RECOGNITION OF SOME CHARACTERISTICALLY SIMPLE GROUPS BY THEIR COMPLEX GROUP ALGEBRAS

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In [20], the following question arose: Which groups can be uniquely determined by the structure of their complex group algebras? In this paper, we prove that the direct product $G^n$ of $n$ copies of a group $G$, where (a) $G \cong A_5$ and $n \leq 5$; (b) $G \cong L_2(7)$ and $n \leq 7$; (c) $G \cong L_3(3)$ and $n \leq 13$; (d) $G \cong L_2(17)$ and $n \leq 17$; are uniquely determined by their order and some information on irreducible character degrees. As a consequence of our results, we show that these groups are uniquely determined by the structure of their complex group algebras.

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1. INTRODUCTION AND PRELIMINARY RESULTS

Let $G$ be a finite group, $\text{Irr}(G)$ be the set of irreducible characters of $G$, and denote by $\text{cd}(G)$, the set of irreducible character degrees of $G$. If $n$ is a natural number, by $G^n$ we mean the direct product of $n$ copies of $G$; that is, $G \times G \times \cdots \times G$.

In [5, Problem 2*], R. Brauer posed the following question: Let $G$ and $H$ are two finite groups. If for all fields $\mathbb{F}$, two group algebras $\mathbb{F}G$ and $\mathbb{F}H$ are isomorphic, can we get that $G$ and $H$ are isomorphic? In [7], E.C. Dade showed that this is false in general.

It was shown in [16, 21] that the symmetric groups are uniquely determined by the structure of their complex group algebras. In [15, 17, 20, 22, 23] it is proved that each nonabelian simple group is uniquely determined by its complex group algebra. In [20], Tong-Viet posed the following question:

QUESTION. Which groups can be uniquely determined by the structure of their complex group algebras?

In [4, 19], it is proved that every quasisimple group $L$ is uniquely determined up to isomorphism by the structure of $\mathbb{C}L$, the complex group algebra
of $L$. In [13] and [14], it is proved that if $q \mid p^2$, where $p > 3$ is an odd prime, $S = L_2(q)$, $M$ is a finite group such that $S < M < \text{Aut}(S)$, $M = \mathbb{Z}_2 \times L_2(q)$ or $M = \text{SL}(2, q)$, then $M$ is uniquely determined by its complex group algebra.

One of the next natural groups to be considered are the characteristically simple groups. Khosravi et al. proved that $L_2(p) \times L_2(p)$ is uniquely determined by its complex group algebra, where $p \geq 5$ is a prime number (see [12]). In [1], we prove that if $M$ is a simple $K_3$-group, then $M \times M$ is uniquely determined by its order and some information on irreducible character degrees. In [2], we proved that the direct product of non-isomorphic Suzuki groups is uniquely determined by its complex group algebra.

In this paper, we prove that the direct product $G^n$ of $n$ copies of a group $G$, where (a) $G \cong A_5$ and $n \leq 5$; (b) $G \cong L_2(7)$ and $n \leq 7$; (c) $G \cong L_3(3)$ and $n \leq 13$; (d) $G \cong L_2(17)$ and $n \leq 17$; are uniquely determined by their order and some information on irreducible character degrees. As a consequence of our results we show that these groups are uniquely determined by the structure of their complex group algebras.

If $N \leq G$ and $\theta \in \text{Irr}(N)$, then the inertia group of $\theta$ in $G$ is $I_G(\theta) = \{g \in G \mid \theta^g = \theta\}$. If the character $\chi = \sum_{i=1}^{k} e_i \chi_i$, where for each $1 \leq i \leq k$, $\chi_i \in \text{Irr}(G)$ and $e_i$ is a natural number, then each $\chi_i$ is called an irreducible constituent of $\chi$.

**Lemma 1** ([11, Theorems 6.2, 6.8, 11.29]). Let $N \leq G$ and let $\chi \in \text{Irr}(G)$. Let $\theta$ be an irreducible constituent of $\chi_N$ and suppose $\theta_1 = \theta, \ldots, \theta_t$ are the distinct conjugates of $\theta$ in $G$. Then $\chi_N = e \sum_{i=1}^{t} \theta_i$, where $e = [\chi_N, \theta]$ and $t = |G : I_G(\theta)|$. Also $\theta(1) \mid \chi(1)$ and $\chi(1)/\theta(1) \mid |G : N|$.

