ON INVERSE PROBLEMS FOR SETS IN $\mathbb{Z}_3^n$

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The sets in $G = \mathbb{Z}_3^3$ with maximal cardinality and zero sum free are characterized. Also, the greatest cardinality and structure of subsets of G without zero sum and length ≤ 3 sets, are given. As a result, we find exact values of Olson's constant $O^k(G)$ for $k \in [3, 7]$.

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1. INTRODUCTION

Let G be a finite additive abelian group. $\mathcal{F}(G)$ denotes the free abelian monoid in G whose operation is concatenation and the unit is the empty sequence. Every $S \in \mathcal{F}(G)$ is a *sequence* and it has the form $S = g_1^{v_1} \cdots g_k^{v_k} = \prod_{g \in G} g^{v_g(S)}$ where $v_g(S) \geq 0$ is the multiplicity of g: the number of times of g in S. In case of $v_g(S) = 1$, S is a *set*: a sequence with no repeated elements. A sequence T which consists of members of S is a *subsequence* of S and it is said to be *proper* if $T \neq S$. Also, $T|S$ is the subsequence of S whose members are not in T. Given a sequence S

- $\sigma(S) \in G$ is the sum of all of the members of S.
- $\sum S = \{\sigma(T) : T \text{ is a nonempty subsequence of } S\}$.
- $|S| = \sum_{g \in G} v_g(S)$ is the *length* of S, the many of its members.

The sequence S is:

- *Zero sum* if $\sigma(S)$ is zero in G.
- *Zero sum free* if $0 \notin \sum S$.
- $k^{\leq}$*-zero sum free*, with k a positive integer, if it does not contain zero sum subsequences with length in $[1, k]$.

In 1961, Erdös, Ginzburg and Ziv [4] proved that in a cyclic group with cardinality n, for every sequence with $2n-1$ elements, there are n of them whose sum is zero. This result, among others, gave rise to the well known *zero sum problems*. Since their appearance, zero sum problems have been studied by a lot of researchers, leading to many problems and conjectures, some of which remain open. The study of zero sum problems is within combinatoric number theory and they might be classified in two types. *Direct problems* search for conditions granting the existence of elements in a sequence (with given characteristics), whose sum is zero. *Inverse problems* consist in finding, for a given integer k, the structure of a $k^{\leq}$-zero sum free sequence in $\mathcal{F}(G)$ with the largest length. Several constants have been defined while studying these two types of problems. One of them is known as Davenport's constant, introduced in 1966, denoted $D(G)$ and defined as the least positive integer d such that every sequence with d elements contains a nonempty zero sum subsequence. The corresponding constant for sets, Olson's constant $O(G)$, is defined in an analogous way. Many results on zero sum problems are well known (see [1, 3, 5–8, 11, 14, 15]). However, very few results on the structure of zero sum free sets with cardinality $O(G) - 1$ are known (see [2, 9, 10, 12–14]).

In this work, the inverse problem in $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ is studied and a characterization of the structure of every set with greatest cardinality and without zero sum subsets is given. The same is done for sets without zero sum subsets with cardinality ≤ 3.

2. ZERO SUM FREE SETS IN $\mathbb{Z}_3^n$

For a given prime p, the elementary p-group $G = \mathbb{Z}_p^n$ is a vector space on the finite field $\mathbb{Z}_p$. Thus, we will use notions like *basis* and *independent sets.* In [15], Julio Subocz proved the following:

PROPOSITION 1. *If $n \geq 3$ then $O(\mathbb{Z}_3^n) = 2n + 1$.*

The following important fact, for sequences, was given by Gao and Geroldinger in 1999 [7].

