INEQUALITIES OF GAGLIARDO-NIRENBERG TYPE IN REALIZED HOMOGENEOUS BESOV AND TRIEBEL-LIZORKIN SPACES

MOHAMED BENALLIA and MADANI MOUSSAI

Communicated by Dan Timotin

In the realized homogeneous Besov spaces and the realized homogeneous Triebel-Lizorkin spaces we will give some inequalities of the Gagliardo-Nirenberg type. Then we deduce some embedding properties of certain realized spaces into the Lebesgue spaces.

AMS 2010 Subject Classification: 46E35.

Key words: homogeneous Besov spaces, homogeneous Triebel-Lizorkin spaces, Gagliardo-Nirenberg inequalities, realizations.

1. INTRODUCTION

In this paper, we study some inequalities of the Gagliardo-Nirenberg type in both, homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and homogeneous Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(\mathbb{R}^n)$. We denote by $\dot{A}^s_{p,q}(\mathbb{R}^n)$ for either $\dot{B}^s_{p,q}(\mathbb{R}^n)$ or $\dot{F}^s_{p,q}(\mathbb{R}^n)$, and by $A^s_{p,q}(\mathbb{R}^n)$ for inhomogeneous counterparts that are either $B^s_{p,q}(\mathbb{R}^n)$ or $F^s_{p,q}(\mathbb{R}^n)$, when we have no need to distinguish them. These spaces will be shortened by the initials B and F, respectively. In connection with Gagliardo-Nirenberg estimates type, we give the following example

$$\|f\|_{L_v(\mathbb{R}^n)} \leq c\|f\|^{p/v}_{F^{\beta}_{p,\infty}(\mathbb{R}^n)}\|f\|^{1-p/v}_{B^{-\alpha}_{\infty,\infty}(\mathbb{R}^n)} \tag{1.1}$$

with $v > p$, $\alpha > 0$ and $\beta := \alpha(v/p-1)$, in which we cannot replace $B^{-\alpha}_{\infty,\infty}(\mathbb{R}^n)$ or $F^{\beta}_{p,\infty}(\mathbb{R}^n)$ by $\dot{B}^{-\alpha}_{\infty,\infty}(\mathbb{R}^n)$ or $\dot{F}^{\beta}_{p,\infty}(\mathbb{R}^n)$, since $\|f\|_{\dot{B}^{-\alpha}_{\infty,\infty}(\mathbb{R}^n)} = \|f\|_{\dot{F}^{\beta}_{p,\infty}(\mathbb{R}^n)} = 0$ for all f polynomials on $\mathbb{R}^n$ (we note that (1.1) also holds by replacing $F^{\beta}_{p,\infty}(\mathbb{R}^n)$ by $A^{\beta}_{p,p}(\mathbb{R}^n)$, which is obtained by the embedding property). For this reason, we give some estimates of type (1.1) with the realized homogeneous Besov spaces $\tilde{\dot{B}}^s_{p,q}(\mathbb{R}^n)$ and the realized homogeneous Triebel-Lizorkin spaces $\tilde{\dot{F}}^s_{p,q}(\mathbb{R}^n)$, since these spaces are defined by both, tempered distributions and polynomials of degree less than ν; the parameter ν depends only on n, p, q, s which is characterized by G. Bourdaud [7] (see also Subsection 2.2 below for its definition). Then owing to nonzero polynomials, we note that there are *nontrivial* embeddings of the homogeneous spaces $\dot{A}^s_{p,q}(\mathbb{R}^n)$ into the Lebesgue spaces $L_p(\mathbb{R}^n)$, and by our wanted estimates, we hope to obtain some embeddings of the realized spaces $\tilde{\dot{A}}^s_{p,q}(\mathbb{R}^n)$ into $L_p(\mathbb{R}^n)$. For instance in Section 4, we prove (1.1) in $\tilde{\dot{A}}^s_{p,q}(\mathbb{R}^n)$. We will essentially prove the following result, where we will use the following notation throughout this work: for every tempered distribution f, we denote by $[f]_{\mathcal{P}}$ the equivalence class of f modulo polynomials.

Theorem 1.1. *Let $0 < p, q < \infty$. We put $r := \min(1, p)$ in the B-case and $r := 1$ in the F-case. Then there exists a constant $c > 0$ such that the inequality*

$$\|f\|_{L_v(\mathbb{R}^n)} \le c\, v^{1-1/q}\, 2^{n/v}\, \|[f]_{\mathcal{P}}\|^{p/v}_{\dot{A}^0_{p,r}(\mathbb{R}^n)}\, \|[f]_{\mathcal{P}}\|^{1-p/v}_{\dot{A}^{n/p}_{p,q}(\mathbb{R}^n)} \tag{1.2}$$

holds, for all $v \in [p, \infty[$ and all $f \in \tilde{\dot{A}}^0_{p,r}(\mathbb{R}^n) \cap \tilde{\dot{A}}^{n/p}_{p,q}(\mathbb{R}^n)$.

If $q \le 1$ in the B-case or $p \le 1$ in the F-case, we can avoid $\dot{A}^0_{p,r}(\mathbb{R}^n)$ in the right-hand side of (1.2) by taking the space $L_u(\mathbb{R}^n)$ instead, and the resulting estimate becomes independent of the $\tilde{\dot{A}}^{n/p}_{p,q}(\mathbb{R}^n)$'s quasi-seminorm, that is the following statement:

Theorem 1.2. *Let $0 < p, q < \infty$ with $q \le 1$ in the B-case and $p \le 1$ in the F-case. Then there exists a constant $c > 0$ such that the inequality*

$$\|f\|_{L_v(\mathbb{R}^n)} \le c\, v^{1-1/q}\, 2^{n/v}\, \|f\|^{u/v}_{L_u(\mathbb{R}^n)}\, \|[f]_{\mathcal{P}}\|^{1-u/v}_{\dot{A}^{n/p_1}_{p_1,q}(\mathbb{R}^n)} \tag{1.3}$$

holds, for all $u \in]0, \infty[$, all $v \in]0, \infty[$ such that $v \ge \max(p, u)$, all $p_1 \in [p, \infty[$ and all $f \in L_u(\mathbb{R}^n) \cap \tilde{\dot{A}}^{n/p}_{p,q}(\mathbb{R}^n)$.

An immediate consequence of these results concerns the embedding into the $L_p(\mathbb{R}^n)$ spaces.

Corollary 1.3. (i) *Let p, q, r and v be given as in Theorem 1.1. Then it holds $\tilde{\dot{A}}^0_{p,r}(\mathbb{R}^n) \cap \tilde{\dot{A}}^{n/p}_{p,q}(\mathbb{R}^n) \hookrightarrow L_v(\mathbb{R}^n)$.*

(ii) *Let p, q, u and v be given as in Theorem 1.2. Then it holds $L_u(\mathbb{R}^n) \cap \tilde{\dot{A}}^{n/p}_{p,q}(\mathbb{R}^n) \hookrightarrow L_v(\mathbb{R}^n)$.*

As mentioned before, the estimate (1.2) fails to hold if $\tilde{\dot{A}}^0_{p,r}(\mathbb{R}^n) \cap \tilde{\dot{A}}^{n/p}_{p,q}(\mathbb{R}^n)$ is replaced by $\dot{A}^0_{p,r}(\mathbb{R}^n) \cap \dot{A}^{n/p}_{p,q}(\mathbb{R}^n)$. Contrary to the homogeneous spaces, Theorems 1.1 and 1.2 cover the case of inhomogeneous ones; in other words, we can take $A^0_{p,r}(\mathbb{R}^n) \cap A^{n/p}_{p,q}(\mathbb{R}^n)$ and $L_u(\mathbb{R}^n) \cap A^{n/p}_{p,q}(\mathbb{R}^n)$ instead of $\tilde{\dot{A}}^0_{p,r}(\mathbb{R}^n) \cap \tilde{\dot{A}}^{n/p}_{p,q}(\mathbb{R}^n)$ and $L_u(\mathbb{R}^n) \cap \tilde{\dot{A}}^{n/p}_{p,q}(\mathbb{R}^n)$, respectively, in both theorems.

In Theorem 1.1, the right-hand side of (1.2) is given by the quasi-seminorms of elements in spaces defined, in the tempered distributions space $\mathcal{S}'(\mathbb{R}^n)$, from the homogeneous ones, using the notion of realization, see *e.g.*, [3, 7, 14]. In this context and in the B-case, we can see [24] where it was considered the homogeneous spaces defined in $\mathcal{S}'(\mathbb{R}^n)$, not in spaces defined modulo polynomials of a certain degree, also with other conditions on the parameters.

In the B-case, Theorem 1.2 is an extension, to the case $\max(p,\, u) \le v < 1$, of a result obtained in [23, Theorem 4.14(i)]. Note that in the right-hand side of (1.3) we find the term $\|[f]_{\mathcal{P}}\|_{\dot{A}^{n/p_1}_{p_1,q}(\mathbb{R}^n)}$ in improving $\|[f]_{\mathcal{P}}\|_{\dot{A}^{n/p}_{p,q}(\mathbb{R}^n)}$ (recall that $p \le p_1$).

Of course, the constants in (1.2) or (1.3) can be restricted to $cv^{1-1/q}$ if $p \ge 1$ or $u \ge 1$, respectively, since $2^{n/v} \le 2^n$. Also, in that case we have in (1.2) the optimality of the growth rate $v^{1-1/q}$ as $v \to \infty$ in the B-case and at least if $q \le p$ in the F-case (see Subsection 4.1 below). However, the proofs of the above results are based on some classical inequalities as Bernstein-type ((2.1) below) and an approximation method by suitable smooth functions.

Finally, we recall that these type of estimates on homogeneous Sobolev, Besov and Triebel-Lizorkin spaces, defined as function spaces excluding polynomials or as tempered distributions modulo all polynomials, have been studied in several works *e.g.*, [12, 13, 16, 24, 25].

The paper is organized as follows. In Section 2, we collect definitions and basic properties of the considered function spaces. Section 3 is devoted to the proofs of our main results. In Section 4, we discuss the optimality of the estimates and some extensions.

