PROVING GOWERS' FIN_k THEOREM IN **ZF+DC**

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In this work, we show an alternative proof of Gowers' FIN_k Theorem [10] following some ideas from the well known proof of Hindman's Theorem given by Baumgartner [1] and an infinite version of Hales-Jewett Theorem given by Karagiannis [8]. The crucial step in our proof is based upon a finite version of Gowers' FIN_k Theorem due to Ojeda-Aristizabal [11] which only uses Peano Arithmetic. Therefore, our proof does not depend upon the existence of non principal ultrafilters, that is to say, it is in **ZF+DC** (Zermelo Fraenkel set theory with the principle of dependent choice).

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1. INTRODUCTION

In 1992, Gowers [10] strenghtened James' distortion Theorem for the Banach space c_0. This was done by proving that *every unconditional Lipschitz function* (and not only every unconditional norm function) *from the unit sphere of* c_0 *to* $\mathbb{R}$ *is oscillation stable* (cf. [10, Theorem 6], [2, Theorem 13.18, p. 312], [9, Theorem 11]). The proof is based on a pigeon-hole principle that we shall refer to as Gowers' FIN_k Theorem. Recall that, for every integer $k \geq 1$, the symbol FIN_k denotes the set of functions $p\colon \mathbb{N} \to \{0, 1, \dots, k\}$ with $k \in \mathrm{Im}(p)$ (here $\mathrm{Im}(p)$ denotes the image set of the function p). As usual, we denote $a < b$ for $a, b \in \mathrm{FIN}_k$ whenever $\max(\mathrm{supp}(a)) < \min(\mathrm{supp}(b))$. An infinite sequence $A = (a_0, a_1, \dots)$ contained in FIN_k is a *block sequence* if $a_i < a_{i+1}$, for every i. Here $\mathrm{supp}(p) = \{n \in \mathbb{N} \colon p(n) \neq 0\}$).

$\mathrm{FIN}_k^{[\infty]}$ denotes the collection of infinite block sequences contained in FIN_k. For $m, n \in \mathbb{N}$, $\mathrm{FIN}_k^{[m]}$ is the set of block sequences with length m and $\mathrm{FIN}_k(n)$ is the set of members of FIN_k whose support is less than n. The *tetris map* $T\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is given by $T(p)(n) = \max\{0, p(n) - 1\}$. The *combinatorial space* of A is the set $[A]$ of members in FIN_k having the form

$$T^{j_1} a_{n_1} + T^{j_2} a_{n_2} + \cdots + T^{j_r} a_{n_r} \tag{1}$$

with $n_1 < n_2 < \cdots < n_r$, every j_i is in $\{0, 1, \ldots, k-1\}$ and $\min\{j_i : i \leq r\} = 0$. For $m \in \mathbb{N}$, $[A \restriction m]$ is the set whose members in FIN_k having the form (1) with $n_r < m$.

As a particular case, we have FIN_1 can be identified with FIN, the set of finite non empty subsets of $\mathbb{N}$, via characteristic functions. For an infinite block sequence A in FIN, the combinatorial space $[A]$ is formed by finite unions of members of A. With this notation, Gowers' FIN_k Theorem reads as follows (cf. [10, Theorem 1]):

THEOREM 1 (Gowers). *Given a positive integer r and*

$$f\colon FIN_k \to \{1, 2, \ldots, r\}$$

there exists $A \in FIN_k^{[\infty]}$ such that f is constant on $[A]$.