**Lemma 2** ([24, Lemma 1]). Let $G$ be a nonsolvable group. Then $G$ has a normal series $1 \leq H \leq K \leq G$ such that $K/H$ is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\text{Out}(K/H)|$.

Given a natural number $n$, let $P(n)$ denote the greatest prime factor of $n$, and let $n_r$, where $r$ is a prime, denote the $r$-part of $n$, i.e., the largest power of $r$ that divides $n$. For every integer $a$ coprime to $n$, let $\text{Ord}_n(a)$ denote the smallest positive integer $e$ such that $a^e \equiv 1 \pmod{n}$. If $s$ is a prime number, then we write $s^k \parallel n$, when $s^k \mid n$ but $s^{k+1} \nmid n$.

Using [25] we have the following result:

**Lemma 3.** If $n > 2$ and $a > b > 0$, then $n + 1 \leq P(a^n - b^n)$.

**Lemma 4** ([18, Theorems 3.6 ]). Let $p$ be an odd prime, and let $a \neq \pm 1$ be an integer not divisible by $p$. Let $d$ be the order of a modulo $p$. Let $k_0$ be the largest integer such that $a^d \equiv 1 \pmod{p^{k_0}}$. Then the order of a modulo $p^k$ is $d$ for $k = 1, \ldots, k_0$ and $dp^{k-k_0}$ for $k > k_0$.  


2. THE MAIN RESULTS

Lemma 5. Let $M$ be a finite group such that $p \mid |M|$. If there exists $\chi \in \text{Irr}(G)$, where $\chi(1) = |M|_p$, then $O_p(M) = 1$.

Proof. Let $|M|_p = p^j$. Assume on the contrary $L = O_p(M) \neq 1$ and $|L| = p^i$, where $1 \leq i \leq j$. If $\eta \in \text{Irr}(L)$ such that $[\chi_L, \eta] \neq 0$, then by Lemma 1, $p^j/\eta(1)$ is a divisor of $|M : L| = p^{j-i}$. Since $\eta(1) \mid |L|$, we get that $\eta(1) = p^i$. On the other hand, $\sum_{\nu \in \text{Irr}(L)} \nu^2(1) = |L|$, which is a contradiction. □

Lemma 6. Let $S$ be a finite nonabelian simple group and let $p_0 = \text{P}(|S|)$. If $G$ is an extension of $S^m$ by $S^n$, where $m + n \leq p_0$, then $G \cong S^{m+n}$.

Proof. We claim that $p_0 \nmid |\text{Out}(S)|$. Obviously, $p_0 \geq 5$. If $S$ is an alternating group or a sporadic simple group, then by page ix and Table 1 in [6], we get that $|\text{Out}(S)| \leq 4$. Therefore we assume that $S$ is a simple group of Lie type over $\text{GF}(q)$, where $q = p^f$. By the notations in [6, Page xiv and Table 5], $|\text{Out}(S)| = dfg$, where $d$, $f$ and $g$ are the orders of the diagonal, field and graph automorphisms of $S$, respectively. Let $k$ be the largest integer such that $q^k - 1$ is a divisor of $|S|$. By Lemma 3, $fk + 1 \leq \text{P}(q^k - 1)$.

Assume that $S \cong L_{l+1}(q)$, where $l \geq 2$. Then $k = l + 1$. Hence
\[
\max\{d, f, g\} < lf + f + 1 \leq \text{P}(p^{(l+1)f} - 1) \leq p_0.
\]

Suppose that $S \cong U_{l+1}(q)$, where $l \geq 2$. we know that $f$ is an even number. If $l$ is an even number, then $k = l$. Thus
\[
\max\{d, f, g\} < lf + f + 1 \leq \text{P}(p^{lf} - 1) \leq p_0.
\]

If $l$ is an odd number, then $k = l + 1$. Therefore
\[
\max\{d, f, g\} < lf + f + 1 \leq \text{P}(p^{(l+1)f} - 1) \leq p_0.
\]

Therefore $p_0 \nmid |\text{Out}(S)|$. For other cases, easily we can check that $p_0$ does not divide $|\text{Out}(S)|$. Therefore the claim is proved.