2. NOTATIONS AND PRELIMINARIES

As usual, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $\mathbb{Z}$ the set of integers and $\mathbb{R}$ the set of real numbers. All function spaces occurring in this work are defined on Euclidean space $\mathbb{R}^n$, then we omit $\mathbb{R}^n$ in notations. For $a \in \mathbb{R}$ we put $a_+ := \max(0, a)$. For $t \in \mathbb{R}$, $[t]$ denotes the greatest integer less than or equal to t. The symbol $\hookrightarrow$ indicates a continuous embedding. $\mathcal{S}$ denotes the Schwartz space and $\mathcal{S}'$ its topological dual. For $0 < p \le \infty$ we denote by $\|\cdot\|_p$ the quasi-norm of the Lebesgue space L_p. Corresponding to this, L_p^{loc} means the set of functions satisfying $\int_K |f(x)|^p \mathrm{d}x < \infty$ for all compact sets K of $\mathbb{R}^n$. For $f \in L_1$, we denote by $\widehat{f}(\xi) := \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x)\mathrm{d}x$ the Fourier transform and by $\mathcal{F}^{-1}f(x) := (2\pi)^{-n}\widehat{f}(-x)$ the inverse Fourier transform. They are extended to the whole space $\mathcal{S}'$ in the usual way.

We denote by $\mathcal{P}_\infty$ the set of all polynomials on $\mathbb{R}^n$. We denote by $\mathcal{S}_\infty$ the set of all $\varphi \in \mathcal{S}$ such that $\langle u, \varphi \rangle = 0$ for all $u \in \mathcal{P}_\infty$ and by $\mathcal{S}'_\infty$ its topological dual, which can identified to the quotient space $\mathcal{S}'/\mathcal{P}_\infty$. For all $f \in \mathcal{S}'$, we denote by $[f]_{\mathcal{P}}$ the equivalence class of f modulo $\mathcal{P}_\infty$; this notation has been given before. The mapping which takes any $[f]_{\mathcal{P}}$ to the restriction of f to $\mathcal{S}_\infty$ turns out to be an isomorphism of $\mathcal{S}'/\mathcal{P}_\infty$ onto $\mathcal{S}'_\infty$. Then $\mathcal{S}'_\infty$ is called the space of distributions modulo polynomials.

Finally, the constants $c, c', c_1, \ldots$ are positives and depend only on the fixed parameters n, s, p, q, …, their values probably change from line to line.

Throughout the paper, we will make use of the following well known inequalities:

- For all $a_j \geq 0$ and all $0 < d \leq 1$ it holds $\left(\sum_{j\in\mathbb{Z}} a_j\right)^d \leq \sum_{j\in\mathbb{Z}} a_j^d$.
- Let $0 < p \leq q \leq \infty$. There exists a constant $c > 0$ such that

$$\|f\|_q \leq cR^{n(1/p-1/q)} \|f\|_p \tag{2.1}$$

holds, for all $R > 0$ and all $f \in L_p$ satisfying $\widehat{f}$ is supported by the ball $|\xi| \leq R$. The constant c can be given explicitly, cf. [15, Theorem 4]; in this paper, $c = p_0^{n(1/p-1/q)}$ where p_0 is the smallest integer not less than $p/2$.

2.1. THE LITTLEWOOD-PALEY DECOMPOSITION

The Littlewood-Paley setting is useful for the definition of Besov and Triebel-Lizorkin spaces. This setting has been initiated by *e.g.*, Bergh and Löfström [2], Peetre [17] and Triebel [20, 21]. We will recall: let ρ be a C^∞, radial function such that $0 \leq \rho \leq 1$, with $\rho(\xi) = 1$ if $|\xi| \leq 1$ and $\rho(\xi) = 0$ if $|\xi| \geq 3/2$. We put $\gamma(\xi) := \rho(\xi) - \rho(2\xi)$ which is supported by the annulus $1/2 \leq |\xi| \leq 3/2$, and the following identities hold

$$\sum_{j\in\mathbb{Z}} \gamma(2^j\xi) = 1 \qquad (\forall \xi \in \mathbb{R}^n \setminus \{0\}),$$

$$\rho(2^{-k}\xi) + \sum_{j\geq k+1} \gamma(2^{-j}\xi) = 1 \qquad (\forall k \in \mathbb{Z},\ \forall \xi \in \mathbb{R}^n).$$

The functions ρ and γ will be fixed once and for all. We define the pseudodifferential operators $(S_j)_{j\in\mathbb{Z}}$ and $(Q_j)_{j\in\mathbb{Z}}$ by $\widehat{S_j f}(\xi) := \rho\left(2^{-j}\xi\right) \widehat{f}\,(\xi)$ and $\widehat{Q_j f}(\xi) := \gamma\left(2^{-j}\xi\right) \widehat{f}\,(\xi)$. We also define the operators $(\widetilde{Q}_j)_{j\in\mathbb{N}_0}$ by $\widetilde{Q}_0 := S_0$ and $\widetilde{Q}_j := Q_j$ for $j \geq 1$. The operators S_j and Q_j take values in the space of analytical functions of exponential type, see Paley-Wiener theorem, in [19, Theorem 29.2, p. 311] or [20, Remark 2.3.1/2, p. 45].

It is clear that S_j is defined on $\mathcal{S}'$ and that Q_j is defined on $\mathcal{S}'_\infty$ since $Q_j f(x) = 0$ if, and only if, f is a polynomial. We make use of the following convention:

$$\text{If } f \in \mathcal{S}'_\infty \text{ we define } Q_j f := Q_j f_1 \text{ for all } f_1 \text{ such that } [f_1]_{\mathcal{P}} = f.$$

The convergence of the Littlewood-Paley decomposition of any function is given by: for every $f \in \mathcal{S}_\infty$ ($\mathcal{S}'_\infty$, respectively) one has $f = \sum_{j\in\mathbb{Z}} Q_j f$ with a convergence in $\mathcal{S}_\infty$ ($\mathcal{S}'_\infty$, respectively), also, for every $f \in \mathcal{S}$ ($\mathcal{S}'$, respectively) and every $k \in \mathbb{Z}$, one has $f = S_k f + \sum_{j>k} Q_j f$ with a convergence in $\mathcal{S}$ ($\mathcal{S}'$, respectively). For the proof of these facts we refer to [14, Proposition 2.7].

2.2. THE BESOV AND TRIEBEL-LIZORKIN SPACES

The basic definitions of $\dot{A}^s_{p,q}$ and $A^s_{p,q}$ are given via the Littlewood-Paley decomposition, see *e.g.* [2, 11, 20].

Definition 2.1. Let $s \in \mathbb{R}$ and $0 < q \le \infty$.

(i) Let $0 < p \le \infty$. The homogeneous Besov space $\dot{B}^s_{p,q}$ is the set of $f \in \mathcal{S}'_\infty$ such that

$$\|f\|_{\dot{B}^s_{p,q}} := \Big(\sum_{j\in\mathbb{Z}} 2^{jsq} \|Q_j f\|_p^q\Big)^{1/q} < \infty.$$

(ii) Let $0 < p < \infty$. The homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q}$ is the set of $f \in \mathcal{S}'_\infty$ such that

$$\|f\|_{\dot{F}^s_{p,q}} := \Big\|\Big(\sum_{j\in\mathbb{Z}} 2^{jsq} |Q_j f|^q\Big)^{1/q}\Big\|_p < \infty.$$

Definition 2.2. Let $s \in \mathbb{R}$ and $0 < q \le \infty$.

(i) Let $0 < p \le \infty$. The inhomogeneous Besov space $B^s_{p,q}$ is the set of $f \in \mathcal{S}'$ such that

$$\|f\|_{B^s_{p,q}} := \Big(\sum_{j\ge 0} 2^{jsq} \|\widetilde{Q}_j f\|_p^q\Big)^{1/q} < \infty.$$

(ii) Let $0 < p < \infty$. The inhomogeneous Triebel-Lizorkin space $F^s_{p,q}$ is the set of $f \in \mathcal{S}'$ such that

$$\|f\|_{F^s_{p,q}} := \Big\|\Big(\sum_{j\ge 0} 2^{jsq} |\widetilde{Q}_j f|^q\Big)^{1/q}\Big\|_p < \infty.$$

The spaces $\dot{A}^s_{p,q}$ and $A^s_{p,q}$ are quasi-Banach spaces for the above defined quasi-seminorms and quasi-norms (Definitions 2.1 and 2.2, respectively), which do not depend on the function ρ, see *e.g.*, [10] or [20]. For characterizations and properties of $A^s_{p,q}$ we refer to [2, 18, 20, 21], however for $\dot{A}^s_{p,q}$ we recall the following assertions:

- $\mathcal{S}_\infty \hookrightarrow \dot{A}^s_{p,q} \hookrightarrow \mathcal{S}'_\infty$,
- $\dot{A}^s_{p,q_1} \hookrightarrow \dot{A}^s_{p,q_2}$ if $q_1 < q_2$ and $\dot{B}^s_{p,\min(p,q)} \hookrightarrow \dot{F}^s_{p,q} \hookrightarrow \dot{B}^s_{p,\max(p,q)}$,
- if $s_1 > s_2$, $0 < p_1 < p_2 < \infty$, $0 < q, r \le \infty$ and $s_1 - n/p_1 = s_2 - n/p_2$ then it holds $\dot{B}^{s_1}_{p_1,q} \hookrightarrow \dot{B}^{s_2}_{p_2,q} \hookrightarrow \dot{B}^{s_2-n/p_2}_{\infty,q}$, $\dot{F}^{s_1}_{p_1,q} \hookrightarrow \dot{B}^{s_2}_{p_2,p_1}$ and $\dot{F}^{s_1}_{p_1,q} \hookrightarrow \dot{F}^{s_2}_{p_2,r}$, see [11],
- if $0 < p, q < \infty$ then $\mathcal{S}_\infty$ is a dense subspace in $\dot{A}^s_{p,q}$, see [7, Proposition 3.11] or [11, (1.6)],
- there exist $c_1, c_2 > 0$ such that

$$c_1 \|f\|_{\dot{A}^s_{p,q}} \le \lambda^{s-n/p} \|f(\lambda^{-1}(\cdot))\|_{\dot{A}^s_{p,q}} \le c_2 \|f\|_{\dot{A}^s_{p,q}}$$

for all $f \in \dot{A}^s_{p,q}$ and all $\lambda > 0$, see [6].