By a compactness argument the following finite version of Theorem 1 can be proved:

THEOREM 2. *Given m, k, $r \in \mathbb{N}$ there exists $n = G(m, k, r) \in \mathbb{N}$ such that for every $f\colon FIN_k(n) \to \{1, 2, \ldots, r\}$ there exists $s = (s_0, s_1, \ldots, s_{m-1})$ in $FIN_k^{[m]}$ with $s_{m-1} \in FIN_k(n)$, such that f is constant on $[s]$.*

As far as we know, each proof of Theorem 1 is based upon the existence of certain non principal ultrafilters, thus some set theoretic hypothesis (the Ultrafilter Theorem **UF**) must be assumed in this case (see [13] and [9]). For $k = 1$, Theorem 1 is Hindman's Theorem for finite unions (see [6]) whose proof was simplified by Baumgartner (see [1] or [4]). Based on Baumgartner's proof of Hindman's Theorem, Karagiannis [8] proved infinite versions of the well known Hales-Jewett Theorem. The crucial step in Karagiannis' proof uses the (finite) Hales-Jewett Theorem. Recently, D. Ojeda-Aristizabal gave a proof of Theorem 2 in Peano Arithmetic. In this work, we follow an approach similar to that of Baumgartner and Karagiannis, and use Theorem 2 to give a proof of Theorem 1 for which ultrafilters are not needed. Hence, our proof is in **ZF + DC**.

2. GOWERS' FIN_k THEOREM IN ZF+DC

For $n \in \mathbb{N}$, $a \in \mathrm{FIN}_k$ and $A = (a_0, a_1, \ldots)$ in FIN_k, denote

$$A/n = (a_j, a_{j+1}, \ldots)$$

with $j = \min\{i : n < \min(\mathrm{supp}(a_i))\}$.

$$A/a = A/\max(\mathrm{supp}(a))$$

If $F \subseteq \mathrm{FIN}_k$ is finite denote

$$A/F = \bigcap_{b \in F} A/b$$

For $B \in \mathrm{FIN}_k^{[\infty]}$, denote $B \leq A$ if $[B] \subseteq [A]$. Notice that $\leq$ is reflexive and transitive.

Remark 1. It is well known that there exists an isomorphism between FIN_k and $[B]$ (as well as between $[B \restriction n]$ and $\mathrm{FIN}_k(n)$) for every $B \in FIN_k^{[\infty]}$. Hence Theorem 1 and Theorem 2 are still true for finite partitions of a given $[B]$.

Definition 1. $X \subseteq \mathrm{FIN}_k$ is **large in** $A \in \mathrm{FIN}_k^{[\infty]}$ if $[B] \cap X \neq \emptyset$ for every $B \leq A$.

It is readily verified from Definition 1 the following facts:

(1) If X is large in A and $B \leq A$ then X is large in B.
(2) For every $A \in \mathrm{FIN}_k^{[\infty]}$, $[A]$ is large in A.
(3) If X is large in A then X/n is large in A, for every $n \in \mathbb{N}$.
(4) If X is large in A and $X \subseteq Y$ then Y is large in A.
(5) X is large in A iff $X \cap [A]$ is large in A.
(6) Given $n \in \mathbb{N}$, if X is large in A and $X = X_1 \cup X_2 \cup \cdots \cup X_n$ then there exist $B \leq A$ and $j \in \{1, 2, \ldots, n\}$ such that X_j is large in B.

Notation. For a, $b \in \mathrm{FIN}_k$ with $a < b$, denote

$$\langle a, b\rangle = \{T^i a + T^j b \colon i, j \in \{0, 1, \ldots, k-1\},\ \min\{i, j\} = 0\}$$

Lemma 1. *If X is large in A then there exists a finite subset $F \subseteq [A]$ such that for every $x \in [A/F]$ there exists $b \in F$ such that $\langle b, x\rangle \subseteq X$.*

Proof. Let $A = (a_0, a_1, \ldots)$. Assume on the contrary that for every finite subset $F \subseteq [A]$ there exists some $x \in [A/F]$ for which $\langle b, x\rangle$ is not contained in X for any $b \in F$. We will construct a sequence B' such that $B' \leq A$ and $X \cap [B'] = \emptyset$, contradicting the assumption that X is large in A. Put $b_0 = a_0$. For every $b \in [A/b_0]$, let $Y_b^1 = \langle b_0, b\rangle$. Notice that $|Y_b^1| = 2k - 1$ and in particular, it does not depend on b. Let $m_1 = 2k - 1$. By Remark 1, we can use Theorem 2, in order to choose $N_1 = G(2, k, 2^{m_1})$ such that for every $f \colon [A/b_0 \restriction N_1] \to 2^{m_1}$ there exists a 2-block sequence s in $[A/b_0 \restriction N_1]$ such that f is constant on $[s]$. So, consider the correspondence