By assumptions, there exists a normal subgroup $H_m$ of $G$, which is isomorphic to $S^m$. We know that $\text{Out}(H_m) \cong \frac{\text{Aut}(H_m)}{\text{Inn}(H_m)}$ and $\text{Inn}(H_m) \cong \frac{H_m}{Z(H_m)}$. Therefore $|\text{Aut}(H_m)| = |\text{Out}(H_m)| \frac{|H_m|}{|Z(H_m)|}$. Since $S$ is a non-abelian simple group and $H_m \cong S^m$, we have $Z(H_m) = 1$ and $|\text{Aut}(H_m)| = |\text{Out}(H_m)||H_m|$. On the other hand, by [8, Page 131], we have $\text{Out}(H_m) \cong \text{Out}(S) \wr S_m$. Therefore
\[
\frac{G}{C_G(H_m)} \hookrightarrow \text{Aut}(H_m) \implies \frac{G}{C_G(H_m)} \mid |\text{Aut}(H_m)| = |\text{Out}(H_m)||H_m|
\]
\[
\implies \frac{|G|}{|C_G(H_m)|} \mid |\text{Out}(S^m)||S^m| = |\text{Out}(S)|^m m!|S^m|}
\]
Since $p_0 > m$, $p_0 \mid |S|$ and $p_0 \nmid |\text{Out}(S)|m^m!$, we get that $p_0 \mid |C_G(H_m)|$ and so $|C_G(H_m)| \neq 1$. As $S$ is a nonabelian simple group, $H_m \cap C_G(H_m) = 1$ and it follows that $H_m C_G(H_m) \cong S^m \times C_G(H_m)$. Since $C_G(H_m) \cong H_m C_G(H_m) / H_m \leq G / H_m \cong S^n$, we have $C_G(H_m) \cong S^i$, where $1 \leq i \leq n$. Put $L = H_m C_G(H_m) \cong S^{m+i}$. Now we consider the possibilities for $n$:
- If $n = 1$, then $G \cong S^{m+1}$.
- If $n = 2$, then we have two cases. If $C_G(H_m) \cong S^2$, then $G \cong S^{m+2}$. If $C_G(H_m) \cong S$, then we have $G / L \cong (G / H_m) / (H_m C_G(H_m) / H_m)$ so $G / S^{m+1} \cong S$ and using the case $n = 1$, we get that $G \cong S^{m+2}$.
- If $n = 3$, then we have three cases. If $C_G(H_m) \cong S^3$, then $G \cong S^{m+3}$. If $C_G(H_m) \cong S^2$, then we have $G / L \cong (G / H_m) / (H_m C_G(H_m) / H_m)$ so $G / S^{m+2} \cong S$ and using the case $n = 1$, we get that $G \cong S^{m+3}$. If $C_G(H_m) \cong S$, then we have $G / L \cong (G / H_m) / (H_m C_G(H_m) / H_m)$ so $G / S^{m+1} \cong S^2$ and using the case $n = 2$, we get that $G \cong S^{m+3}$.

By iterating this process we get that $G$ is isomorphic to $S^{m+n}$, where $m + n \leq p_0$. □

**Lemma 7.** Let $G$ be a finite group. Then the following statements hold:
(a) If $|G| = 2^s 3^t 5^n$, where $s + t < 16n/5$, and $5^n \in \text{cd}(G)$, then $G$ is not solvable;
(b) If $|G| = 2^r 3^s 17^n$, where $2r + s < 256n/17$, and $17^n \in \text{cd}(G)$, then $G$ is not solvable.

