RECOGNITION OF SOME CHARACTERISTICALLY SIMPLE GROUPS BY THEIR COMPLEX GROUP ALGEBRAS

MORTEZA BANIASAD AZAD and BEHROOZ KHOSRAVI

Communicated by Lucian Beznea

In [20], the following question arose: *Which groups can be uniquely determined by the structure of their complex group algebras?* In this paper, we prove that the direct product G^n of n copies of a group G, where (a) $G \cong A_5$ and $n \le 5$; (b) $G \cong L_2(7)$ and $n \le 7$; (c) $G \cong L_3(3)$ and $n \le 13$; (d) $G \cong L_2(17)$ and $n \le 17$; are uniquely determined by their order and some information on irreducible character degrees. As a consequence of our results, we show that these groups are uniquely determined by the structure of their complex group algebras.

AMS 2010 Subject Classification: 20C15, 20D05, 20D60.

Key words: character degree, order, complex group algebra.

1. INTRODUCTION AND PRELIMINARY RESULTS

Let G be a finite group, $\mathrm{Irr}(G)$ be the set of irreducible characters of G, and denote by $\mathrm{cd}(G)$, the set of irreducible character degrees of G. If n is a natural number, by G^n we mean the direct product of n copies of G; that is, $G \times G \times \cdots \times G$.

In [5, Problem 2*], R. Brauer posed the following question: Let G and H are two finite groups. If for all fields $\mathbb{F}$, two group algebras $\mathbb{F}G$ and $\mathbb{F}H$ are isomorphic, can we get that G and H are isomorphic? In [7], E.C. Dade showed that this is false in general.

It was shown in [16, 21] that the symmetric groups are uniquely determined by the structure of their complex group algebras. In [15, 17, 20, 22, 23] it is proved that each nonabelian simple group is uniquely determined by its complex group algebra. In [20], Tong-Viet posed the following question:

QUESTION. *Which groups can be uniquely determined by the structure of their complex group algebras?*

In [4, 19], it is proved that every quasisimple group L is uniquely determined up to isomorphism by the structure of $\mathbb{C}L$, the complex group algebra of L. In [13] and [14], it is proved that if $q \mid p^2$, where $p > 3$ is an odd prime, $S = L_2(q)$, M is a finite group such that $S < M < \mathrm{Aut}(S)$, $M = \mathbb{Z}_2 \times L_2(q)$ or $M = \mathrm{SL}(2, q)$, then M is uniquely determined by its complex group algebra.

One of the next natural groups to be considered are the characteristically simple groups. Khosravi *et al.* proved that $L_2(p) \times L_2(p)$ is uniquely determined by its complex group algebra, where $p \geq 5$ is a prime number (see [12]). In [1], we prove that if M is a simple K_3-group, then $M \times M$ is uniquely determined by its order and some information on irreducible character degrees. In [2], we proved that the direct product of non-isomorphic Suzuki groups is uniquely determined by its complex group algebra.

In this paper, we prove that the direct product G^n of n copies of a group G, where (a) $G \cong A_5$ and $n \leq 5$; (b) $G \cong L_2(7)$ and $n \leq 7$; (c) $G \cong L_3(3)$ and $n \leq 13$; (d) $G \cong L_2(17)$ and $n \leq 17$; are uniquely determined by their order and some information on irreducible character degrees. As a consequence of our results we show that these groups are uniquely determined by the structure of their complex group algebras.

If $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$, then the inertia group of θ in G is $I_G(\theta) = \{g \in G \mid \theta^g = \theta\}$. If the character $\chi = \sum_{i=1}^{k} e_i \chi_i$, where for each $1 \leq i \leq k$, $\chi_i \in \mathrm{Irr}(G)$ and e_i is a natural number, then each χ_i is called an irreducible constituent of χ.