We also recall the Nikol'skij type estimates and refer to [8, Proposition 4] and [14, Propositions 2.15, 2.17] for the proofs.

Proposition 2.3. *Let $s \in \mathbb{R}$ and $0 < p, q \le \infty$ (with $p < \infty$ in the F-case). Let $0 < a < b$ and let $(u_j)_{j\in\mathbb{Z}}$ be a sequence in $\mathcal{S}'$ satisfying*

- *$\widehat{u_j}$ is supported by the annulus $a2^j \le |\xi| \le b2^j$,*
- *$A := \big(\sum_{j\in\mathbb{Z}} (2^{js}\|u_j\|_p)^q\big)^{1/q} < \infty$ ($A := \big\|\big(\sum_{j\in\mathbb{Z}} (2^{js}|u_j(\cdot)|)^q\big)^{1/q}\big\|_p < \infty$ in the F-case).*

Then the series $\sum_{j\in\mathbb{Z}} u_j$ converges in $\mathcal{S}'_\infty$ and $\|\sum_{j\in\mathbb{Z}} u_j\|_{\dot{A}^s_{p,q}} \le cA$, where the constant c depends only on n, s, p, q, a and b.

There exists a link between $\dot{A}^s_{p,q}$ and its inhomogeneous counterpart. Namely, we have the following statement, which is proved in [21, p. 98].

Proposition 2.4. *Let $0 < p, q \le \infty$ (with $p < \infty$ in the F-case). Let s be a real such that $s > (n/p - n)_+$. Then $f \in A^s_{p,q}$ if, and only if, $f \in L_p$ and $[f]_{\mathcal{P}} \in \dot{A}^s_{p,q}$. Moreover, $\|f\|_p + \|[f]_{\mathcal{P}}\|_{\dot{A}^s_{p,q}}$ defines an equivalent quasi-norm in $A^s_{p,q}$.*

To define the realized homogeneous spaces of Besov, and of Triebel-Lizorkin, we first give the notion of distributions vanishing at infinity.

Definition 2.5. A distribution f vanishes at infinity in the weak sense if $\lim_{\lambda\to 0} f(\lambda^{-1}(\cdot)) = 0$ in $\mathcal{S}'$. The set of all such distributions is denoted by $\widetilde{C}_0$.

We second recall that if $f \in \dot{A}^s_{p,q}$, then the Littlewood-Paley series $\sum_{j\in\mathbb{Z}} Q_j f$ converges in $\mathcal{S}'_\nu$ to an element denoted $\sigma_\nu(f)$ which satisfies

$$f = [\sigma_\nu(f)]_{\mathcal{P}} \ \text{in} \ \mathcal{S}'_\infty \quad \text{and} \quad \partial^\alpha \sigma_\nu(f) \in \widetilde{C}_0 \quad \text{for all} \ |\alpha| = \nu,$$

where the integer ν (which will be fixed throughout this paper) is defined as the following:

$$\nu := \begin{cases} ([s-n/p]+1)_+ \text{ if } s-n/p \notin \mathbb{N}_0 \text{ or } q>1 \text{ in } B\text{-case } (p>1 \text{ in } F\text{-case}), \\ s-n/p \text{ if } s-n/p \in \mathbb{N}_0 \text{ and } q \leq 1 \text{ in } B\text{-case } (p \leq 1 \text{ in } F\text{-case}), \end{cases}$$

see [7, 14].

Definition 2.6. Let $s \in \mathbb{R}$ and $0 < q \leq \infty$.

(i) Let $0 < p \leq \infty$. The realized homogeneous Besov space $\dot{\widetilde{B}}^s_{p,q}$ is the set of $f \in \mathcal{S}'_\nu$ such that $[f]_{\mathcal{P}} \in \dot{B}^s_{p,q}$ and $f^{(\alpha)} \in \widetilde{C}_0$ for all $|\alpha| = \nu$. The space $\dot{\widetilde{B}}^s_{p,q}$ is endowed with the quasi-seminorm $\|f\|_{\dot{\widetilde{B}}^s_{p,q}} := \|[f]_{\mathcal{P}}\|_{\dot{B}^s_{p,q}}$.

(ii) Let $0 < p < \infty$. The realized homogeneous Triebel-Lizorkin space $\dot{\widetilde{F}}^s_{p,q}$ is the set of $f \in \mathcal{S}'_\nu$ such that $[f]_{\mathcal{P}} \in \dot{F}^s_{p,q}$ and $f^{(\alpha)} \in \widetilde{C}_0$ for all $|\alpha| = \nu$. The space $\dot{\widetilde{F}}^s_{p,q}$ is endowed with the quasi-seminorm $\|f\|_{\dot{\widetilde{F}}^s_{p,q}} := \|[f]_{\mathcal{P}}\|_{\dot{F}^s_{p,q}}$.

Remark 2.7. It is possible to define $\dot{\widetilde{A}}^s_{p,q}$ in $\mathcal{S}'$ by correcting in the Littlewood-Paley decomposition each $Q_k f$ by a polynomial of degree less than ν. In this sense, the construction of $\dot{\widetilde{A}}^s_{p,q}$ in $\mathcal{S}'$ is given as the following: for $f \in \dot{A}^s_{p,q}$ we have

(i) $\sigma_\nu(f) := \sum_{k\in\mathbb{Z}} Q_k f$, if either $s < n/p$ or $s = n/p$ and $q \leq 1$ in B-case ($p \leq 1$ in F-case),

(ii) $\sigma_\nu(f) := \sum_{k\in\mathbb{Z}} \Big(Q_k f - \sum_{|\alpha|<\nu} (Q_k f)^{(\alpha)}(0) x^\alpha/\alpha! \Big)$, if either $s - n/p \in \mathbb{R}^+ \setminus \mathbb{N}_0$ or $s - n/p \in \mathbb{N}$ and $q \leq 1$ in B-case ($p \leq 1$ in F-case),

(iii) $\sigma_\nu(f) := \sum_{j\geq 1} Q_j f + \sum_{k\leq 0} \Big(Q_k f - \sum_{|\alpha|<\nu} (Q_k f)^{(\alpha)}(0) x^\alpha/\alpha! \Big)$, if $s - n/p \in \mathbb{N}_0$ and $q > 1$ in B-case ($p > 1$ in F-case),

where all above series converge in $\mathcal{S}'$, and $\partial^\alpha \sigma_\nu(f) \in \widetilde{C}_0$ for all $|\alpha| = \nu$, and $[\sigma_\nu(f)]_{\mathcal{P}} = f$ in $\mathcal{S}'_\infty$, see [7].

We finish this section by giving some examples of functions in $\widetilde{C}_0$: we begin by the polynomial functions using the following easy lemma proved in [3, p. 46].

LEMMA 2.8. *If f is a polynomial vanishing at infinity in the weak sense, then $f = 0$, i.e., $\widetilde{C}_0 \cap \mathcal{P}_\infty = \{0\}$.*

Example 2.9. (i) Functions in L_p for $1 \le p < \infty$.

(ii) Derivatives of bounded functions.

(iii) Derivatives of the members of $\widetilde{C}_0$.

(iv) The function $g(x) := x^\alpha \mathrm{e}^{\mathrm{i}x\cdot\eta}$ ($\eta \in \mathbb{R}^n \setminus \{0\}$, $\alpha \in \mathbb{N}_0^n$) belongs to $\widetilde{C}_0$. Indeed, for all $\varphi \in \mathcal{S}$ there exists a constant $c > 0$ independent of η, such that the inequality

$$|\langle g(\lambda^{-1}(\cdot)), \varphi\rangle| = \lambda^{-|\alpha|}|(\mathcal{F}^{-1}\varphi)^{(\alpha)}(\lambda^{-1}\eta)| \le c\lambda^{N-|\alpha|}|\eta|^{-N}$$

holds for all $\lambda > 0$, where the positive integer N is large enough. The last term tends to 0 with $\lambda \to 0$. More generally, for any nonzero polynomial $\mathcal{P}$ and any $\eta \in \mathbb{R}^n \setminus \{0\}$, the functions $f(x) := \mathrm{e}^{\mathrm{i}x\cdot\eta}\mathcal{P}(x)$ belong to $\widetilde{C}_0$.

(v) We also give an example of functions $f \in L_p$ (with $0 < p < 1$) such that $f \notin \widetilde{C}_0$. Indeed, let $f_0(x) := |x|^{-n}\rho(x)$; the function ρ is defined in Subsection 2.1. Clearly that $f_0 \in L_p$ (with $0 < p < 1$). For a positive integer N large enough, we have

$$\begin{aligned}\langle f_0(2^N(\cdot)), \rho\rangle &= 2^{-nN}\int_{\mathbb{R}^n} |x|^{-n}\rho(2^N x)\rho(x)\,\mathrm{d}x \\ &\ge 2^{-nN}\int_{r<|x|<2^{-nN}} |x|^{-n}\,\mathrm{d}x \qquad (\text{with } \log r := -2^{N(n+1)}),\end{aligned}$$

and the last term tends to ∞ with $N \to \infty$.

3. PROOFS OF THE MAIN RESULTS

We first prove the following statement.

PROPOSITION 3.1. *Let $0 < p, q \le \infty$ (with $p < \infty$ in the F-case). Then there exists a constant $c > 0$ such that the inequality*

$$\|f\|_v \le c\, v^{1-1/q}\, 2^{n/v}\, \|f\|_{A^{n/p}_{p,q}} \tag{3.1}$$

holds, for all $v \in [p, \infty]$ (with $v < \infty$ in the F-case) and all $f \in A^{n/p}_{p,q}$. The constant c can be chosen such that $c := \max(1, p^{n/p})$ if $p < v$ and $c := 1$ if $p = v$, see (2.1).