THEOREM 1. *Let G be a finite abelian group. If S is a zero sum free sequence in G with $|S| = D(G) - 1$, then $\sum S \cup \{0\} = G$.*

As a consequence, we have the following:

COROLLARY 1. *Let S be a set in $G = \mathbb{Z}_3^n$. If S is zero sum free and $|S| = 2n$, then S contains a basis of G.*

Proof. Let S be a set in $G = \mathbb{Z}_3^n$ as in the hypothesis. Assume that S does not contain a basis of G. If $S = a_1 \cdots a_{n-k} b_1 \cdots b_{n+k}$ with $1 < k < n$ and $B = \{a_1, \ldots, a_{n-k}\}$ is the largest independent subset of S, then $b_i = \sum_{j=1}^{n-k} \beta_{i_j} a_j$ for every $i \in [1, n+k]$. Let $\{g_1, \ldots, g_k\} \subset G - S$ such that $B \cup \{g_1, \ldots, g_k\}$ is a basis of G. By Theorem 1, $\sum S = G - \{0\}$. Thus, $g_j \in \sum S$ for every $j \in [1, k]$ and hence, there exist $I_{j_1} \subset [1, n-k]$ and $I_{j_2} \subset [1, n+k]$ such that $g_j =$

$\sum_{i\in I_{j_1}} a_i + \sum_{i\in I_{j_2}} b_i = \sum_{t=1}^{n-k} \beta_t a_t$. But this contradicts that $B \cup \{g_1, \ldots, g_k\}$ is a basis. $\square$

From these results, the inverse problem for some constants related to $O(\mathbb{Z}_3^n)$ with $n \geq 3$, is settled. That is to say, the aim is to find the structure of sets in $F(\mathbb{Z}_3^n)$ which don't contain certain zero sum sets.

From now on, a zero sum free set $S \in F(\mathbb{Z}_3^n)$ with $|S| = 2n$, will be denoted by $S = a_1 \cdots a_n b_1 \cdots b_n$, where $\{a_1, \ldots, a_n\}$ is a basis and $b_i = \beta_{i_1} a_1 + \cdots + \beta_{i_n} a_n$ for every $i \in [1, n]$. A very useful tool for what follows is the $\mathcal{B}_1$ **property** of the b_i's:

> Given $S_B = b_1 \cdots b_n$, for every $b \in \sum S_B$, if $b = \beta_1 a_1 + \cdots + \beta_n a_n$, there exists (at least one) $j \in [1, n]$ such that $\beta_j = 1$. In particular, every b_i has this property.

Lemma 1. *Let $S = a_1 \cdots a_n b_1 \cdots b_n$ be a zero sum free set in $\mathbb{Z}_3^n$, where $\{a_1, \ldots, a_n\}$ is a basis. If $\sum_{s\in I} b_s = s_1 a_1 + \cdots s_n a_n$, with $I \subseteq [1, n]$, then $1 \in \{s_i : i \in [1, n]\}$.*

Proof. Let $S = a_1 \cdots a_n b_1 \cdots b_n$ be as in the hypothesis. For every nonempty $I \subseteq [1, n]$, if $\sum_{i\in I} b_i = s_1 a_1 + \cdots + s_n a_n$ and $1 \notin \{s_1, \cdots, s_n\}$, then $s_i \in \{0, 2\}$ for every $i \in [1, n]$. Since S is zero sum free, some s_i is not zero. Consider $I_2 = \{i \in [1, n] : s_i = 2\}$. Then $\sum_{i\in I_2} a_i + \sum_{i\in I} b_i = 0$. But this contradicts that S is zero sum free. $\square$

Proposition 2. *Consider a zero sum free set $S \in F(\mathbb{Z}_3^n)$, with $n \geq 3$, $|S| = 2n$ and $\{a_1, \ldots, a_n\} \subset S$ a basis. If $\sigma(S) = \sigma_1 a_1 + \cdots \sigma_n a_n$ then $\{0, 2\} \subset \{\sigma_i\}_{i\in[1,n]}$.*