Lemma 1 ([11, Theorems 6.2, 6.8, 11.29]). *Let $N \trianglelefteq G$ and let $\chi \in \mathrm{Irr(G)}$. Let θ be an irreducible constituent of χ_N and suppose $\theta_1 = \theta, \ldots, \theta_t$ are the distinct conjugates of θ in G. Then $\chi_N = e \sum_{i=1}^{t} \theta_i$, where $e = [\chi_N, \theta]$ and $t = |G : I_G(\theta)|$. Also $\theta(1) \mid \chi(1)$ and $\chi(1)/\theta(1) \mid |G : N|$.*

Lemma 2 ([24, Lemma 1]). *Let G be a nonsolvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$.*

Given a natural number n, let $\mathrm{P}(n)$ denote the greatest prime factor of n, and let n_r, where r is a prime, denote the r-part of n, *i.e.*, the largest power of r that divides n. For every integer a coprime to n, let $\mathrm{Ord}_n(a)$ denote the smallest positive integer e such that $a^e \equiv 1 \pmod{n}$. If s is a prime number, then we write $s^k \| n$, when $s^k \mid n$ but $s^{k+1} \nmid n$.

Using [25] we have the following result:

Lemma 3. *If $n > 2$ and $a > b > 0$, then $n + 1 \leq \mathrm{P}(a^n - b^n)$.*

Lemma 4 ([18, Theorems 3.6]). *Let p be an odd prime, and let $a \neq \pm 1$ be an integer not divisible by p. Let d be the order of a modulo p. Let k_0 be the largest integer such that $a^d \equiv 1 \pmod{p^{k_0}}$. Then the order of a modulo p^k is d for $k = 1, \ldots, k_0$ and dp^{k-k_0} for $k > k_0$.*

2. THE MAIN RESULTS

Lemma 5. *Let M be a finite group such that $p \mid |M|$. If there exists $\chi \in \mathrm{Irr(G)}$, where $\chi(1) = |M|_p$, then $O_p(M) = 1$.*

Proof. Let $|M|_p = p^j$. Assume on the contrary $L = O_p(M) \neq 1$ and $|L| = p^i$, where $1 \leq i \leq j$. If $\eta \in \mathrm{Irr(L)}$ such that $[\chi_L, \eta] \neq 0$, then by Lemma 1, $p^j/\eta(1)$ is a divisor of $|M : L|_p = p^{j-i}$. Since $\eta(1) \mid |L|$, we get that $\eta(1) = p^i$. On the other hand, $\sum_{\nu \in \mathrm{Irr(L)}} \nu^2(1) = |L|$, which is a contradiction. □

Lemma 6. *Let S be a finite nonabelian simple group and let $p_0 = \mathrm{P}(|S|)$. If G is an extension of S^m by S^n, where $m + n \leq p_0$, then $G \cong S^{m+n}$.*

Proof. We claim that $p_0 \nmid |\mathrm{Out(S)}|$. Obviously, $p_0 \geq 5$. If S is an alternating group or a sporadic simple group, then by page *ix* and Table 1 in [6], we get that $|\mathrm{Out}(S)| \leq 4$. Therefore we assume that S is a simple group of Lie type over GF(q), where $q = p^f$. By the notations in [6, Page *xv* and Table 5], $|\mathrm{Out(S)}| = dfg$, where d, f and g are the orders of the diagonal, field and graph automorphisms of S, respectively. Let k be the largest integer such that $q^k - 1$ is a divisor of $|S|$. By Lemma 3, $fk + 1 \leq \mathrm{P}(q^k - 1)$.

Assume that $S \cong L_{l+1}(q)$, where $l \geq 2$. Then $k = l + 1$. Hence

$$\max\{d, f, g\} < lf + f + 1 \leq \mathrm{P}(p^{(l+1)f} - 1) \leq p_0.$$

Suppose that $S \cong U_{l+1}(q)$, where $l \geq 2$. we know that f is an even number. If l is an even number, then $k = l$. Thus

$$\max\{d, f, g\} < lf + 1 \leq \mathrm{P}(p^{lf} - 1) \leq p_0.$$

If l is an odd number, then $k = l + 1$. Therefore

$$\max\{d, f, g\} < lf + f + 1 \leq \mathrm{P}(p^{(l+1)f} - 1) \leq p_0.$$

Therefore $p_0 \nmid |\mathrm{Out}(S)|$. For other cases, easily we can check that p_0 does not divide $|\mathrm{Out}(S)|$. Therefore the claim is proved.