$$\Phi_1 \colon [A/b_0 \restriction N_1] \to 2^{m_1}$$

given by

$$\Phi_1(b)(T^i b_0 + T^j b) = \begin{cases} 1 & \text{if} \quad T^i b_0 + T^j b \in X \\ 0 & \text{if} \quad T^i b_0 + T^j b \notin X \end{cases}$$

By the choice of N_1 and the assumption about the finite subset $\{b_0\} \subseteq [A]$, there exists a 2-block sequence $s = (s_1, s_2)$ in $[A/b_0 \restriction N_1]$ such that $\langle b_0, b\rangle \cap X = \emptyset$ for every $b \in [s]$. Let $b_1 = s_2$. Assume that we have defined b_0, b_1, $\dots$, b_{n-1}. For every $b \in [A/b_{n-1}]$ let $Y^n_b = \bigcup_{x\in[(b_0,b_1\dots,b_{n-1})]} \langle x, b\rangle$. Since $|[(b_0, b_1 \dots, b_{n-1})]| = k^n - (k-1)^n$, we have that

$$m_n = |Y^n_b| = (2k-1)(k^n - (k-1)^n)$$

which does not depend on b. Let $N_n = G(2, k, 2^{m_n})$ as in Theorem 2. As before, define

$$\Phi_n \colon [A/b_{n-1} \restriction N_n] \to 2^{m_n}$$

by

$$\Phi_n(b)(T^i x + T^j b) = \begin{cases} 1 & \text{if} \quad T^i x + T^j b \in X \\ 0 & \text{if} \quad T^i x + T^j b \notin X \end{cases}$$

By the choice of N_n and the assumption about the finite subset

$$[(b_0, b_1 \dots, b_{n-1})] \subseteq [A]$$

there exists a 2-block sequence $u = (u_1, u_2)$ in $[A/b_{n-1} \restriction N_n]$ such that $\langle x, b\rangle \cap X = \emptyset$ for every $x \in [(b_0, b_1 \dots, b_{n-1})]$ and every $b \in [u]$. Let $b_n = u_2$. This concludes the construction of b_0, b_1, $\dots$ Now define $B' = (b'_0, b'_1, \dots)$ by $b'_j = b_{2j} + b_{2j+1}$. Then $B' \leq A$ and, by construction, $X \cap [B'] = \emptyset$, which contradicts that X is large in A. □

LEMMA 2. *If X is large in A then there exist $b \in [A]$ and $B \leq A$ such that $X_b = \{x \in X/b \colon \langle b, x\rangle \subseteq X\}$ is large in B.*

Proof. Let F be as in Lemma 1 and $B_0 = A/F$. Then, if $x \in X \cap [B_0]$ there exists $b \in F$ such that $\langle b, x\rangle \subseteq X$. That is to say, $x \in X_b$. Therefore $X \cap [B_0] \subseteq \bigcup_{b\in F} X_b$. By Properties 4, 5 and 6 from Definition 1, there exist $B \leq B_0$ and $b \in F$ such that X_b is large in B. □

LEMMA 3. *If X is large in A then there exists $B \leq A$ such that $[B] \subseteq X$.*

Proof. By using Lemma 2 we can obtain $B' = (a_n)_{n\in\mathbb{N}} \in \mathrm{FIN}_k^{[\infty]}$, $(X_n)_{n\in\mathbb{N}}$ and $(A_n)_{n\in\mathbb{N}}$ such that $A_0 = A$, $X_0 = X$ an for every n:

(1) $X_{n+1} \subseteq X_n$ and $A_{n+1} \leq A_n$.
(2) $a_n \in [A_n]$.
(3) X_n is large in A_n.
(4) If $x \in X_{n+1}$ then $\langle a_n, x\rangle \subseteq X_n$.