Proof. Step 1: the B-case. Let $f \in B^{n/p}_{p,q}$. We separate the cases according to q and v.

• The case $q \ge 1$ and $v \ge 1$. It is easy to see that by using (2.1) we obtain

$$\|f\|_v \le \sum_{j\ge 0} \|\widetilde{Q}_j f\|_v \le c\sum_{j\ge 0} 2^{j(n/p-n/v)}\|\widetilde{Q}_j f\|_p,$$

where c is independent of v, indeed, we have $c := p_0^{n(1/p-1/v)}$ with $p_0 \in \mathbb{N}$ such that $p_0 > p/2$, cf. [15, Theorem 4], then $c \leq p_0^{n/p} \leq \max(1, p^{n/p})$ if $p < v$, however $c := 1$ if $p = v$, see (2.1) again. By Hölder's inequality, we have

$$\|f\|_v \leq c\Big(\sum_{j\geq 0} 2^{-jq'n/v}\Big)^{1/q'} \|f\|_{B^{n/p}_{p,q}} \qquad (q' := q/(q-1)). \tag{3.2}$$

Using the elementary inequality

$$\sum_{j\geq 0} 2^{-j\beta} = \frac{1}{1-2^{-\beta}} \leq \frac{2^\beta}{\beta \log 2} \qquad (\forall \beta > 0), \tag{3.3}$$

it holds that the right-hand side of (3.2) is bounded by $c\, v^{1-1/q} 2^{n/v} \|f\|_{B^{n/p}_{p,q}}$.

• The case $q \geq 1$ and $0 < v < 1$. By the embedding $\ell_p(\mathbb{N}_0) \hookrightarrow \ell_v(\mathbb{N}_0)$ since $p \leq v$ (here $\ell_p(\mathbb{N}_0)$ means that $\|(a_k)\|_{\ell_p(\mathbb{N}_0)} := (\sum_{k\geq 0} |a_k|^p)^{1/p} < \infty$), one has

$$\begin{aligned}
\|f\|_v &= \Big\|\sum_{j\geq 0} \widetilde{Q}_j f\Big\|_v \leq \Big(\sum_{j\geq 0} \|\widetilde{Q}_j f\|_v^v\Big)^{1/v} \\
&\leq \Big(\sum_{j\geq 0} \|\widetilde{Q}_j f\|_v^p\Big)^{1/p} \leq c\Big(\sum_{j\geq 0} \Big(2^{jn/p}\|\widetilde{Q}_j f\|_p\Big)^p 2^{-jpn/v}\Big)^{1/p} \\
&\leq c\, \|f\|_{B^{n/p}_{p,\infty}} \Big(\sum_{j\geq 0} 2^{-jpn/v}\Big)^{1/p},
\end{aligned}$$

and as in (3.3) we have

$$\|f\|_v \leq c\, v^{1/p} 2^{n/v} \|f\|_{B^{n/p}_{p,\infty}} \leq c\, v^{1-1/q} 2^{n/v} \|f\|_{B^{n/p}_{p,\infty}}, \tag{3.4}$$

where the last inequality is obtained because $1/p > 1 > 1 - 1/q$. We conclude by using the embedding $B^{n/p}_{p,q} \hookrightarrow B^{n/p}_{p,\infty}$.

• The case $0 < q < 1$ and $v \geq 1$. We get

$$\begin{aligned}
\|f\|_v &\leq \Big\|\Big(\sum_{j\geq 0} |\widetilde{Q}_j f|^q\Big)^{1/q}\Big\|_v \\
&\leq \Big(\sum_{j\geq 0} \|\widetilde{Q}_j f\|_v^q\Big)^{1/q} \leq c\Big(\sum_{j\geq 0} \Big(2^{jn/p}\|\widetilde{Q}_j f\|_p\Big)^q 2^{-jqn/v}\Big)^{1/q},
\end{aligned}$$

where

$$\begin{aligned}
2^{-jqn/v} &= \Big(2^{-jn/v}\Big)^{q(1-1/q)} 2^{-jn/v} \qquad \Big(\text{with } 2^{-jn/v} \leq 1\Big) \\
&\leq \Big(\sum_{k\geq 0} 2^{-kn/v}\Big)^{q(1-1/q)} \leq c_1\, (v 2^{n/v})^{q(1-1/q)}
\end{aligned}$$

$$\leq c_2\, v^{q(1-1/q)}\, 2^{qn/v} \qquad (\forall j \geq 0),$$

and the bound $c\, v^{1-1/q} 2^{n/v}\, \|f\|_{B^{n/p}_{p,q}}$ is obtained again.

• The case $0 < q < 1$ and $0 < v < 1$. Recall that $p \leq v$ implies $p^{-1} \log v < 0 < (1 - 1/q) \log v$. Then

$$\begin{aligned} \|f\|_v &\leq \Big(\sum_{j\geq 0} \|\widetilde{Q}_j f\|_v^v\Big)^{1/v} \leq c\Big(\sum_{j\geq 0} \Big(2^{jn/p}\|\widetilde{Q}_j f\|_p\Big)^p\, 2^{-jpn/v}\Big)^{1/p} \\ &\leq c_1\, v^{1/p}\, 2^{n/v}\, \|f\|_{B^{n/p}_{p,\infty}} \leq c_2\, v^{1-1/q}\, 2^{n/v}\, \|f\|_{B^{n/p}_{p,\infty}} . \end{aligned} \tag{3.5}$$

Step 2: the F-case. Let $f \in F^{n/p}_{p,q}$. Here we use the embedding $F^{n/p}_{p,q} \hookrightarrow B^{n/p}_{p,\infty}$, and also separate the cases with respect to q and v.

• The case $q \geq 1$ and $0 < v \leq 1$. See (3.4).

• The case $0 < q < 1$ and $0 < v \leq 1$. See (3.5).

• The case $0 < q \leq \infty$ and $v > 1$. The previous step (with $q = \infty$) implies that $\|f\|_v \leq c\, v^{1-1/\infty}\, 2^{n/v} \|f\|_{B^{n/p}_{p,\infty}}$, then we have

$$\begin{aligned} \|f\|_v &= \|f\|_v^{1-1/q}\, \|f\|_v^{1/q} \\ &\leq \Big(c_1\, v^{1-1/\infty}\, 2^{n/v}\, \|f\|_{B^{n/p}_{p,\infty}}\Big)^{1-1/q} \Big(\sum_{j\geq 0} \|\widetilde{Q}_j f\|_v\Big)^{1/q} \\ &\leq c_2\, v^{1-1/q}\, 2^{n/v}\, \|f\|^{1-1/q}_{F^{n/p}_{p,q}}\, \|f\|^{1/q}_{B^0_{v,1}} \qquad (\text{ recall that } 2^{-n/(vq)} \leq 1), \end{aligned}$$

and we conclude by the embeddings $F^{n/p}_{p,q} \hookrightarrow B^{n/v}_{v,\infty} \hookrightarrow B^0_{v,1}$. $\square$

Remark 3.2. The estimate (3.1) extends the inequality given in [23, (4.93), p. 145] to the case $0 < q \leq 1$.

Proof of Theorem 1.1. Step 1. We prove (1.2) with functions $f \in A^{n/p}_{p,q}$ such that $[f]_{\mathcal{P}} \in \dot{A}^0_{p,r}$ (with $p < \infty$ in the F-case). We recall that the parameter r is defined as $r := \min(1, p)$ in the B-case and $r := 1$ in the F-case. By (3.1) we get

$$\begin{aligned} \|f\|_v &\leq c_1\, v^{1-1/q}\, 2^{n/v} \big(\|f\|_p + \|[f]_{\mathcal{P}}\|_{\dot{A}^{n/p}_{p,q}}\big) \\ &\leq c_2\, v^{1-1/q}\, 2^{n/v} \big(\|[f]_{\mathcal{P}}\|_{\dot{A}^0_{p,r}} + \|[f]_{\mathcal{P}}\|_{\dot{A}^{n/p}_{p,q}}\big), \end{aligned} \tag{3.6}$$

where the estimate

$$\|f\|_p = \Big\|\sum_{j\geq 0} \widetilde{Q}_j f\Big\|_p \leq \Big\|\Big(\sum_{j\geq 0} |\widetilde{Q}_j f|^r\Big)^{1/r}\Big\|_p \leq c\|[f]_{\mathcal{P}}\|_{\dot{A}^0_{p,r}} \tag{3.7}$$

can be directly obtained in the F-case and by Minkowski inequality in the B-case. Now in (3.6) we replace f by $f(\lambda(\cdot))$ with $\lambda > 0$, and we take

$$\lambda := \|[f]_{\mathcal{P}}\|^{p/n}_{\dot{A}^0_{p,r}} \|[f]_{\mathcal{P}}\|^{-p/n}_{\dot{A}^{n/p}_{p,q}}$$

(here we assume that $\|[f]_{\mathcal{P}}\|_{\dot{A}^{n/p}_{p,q}} \neq 0$ since f is not a polynomial function), then (1.2) follows for all $f \in A^{n/p}_{p,q}$ such that $[f]_{\mathcal{P}} \in \dot{A}^0_{p,r}$.

Step 2. We now take $f \in \dot{\widetilde{A}}^0_{p,r} \cap \dot{\widetilde{A}}^{n/p}_{p,q}$. Since we work with a function f in $\dot{\widetilde{A}}^0_{p,r}$, then $f \in \widetilde{C}_0$ because $\nu = ([-n/p]+1)_+ = 0$. Recall that here $p, q < \infty$.