By assumptions, there exists a normal subgroup H_m of G, which is isomorphic to S^m. We know that $\mathrm{Out}(H_m) \cong \frac{\mathrm{Aut}(H_m)}{\mathrm{Inn}(H_m)}$ and $\mathrm{Inn}(H_m) \cong \frac{H_m}{Z(H_m)}$. Therefore $|\mathrm{Aut}(H_m)| = |\mathrm{Out}(H_m)|\frac{|H_m|}{|Z(H_m)|}$. Since S is a non-abelian simple group and $H_m \cong S^m$, we have $Z(H_m) = 1$ and $|\mathrm{Aut}(H_m)| = |\mathrm{Out}(H_m)||H_m|$. On the other hand, by [8, Page 131], we have $\mathrm{Out}(H_m) \cong \mathrm{Out}(S) \wr S_m$. Therefore

$$\begin{aligned} \frac{G}{C_G(H_m)} \hookrightarrow \mathrm{Aut}(H_m) &\Longrightarrow \left|\frac{G}{C_G(H_m)}\right| \mid |\mathrm{Aut}(H_m)| = |\mathrm{Out}(H_m)||H_m| \\ &\Longrightarrow \frac{|G|}{|C_G(H_m)|} \mid |\mathrm{Out}(S^m)||S^m| = |\mathrm{Out}(S)|^m m!|S^m| \end{aligned}$$

$$\Longrightarrow \frac{|S^m||S^n|}{|C_G(H_m)|} \mid |\mathrm{Out}(S)|^m m!|S^m|$$
$$\Longrightarrow \frac{|S^n|}{|C_G(H_m)|} \mid |\mathrm{Out}(S)|^m m!.$$

Since $p_0 > m$, $p_0 \mid |S|$ and $p_0 \nmid |\mathrm{Out}(S)|^m m!$, we get that $p_0 \mid |C_G(H_m)|$ and so $|C_G(H_m)| \neq 1$. As S is a nonabelian simple group, $H_m \cap C_G(H_m) = 1$ and it follows that $H_m C_G(H_m) \cong S^m \times C_G(H_m)$. Since $C_G(H_m) \cong H_m C_G(H_m)/H_m \trianglelefteq G/H_m \cong S^n$, we have $C_G(H_m) \cong S^i$, where $1 \leq i \leq n$.
Put $L = H_m C_G(H_m) \cong S^{m+i}$. Now we consider the possibilities for n:

- If $n = 1$, then $G \cong S^{m+1}$.
- If $n = 2$, then we have two cases. If $C_G(H_m) \cong S^2$, then $G \cong S^{m+2}$. If $C_G(H_m) \cong S$, then we have $G/L \cong (G/H_m)/(H_m C_G(H_m)/H_m)$ so $G/S^{m+1} \cong S$ and using the case $n = 1$, we get that $G \cong S^{m+2}$.
- If $n = 3$, then we have three cases. If $C_G(H_m) \cong S^3$, then $G \cong S^{m+3}$. If $C_G(H_m) \cong S^2$, then we have $G/L \cong (G/H_m)/(H_m C_G(H_m)/H_m)$ so $G/S^{m+2} \cong S$ and using the case $n = 1$, we get that $G \cong S^{m+3}$. If $C_G(H_m) \cong S$, then we have $G/L \cong (G/H_m)/(H_m C_G(H_m)/H_m)$ so $G/S^{m+1} \cong S^2$ and using the case $n = 2$, we get that $G \cong S^{m+3}$.

By iterating this process we get that G is isomorphic to S^{m+n}, where $m + n \leq p_0$. $\square$

Lemma 7. *Let G be a finite group. Then the following statements hold:*

(a) *If $|G| = 2^s 3^t 5^n$, where $s + t < 16n/5$, and $5^n \in \mathrm{cd}(G)$, then G is not solvable;*

(b) *If $|G| = 2^r 3^s 17^n$, where $2r + s < 256n/17$, and $17^n \in \mathrm{cd}(G)$, then G is not solvable.*