**Proof.** (a) On the contrary, let $G$ be a solvable group. Since $5^n \in \text{cd}(G)$, by Lemma 5, $O_5(G) = 1$ and so $\text{Fit}(G) \cong O_2(G) \times O_3(G) \neq 1$. Then $G / C_G(\text{Fit}(G)) \hookrightarrow \text{Aut}(\text{Fit}(G))$ and since $G$ is a solvable group, $C_G(\text{Fit}(G)) \leq \text{Fit}(G)$. Hence $|G| = |\text{Fit}(G)| \cdot |\text{Aut}(\text{Fit}(G))|$. On the other hand, $\text{Aut}(\text{Fit}(G)) \cong \text{Aut}(O_2(G) \times O_3(G)) \cong \text{Aut}(O_2(G)) \times \text{Aut}(O_3(G))$. Also, according to [10, Section 1.3] we obtain that

$$|\text{Aut}(O_2(G))| \cdot |\text{GL}(s, 2)| = (2^s - 1)(2^s - 2) \cdots (2^s - 2^{s-1})$$

and

$$|\text{Aut}(O_3(G))| \cdot |\text{GL}(t, 3)| = (3^t - 1)(3^t - 3) \cdots (3^t - 3^{t-1})$$

Hence $|G| = |\text{Fit}(G)| \cdot |\text{GL}(s, 2)| \cdot |\text{GL}(t, 3)| = |O_2(G)| \cdot |O_3(G)| \cdot |\text{GL}(s, 2)| \cdot |\text{GL}(t, 3)|$. Therefore $5^n \mid |\text{GL}(s, 2)| \cdot |\text{GL}(t, 3)|$. Consequently the power of 5 in $5^n$ is less than or equal to the power of 5 in $|\text{GL}(s, 2)| \cdot |\text{GL}(t, 3)|$.
First, we calculate the multiplicity of the prime 5 in the number
\[ |\text{GL}(s, 2)| = (2^s - 1)(2^s - 2) \cdots (2^s - 2^{s-1}) = 2^{s(s-1)/2}(2^s - 1)(2^{s-1} - 1) \cdots (2-1). \]

We can start counting:

- The number of \((2^l - 1)\), where \(1 \leq l \leq s\) such that \(5 \mid (2^l - 1)\) is equal to the number of multiples of \(\text{Ord}_5(2)\) which are less than or equal to \(s\), i.e. \([s/\text{Ord}_5(2)]\).
- The number of \((2^l - 1)\), where \(1 \leq l \leq s\) such that \(5^2 \mid (2^l - 1)\) is equal to the number of multiples of \(\text{Ord}_{5^2}(2)\) which are less than or equal to \(s\), i.e. \([s/\text{Ord}_{5^2}(2)]\).
- The number of \((2^l - 1)\), where \(1 \leq l \leq s\) such that \(5^m \mid (2^l - 1)\) is equal to the number of multiples of \(\text{Ord}_{5^m}(2)\) which are less than or equal to \(s\), i.e. \([s/\text{Ord}_{5^m}(2)]\).

Putting this all together, the multiplicity of the prime 5 in \(|\text{GL}(s, 2)|\) is
\[
\left[ \frac{s}{\text{Ord}_5(2)} \right] + \left[ \frac{s}{\text{Ord}_{5^2}(2)} \right] + \left[ \frac{s}{\text{Ord}_{5^3}(2)} \right] + \cdots.
\]

Similarly, the multiplicity of the prime 5 in \(|\text{GL}(t, 3)|\) is
\[
\left[ \frac{t}{\text{Ord}_5(3)} \right] + \left[ \frac{t}{\text{Ord}_{5^2}(3)} \right] + \left[ \frac{t}{\text{Ord}_{5^3}(3)} \right] + \cdots.
\]

By Lemma 4, we obtain that \(\text{Ord}_{5^k}(2) = \text{Ord}_5(2) \cdot 5^{k-1}\) and \(\text{Ord}_{5^k}(3) = \text{Ord}_5(3) \cdot 5^{k-1}\), for every \(k \in \mathbb{N}\). Hence
\[
n \leq \left[ \frac{s}{\text{Ord}_5(2)} \right] + \left[ \frac{s}{\text{Ord}_{5^2}(2)} \right] + \cdots + \left[ \frac{t}{\text{Ord}_5(3)} \right] + \left[ \frac{t}{\text{Ord}_{5^2}(3)} \right] + \cdots
\]
\[
= \frac{s}{4} + \frac{s}{20} + \cdots + \frac{t}{4} + \frac{t}{20} + \cdots \leq \frac{s}{4} + \frac{s}{20} + \cdots + \frac{t}{4} + \frac{t}{20} + \cdots
\]
\[
= \frac{s}{4}(1 + 1/5 + 1/5^2 + \cdots) + \frac{t}{4}(1 + 1/5 + 1/5^2 + \cdots) \leq \frac{s + t}{4} \cdot \frac{5}{4} < n,
\]
which is a contradiction.