We introduce a sequence $(g_k)_{k \in \mathbb{N}_0}$ in $\mathcal{S}_\infty$ satisfying $[g_k]_{\mathcal{P}} \to [f]_{\mathcal{P}}$ (with $k \to \infty$) in both $\dot{A}^{n/p}_{p,q}$ and $\dot{A}^0_{p,r}$ simultaneously. By Step 1, we have

$$\begin{aligned}
\|g_k\|_v &\le c_1\, v^{1-1/q}\, 2^{n/v}\, \|[g_k]_{\mathcal{P}}\|^{p/v}_{\dot{A}^0_{p,r}} \|[g_k]_{\mathcal{P}}\|^{1-p/v}_{\dot{A}^{n/p}_{p,q}} \\
&\le c_2\, v^{1-1/q} 2^{n/v} \Big(\|[g_k]_{\mathcal{P}} - [f]_{\mathcal{P}}\|_{\dot{A}^0_{p,r}} + \|[f]_{\mathcal{P}}\|_{\dot{A}^0_{p,r}} \Big)^{p/v} \\
&\quad \times \Big(\|[g_k]_{\mathcal{P}} - [f]_{\mathcal{P}}\|_{\dot{A}^{n/p}_{p,q}} + \|[f]_{\mathcal{P}}\|_{\dot{A}^{n/p}_{p,q}} \Big)^{1-p/v}.
\end{aligned}$$

Then there exists a natural number $k_0 \in \mathbb{N}_0$, such that

$$\|g_k\|_v \le c\, v^{1-1/q}\, 2^{n/v}\, \|[f]_{\mathcal{P}}\|^{p/v}_{\dot{A}^0_{p,r}} \|[f]_{\mathcal{P}}\|^{1-p/v}_{\dot{A}^{n/p}_{p,q}} \qquad (\forall k \ge k_0). \tag{3.8}$$

Now clearly that $g_k \in \dot{\widetilde{A}}^0_{p,r} \cap \dot{\widetilde{A}}^{n/p}_{p,q}$, then we apply the following lemma, which is proved in *e.g.* [4, p. 52]:

LEMMA 3.3. *Let E be a quasi-Banach satisfying $E \hookrightarrow L^{loc}_1$. If a sequence $(f_k)_k$ satisfies that $f_k \to f$ in E, then admits a subsequence $(f_{k_j})_j$ such that $\lim_{j\to\infty} f_{k_j} = f$ almost everywhere.*

Consequently, from the sequence $\{g_{k_0}, g_{k_0+1}, \ldots\}$ we may extract a subsequence $(g_{k_j})_{j\in\mathbb{N}_0}$ such that $\lim_{j\to\infty} g_{k_j} = f$ a.e. Then using (3.8) with $(g_{k_j})_{j\in\mathbb{N}_0}$ and applying Fatou's lemma to the sequence $(|g_{k_j}|^v)_{j\in\mathbb{N}_0}$, the desired result follows.

The rest is to prove $\dot{\widetilde{A}}^0_{p,r} \cap \dot{\widetilde{A}}^{n/p}_{p,q} \hookrightarrow L^{loc}_1$, where it suffices to show that $\dot{\widetilde{A}}^{n/p}_{p,q} \hookrightarrow L^{loc}_1$.

• *The B-case.* Let $f \in \dot{\widetilde{B}}^{n/p}_{p,q}$. We separate the cases with respect to q, so we first assume that $q > 1$. By Remark 2.7(iii), we can split f into $f_1 + f_2$ where $f_1 := \sum_{j\ge 1} Q_j f$ and $f_2 := \sum_{j\le 0}(Q_j f - Q_j f(0))$ since $\nu = 1$. We have

$$|f_2(x)| \le |x| \sum_{j\le 0} \|\nabla Q_j f\|_\infty \le c_1 |x| \sum_{j\le 0} 2^j (2^{jn/p} \|Q_j f\|_p) \le c_2 |x|\, \|f\|_{\dot{B}^{n/p}_{p,\infty}},$$

and $f_2 \in L_1^{loc}$. However, for f_1 we first see the case $p \geq 1$ (recall that $L_p \hookrightarrow L_1^{loc}$), and we get

$$\|f_1\|_p \leq \|f\|_{\dot{B}^{n/p}_{p,\infty}} \sum_{j\geq 1} 2^{-jn/p} \leq c\|f\|_{\dot{B}^{n/p}_{p,\infty}};$$

if $0 < p < 1$, we drive

$$\|f_1\|_1 \leq c_1 \sum_{j\geq 1} 2^{j(n/p-n)} \|Q_j f\|_p \leq c_1 \|f\|_{\dot{B}^{n/p}_{p,\infty}} \sum_{j\geq 1} 2^{-jn} \leq c_2 \|f\|_{\dot{B}^{n/p}_{p,\infty}}.$$

Now we suppose that $0 < q \leq 1$, we have $\nu = 0$, and by both Remark 2.7(i) and the embedding $\dot{B}^{n/p}_{p,q} \hookrightarrow \dot{B}^0_{\infty,1}$, we obtain $f = \sum_{j\in\mathbb{Z}} Q_j f$ and

$$\|f\|_\infty \leq \sum_{j\in\mathbb{Z}} \|Q_j f\|_\infty = \|f\|_{\dot{B}^0_{\infty,1}}, \tag{3.9}$$

which implies that $f \in L_1^{loc}$.

• *The F-case.* If $p > 1$ we proceed as in "the B-case with $q > 1$" since $\dot{F}^{n/p}_{p,q} \hookrightarrow \dot{B}^{n/p}_{p,\infty}$. However, if $0 < p \leq 1$ we use the embeddings $\dot{F}^{n/p}_{p,q} \hookrightarrow \dot{B}^{n/p}_{p,p} \hookrightarrow \dot{B}^0_{\infty,1}$ and the result follows as in (3.9) too. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. Step 1. We first prove the following assertion: There exists a constant $c > 0$ such that the inequality

$$\|f\|_v \leq c\,\|f\|_u^{u/v}\,\|[f]_{\mathcal{P}}\|^{1-u/v}_{\dot{A}^{n/p}_{p,q}} \tag{3.10}$$

holds, for all $u \in]0,\infty[$, all $v \in [u,\infty[$ and all $f \in L_u \cap \dot{\widetilde{A}}^{n/p}_{p,q}$.

By taking into account of the embedding $\dot{A}^{n/p}_{p,q} \hookrightarrow \dot{B}^0_{\infty,1}$ (recall the assumption: $q \leq 1$ in the B-case and $p \leq 1$ in the F-case), it suffices to prove

$$\|f\|_v \leq \|f\|_u^{u/v}\,\|[f]_{\mathcal{P}}\|^{1-u/v}_{\dot{B}^0_{\infty,1}} \qquad (\forall f \in L_u \cap \dot{\widetilde{B}}^0_{\infty,1}). \tag{3.11}$$

Let now $f \in L_u \cap \dot{\widetilde{B}}^0_{\infty,1}$. We have $\nu = 0$ (ν is defined in Subsection 2.2), then $f \in \widetilde{C}_0$. By Lemma 2.8 we have $\sigma(f) := \sum_{j\in\mathbb{Z}} Q_j f = f$ since $\sigma(f) - f \in \widetilde{C}_0 \cap \mathcal{P}_\infty = \{0\}$. Hence, we immediately obtain

$$\begin{aligned}\|f\|_v &= \Big\| |f|^{u/v} \Big|\sum_{j\in\mathbb{Z}} Q_j f\Big|^{1-u/v} \Big\|_v \leq \|f\|_u^{u/v} \Big(\sum_{j\in\mathbb{Z}} \|Q_j f\|_\infty\Big)^{1-u/v} \\ &\leq \|f\|_u^{u/v}\,\|[f]_{\mathcal{P}}\|^{1-u/v}_{\dot{B}^0_{\infty,1}} \qquad \text{(recall that } v < \infty).\end{aligned}$$

Step 2. Let $f \in L_u \cap \dot{\widetilde{A}}^{n/p}_{p,q}$. Let p_1, v be such that $p_1 \geq p$ and $v \geq p$. By (3.11) we have $f \in L_v$. By the embedding $\dot{A}^{n/p}_{p,q} \hookrightarrow \dot{A}^{n/v}_{v,q}$, Proposition 2.4

implies that $f \in A^{n/v}_{v,q}$, and we are able to apply Proposition 3.1 with $v = p$. Then we get

$$\|f\|_v \leq c\, v^{1-1/q}\, 2^{n/v}\, \|f\|_{A^{n/v}_{v,q}},$$

the constant c depends only on n and q; see again Proposition 3.1 for the term $cv^{1-1/q}2^{n/v}$. This gives

$$\|f\|_v \leq c\, v^{1-1/q}\, 2^{n/v} \left(\|f\|_v + \|[f]_{\mathcal{P}}\|_{\dot{A}^{n/v}_{v,q}}\right).$$

Using (3.10) with $\dot{\widetilde{A}}^{n/p_1}_{p_1,q}$ instead of $\dot{\widetilde{A}}^{n/p}_{p,q}$, we get

$$\|f\|_v \leq c_1\, v^{1-1/q}\, 2^{n/v} \left(c_2\, \|f\|_u^{u/v}\, \|[f]_{\mathcal{P}}\|^{1-u/v}_{\dot{A}^{n/p_1}_{p_1,q}} + \|[f]_{\mathcal{P}}\|_{\dot{A}^{n/v}_{v,q}}\right).$$

Now, we change f by $f(\lambda(\cdot))$, with $\lambda > 0$, in the last inequality, then

$$\|f\|_v \leq c_1\, v^{1-1/q}\, 2^{n/v} \left(c_2\, \|f\|_u^{u/v}\, \|[f]_{\mathcal{P}}\|^{1-u/v}_{\dot{A}^{n/p_1}_{p_1,q}} + \lambda^{n/v}\, \|[f]_{\mathcal{P}}\|_{\dot{A}^{n/v}_{v,q}}\right).$$

Finally, we take $\lambda \to 0$ and obtain the desired estimate. The proof of Theorem 1.2 is complete. $\square$

4. CONCLUDING REMARKS

4.1. OPTIMALITY OF THE ESTIMATES

We are interested in the optimality of the estimate (1.2) in both cases B and F. We begin by the following statement:

PROPOSITION 4.1. *Let $1 < p < \infty$ and $1 < q < \infty$ in the B-case. Let $1 \leq q \leq p < \infty$ $(p \neq 1)$ in the F-case. If there exists a function $h : [p, \infty[\to [0, \infty[$ such that*

$$\|f\|_v \leq h(v)\, \|[f]_{\mathcal{P}}\|^{p/v}_{\dot{A}^0_{p,1}}\, \|[f]_{\mathcal{P}}\|^{1-p/v}_{\dot{A}^{n/p}_{p,q}}, \quad \forall f \in \dot{\widetilde{A}}^0_{p,1} \cap \dot{\widetilde{A}}^{n/p}_{p,q}, \tag{4.1}$$

then $h(v) \geq c\, v^{1-1/q}$ for all $v \in [p, \infty[$ and where the positive constant c is independent of v.