Proof. (a) On the contrary, let G be a solvable group. Since $5^n \in \mathrm{cd}(G)$, by Lemma 5, $O_5(G) = 1$ and so $\mathrm{Fit}(G) \cong O_2(G) \times O_3(G) \neq 1$. Then $G/C_G(\mathrm{Fit}(G)) \hookrightarrow \mathrm{Aut}(\mathrm{Fit}(G))$ and since G is a solvable group, $C_G(\mathrm{Fit}(G)) \leq \mathrm{Fit}(G)$. Hence $|G| \mid |\mathrm{Fit}(G)| \cdot |\mathrm{Aut}(\mathrm{Fit}(G))|$. On the other hand, $\mathrm{Aut}(\mathrm{Fit}(G)) \cong \mathrm{Aut}(O_2(G) \times O_3(G)) \cong \mathrm{Aut}(O_2(G)) \times \mathrm{Aut}(O_3(G))$. Also, according to [10, Section 1.3] we obtain that

$$|\mathrm{Aut}(O_2(G))| \mid |\mathrm{GL}(s,2)| = (2^s - 1)(2^s - 2) \cdots (2^s - 2^{s-1})$$

and

$$|\mathrm{Aut}(O_3(G))| \mid |\mathrm{GL}(t,3)| = (3^t - 1)(3^t - 3) \cdots (3^t - 3^{t-1}).$$

Hence $|G| \mid |\mathrm{Fit}(G)| \cdot |\mathrm{GL}(s,2)| \cdot |\mathrm{GL}(t,3)| = |O_2(G)| \cdot |O_3(G)| \cdot |\mathrm{GL}(s,2)| \cdot |\mathrm{GL}(t,3)|$. Therefore $5^n \mid |\mathrm{GL}(s,2)| \cdot |\mathrm{GL}(t,3)|$. Consequently the power of 5 in 5^n is less than or equal to the power of 5 in $|\mathrm{GL}(s,2)| \cdot |\mathrm{GL}(t,3)|$.

First, we calculate the multiplicity of the prime 5 in the number

$$|\mathrm{GL}(s,2)| = (2^s-1)(2^s-2)\cdots(2^s-2^{s-1}) = 2^{s(s-1/2)}(2^s-1)(2^{s-1}-1)\cdots(2-1).$$

We can start counting:
$*$ The number of (2^l-1), where $1 \le l \le s$ such that $5 \mid (2^l-1)$ is equal to the number of multiples of $\mathrm{Ord}_5(2)$ which are less than or equal to s, *i.e.* $[s/\mathrm{Ord}_5(2)]$.
$*$ The number of (2^l-1), where $1 \le l \le s$ such that $5^2 \mid (2^l-1)$ is equal to the number of multiples of $\mathrm{Ord}_{5^2}(2)$ which are less than or equal to s, *i.e.* $[s/\mathrm{Ord}_{5^2}(2)]$.
$*$ The number of (2^l-1), where $1 \le l \le s$ such that $5^m \mid (2^l-1)$ is equal to the number of multiples of $\mathrm{Ord}_{5^m}(2)$ which are less than or equal to s, *i.e.* $[s/\mathrm{Ord}_{5^m}(2)]$.
Putting this all together, the multiplicity of the prime 5 in $|\mathrm{GL}(s,2)|$ is

$$\left[\frac{s}{\mathrm{Ord}_5(2)}\right] + \left[\frac{s}{\mathrm{Ord}_{5^2}(2)}\right] + \left[\frac{s}{\mathrm{Ord}_{5^3}(2)}\right] + \cdots.$$

Similarly, the multiplicity of the prime 5 in $|\mathrm{GL}(t,3)|$ is

$$\left[\frac{t}{\mathrm{Ord}_5(3)}\right] + \left[\frac{t}{\mathrm{Ord}_{5^2}(3)}\right] + \left[\frac{t}{\mathrm{Ord}_{5^3}(3)}\right] + \cdots.$$

By Lemma 4, we obtain that $\mathrm{Ord}_{5^k}(2) = \mathrm{Ord}_5(2)\cdot 5^{k-1}$ and $\mathrm{Ord}_{5^k}(3) = \mathrm{Ord}_5(3)\cdot 5^{k-1}$, for every $k \in \mathbb{N}$. Hence

$$\begin{aligned} n &\le [\frac{s}{\mathrm{Ord}_5(2)}] + [\frac{s}{\mathrm{Ord}_{5^2}(2)}] + \cdots + [\frac{t}{\mathrm{Ord}_5(3)}] + [\frac{t}{\mathrm{Ord}_{5^2}(3)}] + \cdots \\ &\le [\frac{s}{4}] + [\frac{s}{20}] + \cdots + [\frac{t}{4}] + [\frac{t}{20}] + \cdots \le \frac{s}{4} + \frac{s}{20} + \cdots + \frac{t}{4} + \frac{t}{20} + \cdots \\ &= \frac{s}{4}(1+1/5+1/5^2+\cdots) + \frac{t}{4}(1+1/5+1/5^2+\cdots) \le \frac{s+t}{4}\cdot\frac{5}{4} < n, \end{aligned}$$

which is a contradiction.