(b) We know that \(\text{Ord}_{17}(2) = 8\), \(\text{Ord}_{17}(3) = 16\). Now using Lemma 4, we obtain that \(\text{Ord}_{17^k}(2) = \text{Ord}_{17}(2) \cdot 17^{k-1}\) and \(\text{Ord}_{17^k}(3) = \text{Ord}_{17}(3) \cdot 17^{k-1}\), for every \(k \in \mathbb{N}\). So similarly to the above argument, we get the result. □

**Theorem 8.** Let \(G\) be a finite group and \(\alpha \leq 5\). Then \(G \cong A_5^\alpha\) if and only if \(|G| = |A_5|^\alpha\) and \(5^\alpha \in \text{cd}(G)\).

**Proof.** We put \(H_0 = G\). By Lemma 7, it follows that \(G\) is not solvable. According to Lemma 2, \(G = H_0\) has a normal series \(1 \leq H_1 \leq K_1 \leq H_0 = G\) such that \(K_1/H_1\) is a direct product of isomorphic nonabelian simple groups and \(|H_0/K_1| = |\text{Out}(K_1/H_1)|\). If \(H_1\) is not a solvable group, we can proceed
similarly to the above and get a normal series $1 \trianglelefteq H_2 \trianglelefteq K_2 \trianglelefteq H_1$ such that $K_2/H_2$ is a nonabelian characteristically simple group and $|H_1/K_2| \mid |\text{Out}(K_2/H_2)|$. If $H_2$ is not a solvable group, we continue this process and finally we have a subnormal series

$$(1) \quad 1 \trianglelefteq H_m \trianglelefteq K_m \trianglelefteq H_{m-1} \trianglelefteq K_{m-1} \cdots \trianglelefteq H_2 \trianglelefteq K_2 \trianglelefteq H_1 \trianglelefteq K_1 \trianglelefteq H_0 = G$$

of $G$ such that $m$ is the smallest number, where $H_m$ is solvable. Hence

$$|G| = \prod_{i=1}^{m} |K_i/H_i| \cdot \prod_{i=1}^{m} |H_{i-1}/K_i| \cdot |H_m|.$$

By assumptions, $\alpha \leq 5$ and $K_i/H_i$ is a direct product of $\alpha_i$ copies of a nonabelian simple group $S_i$ such that $|H_{i-1}/K_i| \mid |\text{Out}(K_i/H_i)|$. Also $|\text{Out}(S_i^{\alpha_i})| = |\text{Out}(S_i)|^{\alpha_i(\alpha_i!)}$ and $5 = P(|S_i|) \nmid |\text{Out}(S_i)|$. Therefore we get that $5 \nmid \prod_{i=1}^{m} |H_{i-1}/K_i|$.

We show that $5 \nmid |H_m|$. Otherwise, if $5 \mid |H_m|$, let $5\beta \mid |H_m|$. Then $5\beta$ is a divisor of

$$t = |H_m| \cdot \prod_{i=1}^{m} |H_{i-1}/K_i| = \frac{|G|}{\prod_{i=1}^{m} |K_i/H_i|}.$$

On the other hand, $t$ is a divisor of $|A_5|^{\alpha}/|A_5|^\gamma$, where $5^\gamma \mid \prod_{i=1}^{m} |K_i/H_i|$. Hence $\alpha = \beta + \gamma$. Therefore $|H_m| \mid |A_5|^\beta$ and so $|H_m| = 2^\sigma 3^\tau 5^\beta$, where $\sigma \leq 2\beta, \tau \leq \beta$. Also, by successively applying Lemma 1 at every extension, we get that $5^\beta \in \text{cd}(H_m)$ and by Lemma 7, $H_m$ is not solvable, which is a contradiction. Thus $5 \nmid |H_m|$. Hence $5^\alpha \mid \prod_{i=1}^{m} |K_i/H_i|$. Therefore

$$|G|_5 = \left(\prod_{i=1}^{m} |K_i/H_i| \cdot \prod_{i=1}^{m} |H_{i-1}/K_i| \cdot |H_m|\right)_5 = \left(\prod_{i=1}^{m} |K_i/H_i|\right)_5.$$