Now, we build $B = (x_0, x_1, \ldots)$ in the following way: Let $x_0 \in [B'] \cap X$. If we already have $x_0, x_1, \ldots, x_{n-1}$, define $k_n = \min\{j \colon x_{n-1} \in [(a_0, a_1, \ldots, a_j)]\}$ and pick any $x_n \in X_{k_n+1} \cap [B']/k_n$. Then B is as required. In fact, if $x \in [B]$, say

$$x = T^{j_1}x_{i_1} + T^{j_2}x_{i_2} + \cdots + T^{j_n}x_{i_n} + T^{j_l}x_l$$

with $i_1 < i_2 < \cdots < i_n < l$, let $b = T^{j_1}x_{i_1} + T^{j_2}x_{i_2} + \cdots + T^{j_n}x_{i_n}$. Assume that

$$b = T^{r_1}a_{m_1} + T^{r_2}a_{m_2} + \cdots + T^{r_t}a_{m_t}$$

with $m_1 < m_2 < \cdots < m_t$. Then $m_t < k_l$ and, since $x_l \in X_{k_l+1}$, we have that $x_l \in X_{m_t+1}$. Hence, by (4), $T^{r_{m_t}}a_{m_t} + T^{j_l}x_l \in X_{m_t} \subseteq X_{m_{t-1}+1}$. Again, by (4), $T^{r_{m_{t-1}}}a_{m_{t-1}} + T^{r_{m_t}}a_{m_t} + T^{j_l}x_l \in X_{m_{t-1}}$. And so on. Finally, we have $x \in X_{m_1} \subseteq X_0 = X$. $\square$

Proof of Theorem 1. Let $E = (e_0, e_1, \ldots) \in \text{FIN}_k^{[\infty]}$, with $e_j = k\chi_{\{j\}}$ for every $j \in \mathbb{N}$ (here $\chi_{\{j\}}$ denotes the characteristic function of $\{j\}$). Then FIN_k is large in E. By property (6) from Definition 1, there exist $B \leq E$ and $j \in \{1, 2, \ldots, m\}$ such that X_j is large in B. By Lemma 3, there exists $A \leq B$ such that $[A] \subseteq X_j$. $\square$

3. CONCLUDING REMARKS

Recall that the **principle of dependent choice** is (cf. [7]):

> **DC**: If R is a binary relation on a nonempty set $\mathcal{X}$, and if for every $x \in \mathcal{X}$ there exists $y \in \mathcal{X}$ such that yRx, then there exists a sequence
>
> $$x_0, x_1, \ldots, x_n, \ldots$$
>
> in $\mathcal{X}$ such that $x_{n+1}Rx_n$ for every $n \in \mathbb{N}$.

This is a necessary axiom in order to construct the theory of Banach spaces, because it is equivalent to the Baire category theorem (cf. [12, p. 95]). Also, the topological dual of c_0 is the separable Banach space ℓ_1, so there is no need of ultrafilters in order to construct linear functionals on c_0. Notice that we only use **DC** in the proof given above, with $\mathcal{X} = \text{FIN}_k^{[\infty]}$ and $R = \leq$. Since our proof only uses **DC**, and Theorem 2 is in Peano Arithmetic, it is clear that

$$\textbf{ZF} + \textbf{DC} \vdash \text{Gowers' FIN}_k \text{ Theorem}$$

This conclusion was to be expected in view of the works of Baumgartner and Karagiannis. Moreover, these results suggest that a finite version of the pigeon hole principle in the context of Topological Ramsey Spaces (see [13]) is sufficient to obtain the corresponding infinite version.

On the other hand, Feferman built a model of **DC** + ¬**UF** (see [5]). Thus Gowers' FIN_k Theorem is weaker than **UF** and, in particular, it is a principle of choice weaker than **AC**.

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