(b) We know that $\mathrm{Ord}_{17}(2) = 8, \mathrm{Ord}_{17}(3) = 16$. Now using Lemma 4, we obtain that $\mathrm{Ord}_{17^k}(2) = \mathrm{Ord}_{17}(2)\cdot 17^{k-1}$ and $\mathrm{Ord}_{17^k}(3) = \mathrm{Ord}_{17}(3)\cdot 17^{k-1}$, for every $k \in \mathbb{N}$. So similarly to the above argument, we get the result. $\square$

THEOREM 8. *Let G be a finite group and $\alpha \le 5$. Then $G \cong A_5^\alpha$ if and only if $|G| = |A_5|^\alpha$ and $5^\alpha \in \mathrm{cd}(G)$.*

Proof. We put $H_0 = G$. By Lemma 7, it follows that G is not solvable. According to Lemma 2, $G = H_0$ has a normal series $1 \trianglelefteq H_1 \trianglelefteq K_1 \trianglelefteq H_0 = G$ such that K_1/H_1 is a direct product of isomorphic nonabelian simple groups and $|H_0/K_1| \mid |\mathrm{Out}(K_1/H_1)|$. If H_1 is not a solvable group, we can proceed similarly to the above and get a normal series $1 \unlhd H_2 \unlhd K_2 \unlhd H_1$ such that K_2/H_2 is a nonabelian characteristically simple group and $|H_1/K_2| \mid |\mathrm{Out}(K_2/H_2)|$. If H_2 is not a solvable group, we continue this process and finally we have a subnormal series

$$1 \unlhd H_m \unlhd K_m \unlhd H_{m-1} \unlhd K_{m-1} \cdots \unlhd H_2 \unlhd K_2 \unlhd H_1 \unlhd K_1 \unlhd H_0 = G \tag{1}$$

of G such that m is the smallest number, where H_m is solvable. Hence

$$|G| = \prod_{i=1}^{m} |K_i/H_i| \cdot \prod_{i=1}^{m} |H_{i-1}/K_i| \cdot |H_m|.$$

By assumptions, $\alpha \leq 5$ and K_i/H_i is a direct product of α_i copies of a nonabelian simple group S_i such that $|H_{i-1}/K_i| \mid |\mathrm{Out}(K_i/H_i)|$. Also $|\mathrm{Out}({S_i}^{\alpha_i})| = |\mathrm{Out}(S_i)|^{\alpha_i}(\alpha_i!)$ and $5 = \mathrm{P}(|S_i|) \nmid |\mathrm{Out}(S_i)|$. Therefore we get that $5 \nmid \prod_{i=1}^{m} |H_{i-1}/K_i|$.

We show that $5 \nmid |H_m|$. Otherwise, if $5 \mid |H_m|$, let $5^\beta \| |H_m|$. Then 5^β is a divisor of

$$t = |H_m| \cdot \prod_{i=1}^{m} |H_{i-1}/K_i| = \frac{|G|}{\prod_{i=1}^{m} |K_i/H_i|}.$$

On the other hand, t is a divisor of $|A_5|^\alpha / |A_5|^\gamma$, where $5^\gamma \parallel \prod_{i=1}^{m} |K_i/H_i|$. Hence $\alpha = \beta + \gamma$. Therefore $|H_m| \mid |A_5|^\beta$ and so $|H_m| = 2^\sigma 3^\tau 5^\beta$, where $\sigma \leq 2\beta, \tau \leq \beta$. Also, by successively applying Lemma 1 at every extension, we get that $5^\beta \in \mathrm{cd}(H_m)$ and by Lemma 7, H_m is not solvable, which is a contradiction. Thus $5 \nmid |H_m|$. Hence $5^\alpha \parallel \prod_{i=1}^{m} |K_i/H_i|$. Therefore

$$|G|_5 = (\prod_{i=1}^{m} |K_i/H_i| \cdot \prod_{i=1}^{m} |H_{i-1}/K_i| \cdot |H_m|)_5 = (\prod_{i=1}^{m} |K_i/H_i|)_5.$$