Since $G$ is a $\{2, 3, 5\}$-group, $S_i$ is a $\{2, 3, 5\}$-group. If there exists $i$ such that $S_i \not\cong A_5$, then $|G|_2 < (\prod_{i=1}^{m} |K_i/H_i|)_2$, which is a contradiction. Therefore $K_i/H_i \cong A_5^{\alpha_i}$. Now we have

$$(2) \quad |G| = \prod_{i=1}^{m} |A_5|^{\alpha_i} \cdot \prod_{i=1}^{m} |H_{i-1}/K_i| \cdot |H_m|.$$

By eliminating $|A_5|$ from both sides of $(2)$, we have

$$|A_5|^{\alpha - 1} = |A_5|^{\alpha - 1} \prod_{i=1}^{m} |A_5|^{\alpha_i} \cdot \prod_{i=1}^{m} |H_{i-1}/K_i| \cdot |H_m|.$$
Since $|G|_5 = (\prod_{i=1}^{m} |K_i/H_i|)_5$, by iterating this process, we get that

$$\prod_{i=1}^{m} |H_{i-1}/K_i| \cdot |H_m| = 1.$$  

We get that $H_m = 1$, and for each $1 \leq i \leq m$, $H_{i-1} = K_i$ and $K_i/H_i \cong A_5^{\alpha_i}$, where $\alpha_1 + \cdots + \alpha_m = \alpha$. Therefore

$$1 = H_m \leq H_{m-1} \leq H_{m-2} \cdots \leq H_2 \leq H_1 \leq H_0 = G,$$

such that $K_i = H_{i-1}$ and $H_{i-1}/H_i \cong A_5^{\alpha_i}$. Applying Lemma 6, we obtain that $H_{m-1} \cong A_5^{\alpha_m}$, $H_{m-2} \cong A_5^{\alpha_m + \alpha_{m-1}}$ and finally $G \cong A_5^{\alpha_m + \alpha_{m-1} + \cdots + \alpha_1} = A_5^\alpha$. $\square$

Similarly to the above, we have the following theorem and for convenience we omit the proof.

**Theorem 9.** Let $G$ be a finite group and $\alpha \leq 17$. Then $G \cong L_2(17)^\alpha$ if and only if $|G| = |L_2(17)|^\alpha$ and $17^\alpha \in \text{cd}(G)$.

**Remark 10.** In [24, Theorem A], Xu et al. proved that $L_2(7)$ is characterizable by $|G|$ and $7 \in \text{cd}(G)$.

Using the notations of GAP [9], if $A = \text{SmallGroup}(56,11)$ and $H = A^2 \times \mathbb{Z}_9$, then $|H| = |L_2(7)^2|$ and $H$ has an irreducible character of degree $7^2$. Therefore $L_2(7)^\alpha$, where $\alpha \geq 2$, is not characterizable by $|G|$ and $7^\alpha \in \text{cd}(G)$.

If $C = \text{SmallGroup}(3^313,11)$ and $H = C \times \mathbb{Z}_{16}$, then $|H| = |L_3(3)|$ and $H$ has an irreducible character of degree 13. Thus $L_3(3)^\alpha$, where $\alpha \geq 1$, is not characterizable by $|G|$ and $13^\alpha \in \text{cd}(G)$.

So we need more assumptions to characterize $L_2(7)^\alpha$, where $2 \leq \alpha \leq 7$ and $L_3(3)^\alpha$, where $\alpha \leq 13$.

**Lemma 11.** Let $G$ be a finite group. Then the following statements hold:

(a) If $|G| = 2^r3^s7^n$, where $s < 36n/7$, and $2^r, 7^n \in \text{cd}(G)$, then $G$ is not solvable;

(b) If $|G| = 2^r3^s13^n$, where $r < 144n/13$, and $3^s, 13^n \in \text{cd}(G)$, then $G$ is not solvable.