Proof. Consider S as in the hypothesis. By Theorem 1 it is possible to rearrange S in such a way that $S = a_1 \cdots a_n b_1 \cdots b_n$ where $\{a_1, \cdots, a_n\}$ is a basis and $\sum_{i=1}^n b_i = 2\sum_{i\in I} b_i$ for some nonempty $I \subset [1, n]$. Then $\sigma(S) = \sigma_1 a_1 + \cdots \sigma_n a_n = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i = \sum_{i=1}^n a_i + 2\sum_{i\in I} b_i$. By the $\mathcal{B}_1$ property of $\sum_{i=1}^n b_i$ and $\sum_{i=\in I} b_i$, there exist $j, k \in [1, n]$ $(j \neq k)$ such that $\sigma_j = 2$ and $\sigma_k = 0$. $\square$

3. ZERO SUM FREE SETS IN $\mathbb{Z}_3^3$ WITH GREATEST LENGTH

By Theorem 2 above, we solve a well known inverse problem in $\mathbb{Z}_3^3$: what is the structure of a zero sum free set in $\mathbb{Z}_3^3$ with maximal cardinality? The following lemma will be useful for the characterization given in Theorem 2.

Lemma 2. *In $G = \mathbb{Z}_3^3$, every zero sum free set S with $|S| = 6$, has the form*

$$S = a_1a_2a_3b_1b_2b_3$$

where $\{a_1, a_2, a_3\}$ is a basis of G and $b_k = b_i + b_j$ for $\{i, j, k\} = \{1, 2, 3\}$. Furthermore, $b_k + b_h \neq a_l$, for $h \in \{i, j\}$ and $l \in [1, 3]$.

Proof. Let $S = a_1a_2a_3b_1b_2b_3$ be as in the hypothesis. By Corollary 1 and without a loss of generality, assume that $\{a_1, a_2, a_3\}$ is a basis of G. We proceed according to the dependence of the b_i's. If they are dependent, the result holds since if $\{i, j, k\} = \{1, 2, 3\}$ then

$$b_i = 2 \cdot b_j + b_k \Leftrightarrow b_k = b_i + b_j$$

Thus, assume that the b_i's are independent. We have two cases: 1) at least one $b_i = x_i \cdot a_1 + y_i \cdot a_2 + z_i \cdot a_3$ has a coefficient equal to zero. 2) the opposite of 1). In case 1), for fixed $i, j \in \{1, 2, 3\}$ with $i \neq j$, let $b_1 = p \cdot a_i + q \cdot a_j$. Since S is zero sum free, by the B_1 property, $b_1 \in \{a_i + a_j, 2 \cdot a_i + a_j\}$. Thus, we may write, for $b_1 = a_i + a_j$: $S = a_ia_jb_2b_1b_3a_k$ with $a_k = b_1 + b_3$ or $S = a_ia_jb_2b_1a_kb_3$, with $b_3 = b_1 + a_k$ and $\{a_i, a_j, b_2\}$ a basis. And, for $b_1 = 2 \cdot a_i + a_j$, $S = a_ib_1a_ka_jb_2b_3$, with $b_3 = a_j + b_2$ and $\{a_i, b_1, a_k\}$ a basis.

On the other hand, by independence, if for every $b_s = x_s \cdot a_1 + y_s \cdot a_2 + z_s \cdot a_3$ with $s \in [1, 3]$, $0 \notin \{x_s, y_s, z_s\}$ holds; then by B_1 property we may discard: the cases in which two of them have the form $2 \cdot a_i + a_j + a_k$ (since their sum has no B_1 property); also the case in which all of them have the form $2 \cdot a_i + 2 \cdot a_j + a_k$ is discarded. Analogously, if one of them is $a_1 + a_2 + a_3$. This is why the only valid case is $b_1 = 2 \cdot a_i + 2 \cdot a_j + a_k$, $b_2 = 2 \cdot a_i + a_j + 2 \cdot a_k$ and $b_3 = 2 \cdot a_i + a_j + a_k$ with $\{i, j, k\} = \{1, 2, 3\}$, $b_1 + b_2 = a_i$ and $\{b_3, a_j, a_k\}$ a basis. Therefore