The proof is based on the following lemma due to Bourdaud [5, Proposition 2], where we need some notations: Let φ be a C^∞- function on $\mathbb{R}$ such that $\varphi(t) = 1$ for $t \leq e^{-3}$ and $\varphi(t) = 0$ for $t \geq e^{-2}$. For $(\alpha, \beta) \in \mathbb{R}^2$, we define a function f_0 on $\mathbb{R}^n$ by

$$f_0(x) := \big|\log|x|\big|^{\alpha}\big(\log\big|\log|x|\big|\big)^{-\beta}\varphi(|x|), \qquad x \in \mathbb{R}^n.$$

LEMMA 4.2. (i) *If* $1 \leq u, q \leq \infty$, $\alpha := 1 - 1/q$ *and* $\beta > 1/q$, *then* $f_0 \in B^{n/u}_{u,q}$.

(ii) *If* $1 < u < \infty$, $1 \leq q \leq \infty$, $\alpha := 1-1/u$ *and* $\beta > 1/u$, *then* $f_0 \in F^{n/u}_{u,q}$.

Proof of Proposition 4.1. Step 1: preparation. We first show that $f_0 \in \dot{\widetilde{A}}{}^0_{p,1} \cap \dot{\widetilde{A}}{}^{n/p}_{p,q}$, where we will use f_0 as the following:

- in the B-case, $\alpha := 1 - 1/q$ and $\beta q > 1$,
- in the F-case, $\alpha := 1 - 1/p_1$ with $1 < p_1 < \infty$, $q \leq p_1 < p$ and $\beta p_1 > 1$.

Clearly $f_0 \in \widetilde{C}_0$ since f_0 is an integrable function. For the rest we separate B and F cases.

−The B-case: By Lemma 4.2(i) with $u := p$ and Proposition 2.4 (*i.e.*, $f_0 \in B^{n/p}_{p,q}$ implies $[f_0]_{\mathcal{P}} \in \dot{B}^{n/p}_{p,q}$), we get $f_0 \in \dot{\widetilde{B}}{}^{n/p}_{p,q}$. For $f_0 \in \dot{\widetilde{B}}{}^0_{p,1}$, we use Lemma 4.2(i) with $u := 1$ and the embeddings $B^n_{1,q} \hookrightarrow B^{n-n/p}_{1,1} \hookrightarrow \dot{B}^{n-n/p}_{1,1} \hookrightarrow \dot{B}^0_{p,1}$ (recall that by assumption $1 < p < \infty$).

−The F-case: We use Lemma 4.2(ii) with $u := p_1 < p$ and Proposition 2.4 (*i.e.*, $f_0 \in F^{n/p_1}_{p_1,q}$ implies $[f_0]_{\mathcal{P}} \in \dot{F}^{n/p_1}_{p_1,q}$), then we apply the embedding $\dot{F}^{n/p_1}_{p_1,q} \hookrightarrow \dot{F}^{n/p}_{p,q}$. As above for $f_0 \in \dot{\widetilde{F}}{}^0_{p,1}$, we employ the embeddings $F^{n/p_1}_{p_1,q} \hookrightarrow F^{n/p_1-n/p}_{p_1,1} \hookrightarrow \dot{F}^{n/p_1-n/p}_{p_1,1} \hookrightarrow \dot{F}^0_{p,1}$ (here also $1 < p < \infty$ implies that $1 < p_1 < p < \infty$ is possible).

Now, we want to see the L_v-norm of f_0. To that purpose, we introduce the function

$$f_1(x) := \big|\log|x|\big|^{\alpha}\big(\log\big(1 - \log|x|\big)\big)^{-\beta}\varphi(|x|), \qquad x \in \mathbb{R}^n,$$

where α and β are given above. By [9, Theorem 2.7.1, p. 82] or [22] the function f_1 satisfies

$$ck^{\alpha} \leq \|f_1\|_k \leq c'k^{\alpha}, \qquad \forall k \in \mathbb{N}, \tag{4.2}$$

where the positive constants c, c' are independent of k. On the other hand, by the following elementary inequality

$$\log 2 \leq \log\big|\log|x|\big| \leq \log\big(1 - \log|x|\big), \qquad \forall |x| \leq e^{-2}\ (\textit{i.e.}, |x| \in \operatorname{supp}\varphi),$$

and since $\beta > 0$, we have $|f_0(x)| \geq |f_1(x)|$ for all $x \in \mathbb{R}^n$. Then by using (4.2), we obtain

$$\|f_0\|_k \geq c_1 k^{\alpha}, \qquad \forall k \in \mathbb{N}, \tag{4.3}$$

with a constant $c_1 > 0$ independent of k.

Step 2. We will proceed by contradiction. Let us assume that $h(v) < cv^{1-1/q}$ $(\forall v \in [p, \infty[)$, and

$$\lim_{v\to\infty} v^{1/q-1}h(v) = 0; \tag{4.4}$$

indeed to justify (4.4), if $\lim_{v\to\infty} v^{1/q-1}h(v) \neq 0$ say $2c_0$, then there exists $v_0 \in [p,\infty[$ such that $\left|v^{1/q-1}h(v) - 2c_0\right| \leq c_0$ for all $v \geq v_0$, implies that $h(v) \geq c_0 v^{1-1/q}$.

Observe that $\|[f_0]_{\mathcal{P}}\|_{\dot{A}^0_{p,1}} \|[f_0]_{\mathcal{P}}\|_{\dot{A}^{n/p}_{p,q}} \neq 0$ since f_0 is not a polynomial, then it holds that $\lim_{k\to\infty} \|[f_0]_{\mathcal{P}}\|^{p/k}_{\dot{A}^0_{p,1}} \|[f_0]_{\mathcal{P}}\|^{1-p/k}_{\dot{A}^{n/p}_{p,q}} = \|[f_0]_{\mathcal{P}}\|_{\dot{A}^{n/p}_{p,q}}$, *i.e.*, there exists an integer $k_0 \in \mathbb{N}$ such that

$$\|[f_0]_{\mathcal{P}}\|^{p/k}_{\dot{A}^0_{p,1}} \|[f_0]_{\mathcal{P}}\|^{1-p/k}_{\dot{A}^{n/p}_{p,q}} \leq 1 + \|[f_0]_{\mathcal{P}}\|_{\dot{A}^{n/p}_{p,q}}, \quad \forall k \geq k_0.$$

Now, using the inequality (4.1) with both $f = f_0$ and $v = k$, where the integer k is chosen satisfying $k \geq \max(k_0, p)$, and taking into account of (4.3), we find $c_1 k^\alpha \leq h(k)\big(1+\|[f_0]_{\mathcal{P}}\|_{\dot{A}^{n/p}_{p,q}}\big)$, the constant c_1 is defined in (4.3). Consequently,

$$\lim_{k\to\infty} k^{-\alpha}h(k) \geq c_1 \left(1 + \|[f_0]_{\mathcal{P}}\|_{\dot{A}^{n/p}_{p,q}}\right)^{-1} > 0.$$

But this is impossible with (4.4); recall that

- $k^{-\alpha} = k^{1/q-1}$ in the B-case,
- $k^{-\alpha} = k^{1/p_1-1} \leq k^{1/q-1}$ in the F-case.

The proof of Proposition 4.1 is complete. $\square$

Remark 4.3. By Proposition 4.1 we obtain the optimality of the growth rate as $v^{1-1/q}$ in (1.2) with $v \to \infty$, for $1 < p,q < \infty$ in the B-case and $1 \leq q \leq p < \infty$ $(p \neq 1)$ in the F-case. On the other hand, it would be also interesting to extend the validity of this proposition to $p < 1$ or $q < 1$, and to show that the growth of the constant $v^{1-1/q}$ in (1.3) with $v \to \infty$ is optimal at least for the B-case.

4.2. SOME EXTENSIONS

We first generalize Theorem 1.1 in the following sense.

THEOREM 4.4. *Let* $0 < p,q < \infty$. *Let* $m \geq 0$ *be such that one of the following two conditions is satisfied:*

(i) $m \in \mathbb{N}_0$.

(ii) $m > (n/p - n)_+$ *and either* $m < n/p$ *or* $m - n/p \in \mathbb{N}_0$.

We put $r := \min(1, p)$ *in the* B*-case and* $r := 1$ *in the* F*-case. We also put* $w := (m + n/v)^{1/q-1}2^{m+n/v}$. *Then there exists a constant* $c = c(n,p,q) > 0$ *such that the inequality*

$$\|f\|_v \leq c\, w \,\|[f]_{\mathcal{P}}\|^{p/v+mp/n}_{\dot{A}^m_{p,r}} \|[f]_{\mathcal{P}}\|^{1-p/v-mp/n}_{\dot{A}^{m+n/p}_{p,q}} \tag{4.5}$$

holds, for all $v \in [p,\infty[$ *and all* $f \in \dot{\widetilde{A}}^m_{p,r} \cap \dot{\widetilde{A}}^{m+n/p}_{p,q}$.