**Proof.** (a) Let $G$ be a solvable group. By Lemma 5, $O_2(G) = 1$ and $O_7(G) = 1$. Therefore $\text{Fit}(G) \cong O_3(G) \neq 1$. Since $G$ is a solvable group, $C_G(\text{Fit}(G)) \leq \text{Fit}(G)$. Hence $|G|$ divides $|\text{Fit}(G)| \cdot |\text{Aut(Fit}(G))|$. So $7^n \mid |\text{GL}(t,3)|$ and by Lemma 4, $\text{Ord}_{7^k}(3) = \text{Ord}_7(3) \cdot 7^{k-1}$. Hence

$$n \leq \left[ \frac{s}{\text{Ord}_7(3)} \right] + \left[ \frac{s}{\text{Ord}_{7^2}(3)} \right] + \cdots \leq \frac{s}{6} \left( 1 + \frac{1}{7} + \frac{1}{7^2} + \cdots \right) \leq \frac{s}{6} \cdot \frac{7}{6} < n,$$
which is a contradiction.

(b) Similarly to the above, the result holds. □

**Theorem 12.** Let $G$ be a finite group and $\alpha \leq 7$. Then $G \cong L_2(7)^\alpha$ if and only if $|G| = |L_2(7)|^\alpha$ and $2^{3\alpha}, 7^\alpha \in \text{cd}(G)$.

**Proof.** By Lemma 11, it follows that $G$ is not solvable. So, similarly to the proof of Theorem 8, we get that a subnormal series, like (1), where $K_i/H_i$ is a direct product of $\alpha_i$ copies of a nonabelian simple group $S_i$. Also $|H_{i-1}/K_i| \cdot |\text{Out}(K_i/H_i)|$ and $m$ is the smallest number where $H_m$ is solvable. We show that $7 \nmid |H_m|$. On the contrary, let $7^\beta \mid |H_m|$. Then $7^\beta$ is a divisor of $t = |H_m| \cdot \prod_{i=1}^{m} |H_{i-1}/K_i| = \frac{|G| \cdot \prod_{i=1}^{m} |K_i/H_i|}{|H_m|}$. On the other hand, $t$ is a divisor of $|L_2(7)|^\alpha / |L_2(7)|^\gamma$, where $7^\gamma \mid \prod_{i=1}^{m} |K_i/H_i|$. By Lemma 1, $7^\beta \in \text{cd}(H_m)$ and using Lemma 11, $H_m$ is not solvable which is a contradiction. Thus $7 \nmid |H_m|$. We conclude that $7^\alpha \mid \prod_{i=1}^{m} |K_i/H_i|$. As $K_i/H_i \cong S_i^{\alpha_i}$, where $S_i$ is a nonabelian simple $\{2, 3, 7\}$-group, it must hold that $H_m = 1$ and for each $i$, $H_{i-1} = K_i$. We obtain that $|G| = \prod_{i=1}^{m} |K_i/H_i|$, therefore for each $i$, $K_i/H_i \cong L_2(7)^{\alpha_i}$, where $\alpha_1 + \cdots + \alpha_m = \alpha$. Applying Lemma 6, we get that $G \cong L_2(7)^\alpha$. □

**Remark 13.** Theorems 8, 9 and 12 are generalizations of Theorem 2.4 in [12], for special cases $p = 5, 7$ and 17.

Similarly to the above theorem we have the following theorem:

**Theorem 14.** Let $G$ be a finite group and $\alpha \leq 13$. Then $G \cong L_3(3)^\alpha$ if and only if $|G| = |L_3(3)|^\alpha$ and $3^{3\alpha}, 13^\alpha \in \text{cd}(G)$.

As a consequence of the above theorems, by [3, Theorem 2.13] we have the following result which is a partial answer to the question arose in [20].

**Corollary 15.** Let $M \in \{A_n^n | n \leq 5\} \cup \{L_2(7)^n | n \leq 7\} \cup \{L_3(3)^n | n \leq 13\} \cup \{L_2(17)^n | n \leq 17\}$. If $G$ is a group such that $\mathbb{C}G \cong \mathbb{C}M$, then $G \cong M$. Thus $M$ is uniquely determined by the structure of its complex group algebra.

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**REFERENCES**


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