+2)\omega_\varphi(\ln\lambda)-4n\mid grad\ \ln\lambda\mid^2\end{aligned}$$

$$
\begin{aligned}
&+ 2n\Delta \ln \lambda)(grad\, f^2, 0) \circ \phi_{x_0} + \frac{n^2}{8}\lambda^4(grad \mid grad\, f^2 \mid^2, 0) \circ \phi_{x_0} \\
&- \frac{n}{2}\lambda^2(Ricci^M grad\ f^2, 0) + \lambda^2 \mid grad\, f \mid^2 (0, \mathrm{d}\phi(grad\ \ln \lambda^{n(n-4)} \\
&+ nT)) - (\lambda^2 \mid grad\, f \mid^2 (1 + 2(n-1)\lambda^2 f^2) + \Delta\ \ln f \\
&+ grad\ \ln\ f)(0, \mathrm{d}\varphi(\omega_\varphi)) \qquad (5.8)
\end{aligned}
$$

Thus, biharmonicity of $\hat{\phi}$ implies that

$$
\begin{aligned}
0 = &(-div\ \omega_\varphi - div^{\mathcal{H}}\omega_\varphi - (n+2)\omega_\varphi(\ln \lambda) - 4n \mid grad\ \ln \lambda \mid^2 \\
&+ 2n\Delta \ln \lambda) grad\, f^2 - \frac{n^2}{4}\lambda^2 grad \mid grad\, f^2 \mid^2 - nRicci^M grad\ f^2 \qquad (5.9)
\end{aligned}
$$

and

$$
\begin{aligned}
0 = &(\lambda^2 \mid grad\, f \mid^2 (1 + 2(n-1)\lambda^2 f^2) + \Delta \ln f + grad\ \ln f)\omega_\varphi \\
&- \lambda^2 \mid grad\, f \mid^2 (\mathcal{H}\, grad\ \ln \lambda^{n(n-6)} + nT), \qquad (5.10)
\end{aligned}
$$

This completes the proof. □

REFERENCES


[1] P. Baird and D. Kamissoko, *On constructing biharmonic maps and metrics.* Ann. Global Anal. Geom. **23** (2003), 65–75.

[2] P. Baird, A. Fardoun and S. Ouakkas, *Biharmonic maps from biconformal deformations with respect to isoparametric functions.* Differential Geom. Appl. **50** (2017), 155-166.

[3] P. Baird, A. Fardoun and S. Ouakkas, *Conformal and semi-conformal biharmonic maps.* Ann. Global Anal. Geom. **34** (2008), 403–414.

[4] P. Baird and J.C. Wood, *Harmonic Morphisms Between Riemannian Manifolds.* Lond. Math. Soc. Monographs. New Series **29**, Oxford: Oxford Univ. Press, 2003.

[5] A. Balmus, *Biharmonic non-harmonicities and conformal changes.* An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **50** (2004), 361–372.

[6] A. Balmus, S. Montaldo and C. Oniciuc, *Biharmonic maps between warped product manifolds.* J. Geom. Phys. **57** (2007), 449–466.

[7] J.K. Beem, P. Ehrlich and T.G. Powell, *Warped product manifolds in relativity. Selected Studies.* A Volume Dedicated to the Memory of Albert Einstein. North-Holland, Armsterdam, 41–56, 1982.

[8] R.L. Bishop and B. O'Neill, *Manifolds of negative curvature.* Trans. Amer. Math. Soc. **145** (1969), 1–49.

[9] J. Eells and J.H. Sampson, *Harmonic mappings of Riemannian manifolds.* Amer. J. Math. **86** (1964), 109–160.

[10] G.Y. Jiang, *2-harmonic maps and their first and second variation formulas.* Chinese Ann. Math. Ser. A **7** (1986), 389–402.

[11] B. O'Neill, *Semi-Riemannian geometry with applications to relativity.* Pure Appl. Math. **103**, New York-London etc.: Academic Press, 1983.

[12] S. Ouakkas and D. Djebbouri, *Conformal maps, biharmonic maps, and the warped product.* Mathematics **4** (2016), *1*, 1–17.

[13] Premalatha and N. Nathiya, *Unique integral representation for the class of balanced separately super harmonic functions on a product network.* Math. Rep. (Bucur.) **18(68)** (2016), *3*, 299–313.

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