Since G is a $\{2,3,5\}$-group, S_i is a $\{2,3,5\}$-group. If there exists i such that $S_i \ncong A_5$, then $|G|_2 < (\prod_{i=1}^{m} |K_i/H_i|)_2$, which is a contradiction. Therefore $K_i/H_i \cong A_5^{\alpha_i}$. Now we have

$$|G| = \prod_{i=1}^{m} |A_5|^{\alpha_i} \cdot \prod_{i=1}^{m} |H_{i-1}/K_i| \cdot |H_m|. \tag{2}$$

By eliminating $|A_5|$ from both sides of (2), we have

$$|A_5|^{\alpha-1} = |A_5|^{\alpha_j - 1} \prod_{\substack{i=1 \\ i \neq j}}^{m} |A_5|^{\alpha_i} \cdot \prod_{i=1}^{m} |H_{i-1}/K_i| \cdot |H_m|.$$

Since $|G|_5 = (\prod_{i=1}^{m} |K_i/H_i|)_5$, by iterating this process, we get that

$$\prod_{i=1}^{m} |H_{i-1}/K_i| \cdot |H_m| = 1.$$

We get that $H_m = 1$, and for each $1 \le i \le m$, $H_{i-1} = K_i$ and $K_i/H_i \cong A_5^{\alpha_i}$, where $\alpha_1 + \cdots + \alpha_m = \alpha$. Therefore

$$1 = H_m \trianglelefteq H_{m-1} \trianglelefteq H_{m-2} \cdots \trianglelefteq H_2 \trianglelefteq H_1 \trianglelefteq H_0 = G,$$

such that $K_i = H_{i-1}$ and $H_{i-1}/H_i \cong A_5^{\alpha_i}$. Applying Lemma 6, we obtain that $H_{m-1} \cong A_5^{\alpha_m}$, $H_{m-2} \cong A_5^{\alpha_m+\alpha_{m-1}}$ and finally $G \cong A_5^{\alpha_m+\alpha_{m-1}+\cdots+\alpha_1} = A_5^{\alpha}$. $\square$

Similarly to the above, we have the following theorem and for convenience we omit the proof.

THEOREM 9. *Let G be a finite group and $\alpha \le 17$. Then $G \cong L_2(17)^\alpha$ if and only if $|G| = |L_2(17)|^\alpha$ and $17^\alpha \in \mathrm{cd}(G)$.*

Remark 10. In [24, Theorem A], Xu *et al.* proved that $L_2(7)$ is characterizable by $|G|$ and $7 \in \mathrm{cd}(G)$.

Using the notations of GAP [9], if $A = \mathrm{SmallGroup}(56, 11)$ and $H = A^2 \times \mathbb{Z}_9$, then $|H| = |L_2(7)|^2$ and H has an irreducible character of degree 7^2. Therefore $L_2(7)^\alpha$, where $\alpha \ge 2$, is not characterizable by $|G|$ and $7^\alpha \in \mathrm{cd}(G)$.

If $C = \mathrm{SmallGroup}(3^3 13, 11)$ and $H = C \times \mathbb{Z}_{16}$, then $|H| = |L_3(3)|$ and H has an irreducible character of degree 13. Thus $L_3(3)^\alpha$, where $\alpha \ge 1$, is not characterizable by $|G|$ and $13^\alpha \in \mathrm{cd}(G)$.

So we need more assumptions to characterize $L_2(7)^\alpha$, where $2 \le \alpha \le 7$ and $L_3(3)^\alpha$, where $\alpha \le 13$.

LEMMA 11. *Let G be a finite group. Then the following statements hold:*

(a) *If $|G| = 2^r 3^s 7^n$, where $s < 36n/7$, and $2^r, 7^n \in \mathrm{cd}(G)$, then G is not solvable;*

(b) *If $|G| = 2^r 3^s 13^n$, where $r < 144n/13$, and $3^s, 13^n \in \mathrm{cd}(G)$, then G is not solvable.*