Proof. We only make a check since the proof is similar to that of Theorem 1.1. First, the inequality (3.1) becomes

$$\|f\|_v \le c\,w\,\|f\|_{A^{m+n/p}_{p,q}}\,,$$

for all $v \in [p,\infty]$ (with $v<\infty$ in the F-case) and all $f \in A^{m+n/p}_{p,q}$. Second by proceeding as in (3.6), then similar to (3.7) we have

$$\|f\|_p = \Big\| \sum_{j\ge 0} 2^{-jm} 2^{jm} \widetilde{Q}_j f \Big\|_p \le \Big\| \Big(\sum_{j\ge 0} |2^{jm} \widetilde{Q}_j f|^r \Big)^{1/r} \Big\|_p \le c \|[f]_{\mathcal{P}}\|_{\dot{A}^m_{p,r}}.$$

Now to make sure that the approximation method will be done as in Step 2 in the proof of Theorem 1.1, it suffices to see that $\dot{\widetilde{A}}{}^m_{p,r} \cap \dot{\widetilde{A}}{}^{m+n/p}_{p,q} \hookrightarrow L_1^{loc}$, wherein we note that the case $m=0$ has been studied before. Then we prove this embedding with respect to the cases (i) and (ii) separately.

Step 1: proof of $\dot{\widetilde{A}}{}^{m+n/p}_{p,q} \hookrightarrow L_1^{loc}$ *under the assumption* (i). We begin with $q \le 1$ in B-case ($p \le 1$ in F-case) (here $\nu := m$). Let $f \in \dot{\widetilde{A}}{}^{m+n/p}_{p,q}$. By Taylor's formula we write, (ii) of Remark 2.7, as

$$f(x) = m \sum_{k\in\mathbb{Z}} \sum_{|\alpha|=m} \frac{x^\alpha}{\alpha!} \int_0^1 (1-t)^{m-1} (Q_k f)^{(\alpha)}(tx)\,\mathrm{d}t, \tag{4.6}$$

which implies

$$\begin{aligned} |f(x)| &\le c_1 |x|^m \sum_{k\in\mathbb{Z}} \sum_{|\alpha|=m} \|(Q_k f)^{(\alpha)}\|_\infty \le c_2 |x|^m \sum_{k\in\mathbb{Z}} 2^{k(m+n/p)} \|Q_k f\|_p \\ &\le c_3 |x|^m \|[f]_{\mathcal{P}}\|_{\dot{B}^{m+n/p}_{p,1}} \qquad (\forall x \in \mathbb{R}^n), \end{aligned}$$

and by the embedding $\dot{A}^{m+n/p}_{p,q} \hookrightarrow \dot{B}^{m+n/p}_{p,1}$ we deduce that $f \in L_1^{loc}$.

Now, we assume that $q>1$ in B-case ($p>1$ in F-case) (here $\nu := m+1$). As in the previous case we write, (iii) of Remark 2.7, as

$$f(x) = \sum_{j\ge 1} Q_j f(x) + (m+1) \sum_{k\le 0} \sum_{|\alpha|=m+1} \frac{x^\alpha}{\alpha!} \int_0^1 (1-t)^m (Q_k f)^{(\alpha)}(tx)\,\mathrm{d}t. \tag{4.7}$$

We put $f_1 := \sum_{j\ge 1} Q_j f$ and $f_2 := f - f_1$. By Hölder's inequality we have

$$|f_1(x)| \le c_1 \sum_{j\ge 1} 2^{-jm} (2^{j(m+n/p)} \|Q_j f\|_p) \ \le\ c_2 \|[f]_{\mathcal{P}}\|_{\dot{A}^{m+n/p}_{p,q}} \qquad (\forall x \in \mathbb{R}^n),$$

and $f_1 \in L_\infty$. We also have

$$|f_2(x)| \le c_1 |x|^{m+1} \sum_{k\le 0} \sum_{|\alpha|=m+1} \|(Q_k f)^{(\alpha)}\|_\infty$$

$$\leq c_2|x|^{m+1}\sum_{k\leq 0}2^{k(m+1+n/p)}\|Q_k f\|_p$$
$$\leq c_3|x|^{m+1}\|[f]_{\mathcal{P}}\|_{\dot{B}^{m+n/p}_{p,\infty}}\sum_{k\leq 0}2^k \ \leq c_4|x|^{m+1}\|[f]_{\mathcal{P}}\|_{\dot{A}^{m+n/p}_{p,q}},$$

and $f_2 \in L_1^{loc}$. All these facts imply $\dot{\widetilde{A}}^{m+n/p}_{p,q} \hookrightarrow L_1^{loc}$.

Step 2: proof of $\dot{\widetilde{A}}^{m}_{p,q} \hookrightarrow L_1^{loc}$ *under the assumption* (ii). Let $f \in \dot{\widetilde{A}}^{m}_{p,q}$.

• The case $m = n/p$ has been studied in Step 2 of the proof of Theorem 1.1.

• The case $m < n/p$: The function f coincides with $\sum_{j\in\mathbb{Z}} Q_j f$, cf. Remark 2.7(i). We write $f = f_1 + f_2$ where $f_1 := \sum_{j\geq 1} Q_j f$ and $f_2 := \sum_{j\leq 0} Q_j f$. We have

$$|f_2(x)| \leq c_1\sum_{j\leq 0}2^{j(n/p-m)}(2^{jm}\|Q_j f\|_p)$$
$$\leq c_2\|[f]_{\mathcal{P}}\|_{\dot{B}^{m}_{p,\infty}}\sum_{j\leq 0}2^{j(n/p-m)} \leq c_3\|[f]_{\mathcal{P}}\|_{\dot{A}^{m}_{p,q}},$$

and $f_2 \in L_\infty$ (recall that $\dot{A}^m_{p,q} \hookrightarrow \dot{B}^m_{p,\infty}$ and $L_\infty \hookrightarrow L_1^{loc}$). For f_1 we first see the case $p \geq 1$ ($L_p \hookrightarrow L_1^{loc}$), and we get

$$\|f_1\|_p \leq c_1\sum_{j\geq 1}2^{-jm}(2^{jm}\|Q_j f\|_p) \leq c_2\|[f]_{\mathcal{P}}\|_{\dot{B}^{m}_{p,\infty}};$$

for $0 < p < 1$, we have

$$\|f_1\|_1 \leq c_1\sum_{j\geq 1}2^{j(n/p-n-m)}(2^{jm}\|Q_j f\|_p)$$
$$\leq c_1\|[f]_{\mathcal{P}}\|_{\dot{B}^{m}_{p,\infty}}\sum_{j\geq 1}2^{j(n/p-n-m)} \ \leq c_2\|[f]_{\mathcal{P}}\|_{\dot{A}^{m}_{p,q}},$$

which implies $f \in L_1^{loc}$.

• The case $m - n/p =: m_1 \in \mathbb{N}$. Here we treat the cases $p \leq 1$ and $p > 1$ separately, as in (4.6) and (4.7) (by replacing m by m_1), respectively. We omit the details and the proof of Theorem 4.4 is finished. $\square$

Now, we turn to give the estimate (1.1) in the realized spaces with a more general case.

THEOREM 4.5. *Let* $0 < p < v < \infty$ *and* $0 < q < \infty$. *Let* $\alpha > 0$ *be such that the number* $\beta := \alpha(v/p - 1)$ *satisfies*

$$\beta > (n/p-n)_+ \quad \textit{and} \quad \textit{either } \beta < n/p \quad \textit{or } \beta - n/p \in \mathbb{N}_0.$$

Then there exists a constant $c > 0$ such that the inequality

$$\|f\|_v \le c\|[f]_{\mathcal{P}}\|^{p/v}_{\dot{F}^{\beta}_{p,q}} \|[f]_{\mathcal{P}}\|^{1-p/v}_{\dot{B}^{-\alpha}_{\infty,\infty}} \tag{4.8}$$

holds, for all $f \in \dot{\widetilde{B}}{}^{-\alpha}_{\infty,\infty} \cap \dot{\widetilde{F}}{}^{\beta}_{p,q}$.

We first prove the following assertion.

Proposition 4.6. *Let $0 < p < v < \infty$ and $\alpha > 0$. We put $\beta := \alpha(v/p - 1)$. Then there exists a constant $c > 0$ such that the inequality*

$$\|f\|_v \le c\|[f]_{\mathcal{P}}\|^{p/v}_{\dot{F}^{\beta}_{p,\infty}} \|[f]_{\mathcal{P}}\|^{1-p/v}_{\dot{B}^{-\alpha}_{\infty,\infty}} \tag{4.9}$$

holds, for all $f \in F^{\beta}_{p,\infty}$ such that $[f]_{\mathcal{P}} \in \dot{B}^{-\alpha}_{\infty,\infty}$.

Proof. If $\|[f]_{\mathcal{P}}\|_{\dot{B}^{-\alpha}_{\infty,\infty}} = 0$ the inequality (4.9) is trivial since the assumptions imply that $f = 0$. Thus, we assume that $\|[f]_{\mathcal{P}}\|_{\dot{B}^{-\alpha}_{\infty,\infty}} = 1$, and write

$$\|f\|_v^v = v\int_0^\infty t^{v-1}|\{x : |f(x)| > t\}|\mathrm{d}t = v\int_0^\infty t^{v-1}\Big|\Big\{x : \Big|\sum_{j\ge 0} \widetilde{Q}_j f(x)\Big| > t\Big\}\Big|\mathrm{d}t,$$

where $|\{x : \ldots\}|$ is the Lebesgue measure of the set $\{x : \ldots\}$. Then the estimate of the last-hand side is similar to that of the proof given in [20, pp. 129–130], we obtain

$$\|f\|_v \le c\|f\|^{p/v}_{F^{\beta}_{p,\infty}}.$$

Now, we suppose that $\|[f]_{\mathcal{P}}\|_{\dot{B}^{-\alpha}_{\infty,\infty}} \ne 0$ and change f by $(\|[f]_{\mathcal{P}}\|^{-1}_{\dot{B}^{-\alpha}_{\infty,\infty}})f$ in the last inequality, then we deduce that the following estimate

$$\|f\|_v \le c\|f\|^{p/v}_{F^{\beta}_{p,\infty}} \|[f]_{\mathcal{P}}\|^{1-p/v}_{\dot{B}^{-\alpha}_{\infty,\infty}} \tag{4.10}$$

holds, for all $f \in F^{\beta}_{p,\infty}$ such that $[f]_{\mathcal{P}} \in \dot{B}^{-\alpha}_{\infty,\infty}$. Again in (4.10), we replace f by $f(\lambda(\cdot))$, with $\lambda > 0$, and use the fact that $(\beta - n/p)p/v - \alpha(1-p/v) = -n/v$ and $\|f\|_{F^{\beta}_{p,\infty}} \sim \|f\|_p + \|[f]_{\mathcal{P}}\|_{\dot{F}^{\beta}_{p,\infty}}$, then we obtain

$$\|f\|_v \le c\Big(\lambda^{-\beta}\|f\|_p + \|[f]_{\mathcal{P}}\|_{\dot{F}^{\beta}_{p,\infty}}\Big)^{p/v} \|[f]_{\mathcal{P}}\|^{1-p/v}_{\dot{B}^{-\alpha}_{\infty,\infty}},$$

and by taking $\lambda \to \infty$, the inequality (4.9) follows. $\square$

Proof of Theorem 4.5. Step 1. Let $f \in \dot{\widetilde{B}}{}^{-\alpha}_{\infty,\infty} \cap \dot{\widetilde{F}}{}^{\beta}_{p,q}$. We set $g_k := \sum_{j=-k}^{k} Q_j f$ for all $k \in \mathbb{N}_0$. Then the sequence $(g_k)_{k\in\mathbb{N}_0}$ has the following properties:

(I) $\|[g_k]_{\mathcal{P}}\|_{\dot{F}^{\beta}_{p,q}} \le c\|[f]_{\mathcal{P}}\|_{\dot{F}^{\beta}_{p,q}}$ and $\|[g_k]_{\mathcal{P}}\|_{\dot{B}^{-\alpha}_{\infty,\infty}} \le c\|[f]_{\mathcal{P}}\|_{\dot{B}^{-\alpha}_{\infty,\infty}}$ for all $k \in \mathbb{N}_0$, see Proposition 2.3.

(II) $g_k \in L_p$ for all $k \in \mathbb{N}_0$; indeed, we introduce the parameter $r := \min(1, p)$, then it holds

$$\|g_k\|_p \leq \Big\| \Big(\sum_{-k \leq j \leq k} |Q_j f|^r \Big)^{1/r} \Big\|_p \leq \|[f]_{\mathcal{P}}\|_{\dot{F}^\beta_{p,q}} \Big(\sum_{-k \leq j \leq k} 2^{-jr\beta} \Big)^{1/r} \leq c(k) \|[f]_{\mathcal{P}}\|_{\dot{F}^\beta_{p,q}}.$$

(III) g_k tends to f in $\dot{\widetilde{F}}^\beta_{p,q}$; indeed, by Proposition 2.3 we have

$$\|[g_k]_{\mathcal{P}} - [f]_{\mathcal{P}}\|_{\dot{F}^\beta_{p,q}} \leq \Big\| \Big(\sum_{|j|>k} 2^{jq\beta} |Q_j f|^q \Big)^{1/q} \Big\|_p \qquad (\forall k \in \mathbb{N}_0),$$

and because $q < \infty$ the last term tends to 0 with $k \to \infty$.

By (I) and (II) we can apply Proposition 4.6 to $(g_k)_{k \in \mathbb{N}_0}$ and obtain

$$\|g_k\|_v \leq c \|[f]_{\mathcal{P}}\|^{p/v}_{\dot{F}^\beta_{p,q}} \|[f]_{\mathcal{P}}\|^{1-p/v}_{\dot{B}^{-\alpha}_{\infty,\infty}} \qquad (\forall k \in \mathbb{N}_0). \tag{4.11}$$

On the other hand, if we assume for a moment that the following embedding holds

$$\dot{\widetilde{F}}^\beta_{p,q} \hookrightarrow L_1^{loc}, \tag{4.12}$$

then by (III) and Lemma 3.3 we may extract a subsequence $(g_{k_j})_{j \in \mathbb{N}_0}$ such that $\lim_{j \to \infty} g_{k_j} = f$ a.e. Now the inequality (4.11) with $(g_{k_j})_{j \in \mathbb{N}_0}$ and an application of Fatou's lemma to the sequence $(|g_{k_j}|^v)_{j \in \mathbb{N}_0}$ yield the desired result.

Step 2: proof of (4.12). It is similar to that of the proof given in Step 2 of Theorem 4.4 in the F-case, wherein we just change m by β. The proof of Theorem 4.5 is therefore complete. □

Using the embedding properties of homogeneous spaces *i.e.*, $\dot{F}^\beta_{p,q} \hookrightarrow \dot{F}^\beta_{p,\infty}$ and the fact that if $q \leq p$ then $\dot{B}^\beta_{p,q} \hookrightarrow \dot{B}^\beta_{p,p} = \dot{F}^\beta_{p,p}$, we drive the following statement.

Corollary 4.7. *Let p, q, v, α and β be given as in Theorem 4.5. Let in addition $q \leq p$ in the B-case. Then there exists a constant $c > 0$ such that the inequality*

$$\|f\|_v \leq c \|[f]_{\mathcal{P}}\|^{p/v}_{\dot{A}^\beta_{p,q}} \|[f]_{\mathcal{P}}\|^{1-p/v}_{\dot{B}^{-\alpha}_{\infty,\infty}} \tag{4.13}$$

holds, for all $f \in \dot{\widetilde{B}}^{-\alpha}_{\infty,\infty} \cap \dot{\widetilde{A}}^\beta_{p,q}$.

Remark 4.8. Corollary 4.7 covers the result given in [1, Theorem 2.42, p. 82].

Remark 4.9. As in Corollary 1.3, from the inequalities (4.5), (4.8) and (4.13) we have the intersection of certain realized spaces are embedded in the spaces L_v.

Acknowledgements. We would like to thank the referee for his valuable comments and suggestions.

REFERENCES

[1] H. Bahouri, J.Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations.* Grundlehren Math. Wiss. **343**, Springer, Berlin, 2011.

[2] G. Bergh and J. Löfström, *Interpolation Theory, An Introduction.* Grundlehren Math. Wiss. **223**, Springer, 1976.

[3] G. Bourdaud, *Réalisations des espaces de Besov homogènes.* Ark. Mat. **26** (1988), 41–54.

[4] G. Bourdaud, *Analyse Fonctionnelle dans l'Espace Euclidien.* 2ième édition, Pub. Math. Univ. Paris 7 **23**, 1995.

[5] G. Bourdaud, *A sharpness result for powers of Besov functions.* J. Funct. Spaces Appl. **2** (2004), 267–277.

[6] G. Bourdaud, *Ce qu'il faut savoir sur les espaces de Besov.* Preprint, Paris, 2009.

[7] G. Bourdaud, *Realizations of homogeneous Besov and Lizorkin-Triebel spaces.* Math. Nachr. **286** (2013), 476–491.

[8] G. Bourdaud, M. Moussai and W. Sickel, *Composition operators in Lizorkin-Triebel spaces.* J. Funct. Anal. **259** (2010), 1098–1128.

[9] D.E. Edmunds and H. Triebel, *Function Spaces, Entropy Numbers, Differential Operators.* Cambridge Univ. Press, Cambridge, 1996.

[10] M. Frazier and B. Jawerth, *A discrete transform and decomposition of distribution spaces.* J. Funct. Anal. **93** (1990), 34–170.

[11] B. Jawerth, *Some observations on Besov and Lizorkin-Triebel spaces.* Math. Scand. **40** (1977), 94–104.

[12] V.I. Kolyada and F.J. Pérez Lázaro, *On Gagliardo-Nirenberg type inequalities.* J. Fourier Anal. Appl. **20** (2014), 577–607.

[13] H. Kozono and H. Wadade, *Remarks on Gagliardo-Nirenberg type inequality with critical Sobolev space and BMO.* Math. Z. **259** (2008), 935–950.

[14] M. Moussai, *Realizations of homogeneous Besov and Triebel-Lizorkin spaces and an application to pointwise multipliers.* Anal. Appl. (Singap.) **13** (2015), 149–183.

[15] R.J. Nessel and G. Wilmes, *Nikolskii-type inequalities for trigonometric polynomials and entire functions of exponential type.* J. Austral. Math. Soc. **25** (1978), 7–18.

[16] T. Ozawa, *On critical cases of Sobolev's inequalities.* J. Funct. Anal. **127** (1995), 259–269.

[17] J. Peetre, *New Thoughts on Besov Spaces.* Duke Univ. Math. Series I, Durham, N.C., 1976.

[18] T. Runst and W. Sickel, *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations.* De Gruyter Ser. Nonlinear Anal. Appl. **3**, de Gruyter, Berlin, 1996.

[19] F. Treves, *Topological Vector Spaces, Distributions and Kernels.* Academic Press, New York-London, 1967.

[20] H. Triebel, *Theory of Function Spaces.* Monog. Math. **78**. Birkhäuser, Basel, 1983.

[21] H. Triebel, *Theory of Function Spaces II.* Monog. Math. **84**. Birkhäuser, Basel, 1992.

[22] H. Triebel, *Approximation numbers and entropy numbers of embeddings of fractional Besov-Sobolev spaces in Orlicz spaces.* Proc. London Math. Soc. **66** (1993), 589–618.

[23] H. Triebel, *Local Function Spaces, Heat and Navier-Stokes Equations.* EMS Tracts Math. **20**, European Math. Soc. Publishing House, Zürich, 2013.

[24] H. Wadade, *Remarks on the Gagliardo-Nirenberg type inequality in the Besov and the Triebel-Lizorkin spaces in the limiting case.* J. Fourier Anal. Appl. **15** (2009), 857–870.

[25] H. Wadade, *Quantitative estimates of embedding constants for Gagliardo-Nirenberg inequalities on critical Sobolev-Besov-Lorentz spaces.* J. Fourier Anal. Appl. **19** (2013), 1029–1059.

Received 6 March 2017

Mohamed Boudiaf University of M'Sila,
Department of Mathematics,
Labo. Funct. Anal. Geom. Spaces,
28000 M'Sila, Algeria
benalliam@yahoo.fr

Mohamed Boudiaf University of M'Sila,
Department of Mathematics,
Labo. Funct. Anal. Geom. Spaces,
28000 M'Sila, Algeria
mmoussai@yahoo.fr