$$S = b_3a_ja_kb_1b_2a_i.$$

For $b_k = b_i + b_j$ with $\{i, j, k\} = \{1, 2, 3\}$ assume that $b_k + b_h = a_l$ for some $h \in \{i, j\}$ and $l \in [1, 3]$; then $a_l = (b_i + b_j) + b_h = 2 \cdot b_i + b_j$ (o $b_i + 2 \cdot b_j$). Hence

$$a_l + b_k + b_j = (2 \cdot b_i + b_j) + (b_i + b_j) + b_j = 0.$$

But this is a contradiction. □

Theorem 2. *Let $S \in \mathcal{F}(\mathbb{Z}_3^3)$ be a set with $|S| = 6$, then S is zero sum free if and only if it has the following structure*

$$S = a_1a_2a_3(a_1 + a_2)(a_1 + a_3)(2a_1 + a_2 + a_3)$$

where $\{a_i\}_{i=1}^3$ is a basis.

Proof. ($\Leftarrow$) Let $S = a_1a_2a_3(a_1 + a_2)(a_1 + a_3)(2a_1 + a_2 + a_3) \in \mathcal{F}(\mathbb{Z}_3^3)$ where $\{a_i\}_{i=1}^3$ is a basis of $\mathbb{Z}_3^3$. First, we prove that S is zero sum free. For this, we shall see that the only solution of

$$\alpha_1 \cdot a_1 + \alpha_2 \cdot a_2 + \alpha_3 \cdot a_3 + \alpha_4 \cdot (a_1 + a_2) + \alpha_5 \cdot (a_1 + a_3) + \alpha_6 \cdot (2a_1 + a_2 + a_3) = 0$$

with $\alpha_i \in \{0, 1\}$ for $i \in [1, 6]$, is the trivial. We have the following homogeneous system

$$\begin{cases} \alpha_1 + \alpha_4 + \alpha_5 + 2\alpha_6 = 0 \\ \alpha_2 + \alpha_4 + \alpha_6 = 0 \\ \alpha_3 + \alpha_5 + \alpha_6 = 0. \end{cases}$$

If there is a non trivial solution, from the second and third equations we conclude that $\alpha_2 = \cdots = \alpha_6 = 1$ and from the first equation, we have $\alpha_1 + 1 = 0$, which is not possible since $\alpha_1 \in \{0, 1\}$.

($\Rightarrow$) Let $S \in \mathcal{F}(\mathbb{Z}_3^3)$ be a zero sum free with $|S| = 6$. By Corollary 1, S contains a basis $B = \{a_i\}_{i=1}^3$. For $S - B = \{b_1, b_2, b_3\}$ each element has the form $b_i = \alpha_i \cdot a_1 + \beta_i \cdot a_2 + \gamma_i \cdot a_3$. By Lemma 5, without a loss of generality, we may assume that $b_3 = b_1 + b_2$. Then for $\sigma(S) = \alpha \cdot a_1 + \beta \cdot a_2 + \gamma \cdot a_3$, we have

$$b_3 = 2\sigma(S) + \sum_{i=1}^{3} a_i \Rightarrow \begin{array}{l} \alpha_3 = 2\alpha + 1 \\ \beta_3 = 2\beta + 1 \\ \gamma_3 = 2\gamma + 1 \end{array}$$

By Proposition 2, $\{0, 2\} \subset \{\alpha, \beta, \gamma\}$ and by symmetry of the system, without a loss of generality, assume that $\alpha = 2$ and $\beta = 0$. Then, $\alpha_3 = 2$ and $\beta_3 = 1$. Thus, for $\gamma_3 = 2\gamma + 1$ the possible solutions for (γ_3, γ) are $\{(0, 1), (1, 0), (2, 2)\}$.