Proof. (a) Let G be a solvable group. By Lemma 5, $O_2(G) = 1$ and $O_7(G) = 1$. Therefore $\mathrm{Fit}(G) \cong O_3(G) \ne 1$. Since G is a solvable group, $C_G(\mathrm{Fit}(G)) \le \mathrm{Fit}(G)$. Hence $|G|$ divides $|\mathrm{Fit}(G)| \cdot |\mathrm{Aut}(\mathrm{Fit}(G))|$. So $7^n \mid |\mathrm{GL}(t, 3)|$ and by Lemma 4, $\mathrm{Ord}_{7^k}(3) = \mathrm{Ord}_7(3) \cdot 7^{k-1}$. Hence

$$n \le [\frac{s}{\mathrm{Ord}_7(3)}] + [\frac{s}{\mathrm{Ord}_{7^2}(3)}] + \cdots \le \frac{s}{6}(1 + 1/7 + 1/7^2 + \cdots) \le \frac{s}{6} \cdot \frac{7}{6} < n,$$

which is a contradiction.

(b) Similarly to the above, the result holds. □

Theorem 12. *Let G be a finite group and $\alpha \leq 7$. Then $G \cong L_2(7)^\alpha$ if and only if $|G| = |L_2(7)|^\alpha$ and $2^{3\alpha}, 7^\alpha \in \mathrm{cd}(G)$.*

Proof. By Lemma 11, it follows that G is not solvable. So, similarly to the proof of Theorem 8, we get that a subnormal series, like (1), where K_i/H_i is a direct product of α_i copies of a nonabelian simple group S_i. Also $|H_{i-1}/K_i| \mid |\mathrm{Out}(K_i/H_i)|$ and m is the smallest number where H_m is solvable. We show that $7 \nmid |H_m|$. On the contrary, let $7^\beta \| |H_m|$. Then 7^β is a divisor of

$$t = |H_m| \cdot \prod_{i=1}^{m} |H_{i-1}/K_i| = \frac{|G|}{\prod_{i=1}^{m} |K_i/H_i|}.$$

On the other hand, t is a divisor of $|L_2(7)|^\alpha / |L_2(7)|^\gamma$, where $7^\gamma \parallel \prod_{i=1}^{m} |K_i/H_i|$. By Lemma 1, $7^\beta \in \mathrm{cd}(H_m)$ and using Lemma 11, H_m is not solvable which is a contradiction. Thus $7 \nmid |H_m|$. We conclude that $7^\alpha \parallel \prod_{i=1}^{m} |K_i/H_i|$. As $K_i/H_i \cong {S_i}^{\alpha_i}$, where S_i is a nonabelian simple $\{2,3,7\}$-group, it must hold that $H_m = 1$ and for each i, $H_{i-1} = K_i$. We obtain that $|G| = \prod_{i=1}^{m} |K_i/H_i|$, therefore for each i, $K_i/H_i \cong L_2(7)^{\alpha_i}$, where $\alpha_1 + \cdots + \alpha_m = \alpha$. Applying Lemma 6, we get that $G \cong L_2(7)^\alpha$. □

Remark 13. Theorems 8, 9 and 12 are generalizations of Theorem 2.4 in [12], for special cases $p = 5, 7$ and 17.

Similarly to the above theorem we have the following theorem:

Theorem 14. *Let G be a finite group and $\alpha \leq 13$. Then $G \cong L_3(3)^\alpha$ if and only if $|G| = |L_3(3)|^\alpha$ and $3^{3\alpha}, 13^\alpha \in \mathrm{cd}(G)$.*

As a consequence of the above theorems, by [3, Theorem 2.13] we have the following result which is a partial answer to the question arose in [20].

Corollary 15. *Let $M \in \{A_5^n | n \leq 5\} \cup \{L_2(7)^n | n \leq 7\} \cup \{L_3(3)^n | n \leq 13 \cup \{L_2(17)^n | n \leq 17\}$. If G is a group such that $\mathbb{C}G \cong \mathbb{C}M$, then $G \cong M$. Thus M is uniquely determined by the structure of its complex group algebra.*

Acknowledgements. The authors would like to thank the referee for valuable comments and suggestions.

REFERENCES

[1] M. Baniasad Azad and B. Khosravi, *Recognition of $M \times M$ by its complex group algebra where M is a simple K_3-group.* Mathematics **6(7)** (2018), 107.

[2] M. Baniasad Azad and B. Khosravi, *Complex group algebras of the direct product of non-isomorphic Suzuki groups.* J. Algebra Appl. (2019), 2050036 (8 pages).