For every (γ_3, γ) the corresponding values of b_3 are $\{2a_1 + a_2, 2a_1 + a_2 + a_3, 2a_1 + a_2 + 2a_3\}$.

On the other hand, since $b_3 = b_1 + b_2$, then $b_1 + b_2 + b_3 = 2b_3$. Hence, for every value of b_3 there is an associated system, which is represented as follows:

$$\begin{array}{l} \alpha_1 + \alpha_2 + 2 = 1 \\ \beta_1 + \beta_2 + 1 = 2 \\ \gamma_1 + \gamma_2 + 0 = 0 \end{array}, \quad \begin{array}{l} \alpha_1 + \alpha_2 + 2 = 1 \\ \beta_1 + \beta_2 + 1 = 2 \\ \gamma_1 + \gamma_2 + 1 = 2 \end{array}, \quad \begin{array}{l} \alpha_1 + \alpha_2 + 2 = 1 \\ \beta_1 + \beta_2 + 1 = 2 \\ \gamma_1 + \gamma_2 + 2 = 1 \end{array}$$

Before solving it, we note that the three have in common $\alpha_1 + \alpha_2 + 2 = 1 \Rightarrow \alpha_1 + \alpha_2 = 2$ and then $(\alpha_1, \alpha_2) \in \{(0, 2), (2, 0), (1, 1)\}$. Since $b_3 = b_1 + b_2 = b_2 + b_1$, then $(0, 2)$ and $(2, 0)$ are equivalent. Therefore, to solve each system we study only the cases $(\alpha_1, \alpha_2) = (0, 2)$ and $(\alpha_1, \alpha_2) = (1, 1)$. Thus:

- For $(\alpha_1, \alpha_2) = (0, 2)$ we have $\begin{array}{l} 0 + 2 + 2 = 1 \\ \beta_1 + \beta_2 + 1 = 2 \\ \gamma_1 + \gamma_2 + \gamma_3 = 2\gamma_3. \end{array}$

Since S is zero sum free, $0 \notin \{\beta_1, \gamma_1\}$. And $\beta_1 + \beta_2 = 1$, then $(\beta_1, \beta_2) = (1, 0)$ or $(2, 2)$. In the first case $b_2 = 2a_1 + \gamma_2 a_3$ and the B_1 property implies that $\gamma_2 = 1$. In addition, $b_1 = a_2 + \gamma_1 a_3$ and $b_1 + b_3 = 2a_1 + 2a_2 + (\gamma_1 + \gamma_3)a_3$. By B_1 property it must be $\gamma_1 + \gamma_3 = 1$ and since $\gamma_1 + 1 = \gamma_3$ then $2\gamma_1 + 1 = 1$ and therefore $\gamma_1 = 0$, which is a contradiction. For the case $(\beta_1, \beta_2) = (2, 2)$ it must be $\gamma_1 = \gamma_2 = 1$, then $\gamma_3 = 2$ and $b_1 + b_3 = 2a_1$ does not have the B_1 property.

- For $(\alpha_1, \alpha_2) = (1, 1)$ we have $\begin{array}{l} 1+1+2=1 \\ \beta_1+\beta_2+1=2 \\ \gamma_1+\gamma_2+\gamma_3=2\gamma_3. \end{array}$

Since $\beta_1 + \beta_2 = 1$ then $(\beta_1, \beta_2) = (0, 1)$ or $(2, 2)$. In the first case, $b_1 = a_1 + \gamma_1 a_3$ and $b_2 = a_1 + a_2 + \gamma_2 a_3$. Since S is a set then $\gamma_1 \neq 0$. On the other hand, $b_2 + b_3 = 2a_2 + (\gamma_2 + \gamma_3)a_3$ and $b_1 + b_3 = a_2 + (\gamma_1 + \gamma_3)a_3$. By the B_1 property $\gamma_2 + \gamma_3 = 1$ and since S is zero sum free, $\gamma_1 + \gamma_3 \neq 0$. Since $\gamma_1 + \gamma_2 = \gamma_3$ it follows that $\gamma_1 \neq \gamma_2$ and $\gamma_1 + 2\gamma_2 = 1$. Then we have the cases $(\gamma_1, \gamma_2) \in \{(1, 0), (2, 1)\}$.