[3] Y.G. Berkovich and E.M. Zhmud', *Characters of Finite Groups.* Part 1. Transl. Math. Monogr., American Mathematical Society, Rhode Island, 1998.

[4] C. Bessenrodt, H. Nguyen, J. Olsson and H. Tong-Viet, *Complex group algebras of the double covers of the symmetric and alternating groups.* Algebra Number Theory **9** (2015), *3*, 601–628.

[5] R. Brauer, *Representations of finite groups.* Vol. I. Lectures on Modern Mathematics, 133–175, 1963.

[6] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *Atlas of Finite Groups.* Oxford Univ. Press, Oxford, 1985.

[7] E.C. Dade, *Deux groupes finis distincts ayant la meme algèbre de groupe sur tout corps.* Math. Z. **119** (1971), 345–348.

[8] J.D. Dixon and B. Mortimer, *Permutation Groups.* Grad. Texts in Math. **163**, Springer, New York, 1996.

[9] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.12, 2008. http:www.gap-system.org/gap.

[10] P. Hall, *A contribution to the theory of groups of prime power order.* Proc. London Math. Soc. **36** (1933), 29–95.

[11] I.M. Isaacs, *Character theory of finite groups.* Pure Appl. Math. **69**, Academic Press, New York, 1976.

[12] B. Khosravi, B. Khosravi, B. Khosravi and Z. Momen, *Recognition of* $\mathrm{PSL}(2,p) \times \mathrm{PSL}(2,p)$ *by its complex group Algebra.* J. Algebra Appl. **16** (2017), *1*, 1750036 (9 pages).

[13] B. Khosravi, B. Khosravi and B. Khosravi, *Some extensions of* $\mathrm{PSL}(2,p^2)$ *are uniquely determined by their complex group algebras.* Comm. Algebra. **43** (2015), *8*, 3330–3341.

[14] B. Khosravi, B. Khosravi and B. Khosravi, *A new characterization for some extensions of* $\mathrm{PSL}(2,q)$ *for some* q *by some character degrees.* Proc. Indian Acad. Sci. Math. Sci. **126** (2016), *1*, 49–59.

[15] W. Kimmerle, *Group rings of finite simple groups.* Resenhas do Instituto de Matemática e Estatística da Universidade de São Paulo **5** (2002), *4*, 261–278.

[16] M. Nagl, *Charakterisierung der Symmetrischen Gruppen durch ihre komplexe Gruppenalgebra.* Stuttgarter Mathematische Berichte 2011. http://www.mathematik.uni-stuttgart.de/preprints/downloads/2011/2011-007.pdf

[17] M. Nagl, *Über das Isomorphieproblem von Gruppenalgebren endlicher einfacher Gruppen.* PhD Diplomarbeit, Universität Stuttgart, 2008.

[18] M.B. Nathanson, *Elementary Methods in Number Theory.* Grad. Texts in Math. **195**, Springer, New York, 2000.

[19] H.N. Nguyen and H.P. Tong-Viet, *Characterizing finite quasisimple groups by their complex group algebras.* Algebr. Represent. Theory **17** (2014), *1*, 305–320.

[20] H.P. Tong-Viet, *Simple classical groups of Lie type are determined by their character degrees.* J. Algebra **357** (2012), 61–68.

[21] H.P. Tong-Viet, *Symmetric groups are determined by their character degrees.* J. Algebra **334** (2011), *1*, 275–284.

[22] H.P. Tong-Viet, *Alternating and sporadic simple groups are determined by their character degrees.* Algebr. Represent. Theory **15** (2012), *2*, 379–389.

[23] H.P. Tong-Viet, *Simple exceptional groups of Lie type are determined by their character degrees.* Monatsh. Math. **166** (2012), 559–577.

[24] H. Xu, G.Y. Chen and Y. Yan, *A new characterization of simple K_3-groups by their orders and large degrees of their irreducible characters.* Comm. Algebra **42** (2014), 5374–5380.

[25] K. Zsigmondy, *Zur Theorie der Potenzreste.* Montash. Math. **3** (1892), 265–284.

Received 27 February 2017

Amirkabir University of Technology
(Tehran Polytechnic),
Faculty of Math. and Computer Sci.,
Dept. of Pure Math.,
424, Hafez Ave., Tehran 15914, Iran
baniasad84@gmail.com
khosravibbb@yahoo.com