– For $(\gamma_1, \gamma_2) = (1, 0)$, $\gamma_3 = 1$. Which gives us the set

$$S = a_1 a_2 a_3 (a_1 + a_2)(a_1 + a_3)(2a_1 + a_2 + a_3)$$

– If $(\gamma_1, \gamma_2) = (2, 1)$, then $\gamma_3 = 0$. In this case the set is

$$S = a_1 a_2 a_3 (a_1 + 2a_3)(a_1 + a_2 + a_3)(2a_1 + a_2)$$

For those, consider

$$\begin{aligned} S &= \underbrace{a_1}_{\overline{b_2}}\ \underbrace{a_2}_{\overline{b_3}}\ \underbrace{a_3}_{\overline{a_3}}\ (\underbrace{a_1 + a_2 + a_3}_{\overline{a_2}})(\underbrace{a_1 + 2a_3}_{\overline{a_1}})(\underbrace{2a_1 + a_2}_{\overline{b_1}}) \\ &= (\underbrace{a_1 + 2a_3}_{\overline{a_1}})(\underbrace{a_1 + a_2 + a_3}_{\overline{a_2}})\ \underbrace{a_3}_{\overline{a_3}}\ (\underbrace{2a_1 + a_2}_{\overline{b_1}})\ \underbrace{a_1}_{\overline{b_2}}\ \underbrace{a_2}_{\overline{b_3}} \\ S &= \overline{a_1}\ \overline{a_2}\ \overline{a_3}(\overline{a_1} + \overline{a_2})(\overline{a_1} + \overline{a_3})(2\overline{a_1} + \overline{a_2} + \overline{a_3}) \end{aligned}$$

which has the expected structure.

It remains the case $\beta_1 = \beta_2 = 2$, for which $b_1 + b_3 = (\gamma_1 + \gamma_3)a_3$. But this contradicts that S is zero sum free. □

4. $3^{\leq}$-ZERO SUM FREE SETS IN $\mathbb{Z}_3^3$ WITH GREATEST CARDINALITY

In this section, we give the structure of every set S in $\mathbb{Z}_3^3$ which does not have any zero sum set with cardinality in $[1, 3]$, and $|S|$ is the greatest. In order to do that, the following (which is an interesting result) will be used:

PROPOSITION 3. *Every zero sum set $S \in \mathcal{F}(\mathbb{Z}_3^3)$ with $|S| = 8$ and $3^{\leq}$-zero sum free, has a zero sum free subset with cardinality 6.*

Proof. Let $S \in \mathcal{F}(\mathbb{Z}_3^3)$ be as in the hypothesis. Since $O(\mathbb{Z}_3^3) = 7$, any zero sum subset of S has cardinality 4. Furthermore, S contains two disjoint zero sum sets with cardinality 4.

Let $C_1 = x_1x_2x_3x_4$ be a zero sum subset of S, then $C_1|S$ is zero sum. For every $x_i, x_j, x_k \in C_1$ with $i, j, k \in [1, 4]$ (distinct) and $u \in C_1|S$; since $x_1 + x_2 + x_3 + x_4 = 0$, if $\sigma(x_ix_jx_ku) = 0$ then $u = x_ix_jx_k|C_1$. That is, u is a member of C_1, but this is not possible since S is a set. Hence $x_i+x_j+x_k+u \neq 0$. Thus, if S contains a zero sum set other than C_1 or $C_1|S$ then it consists of two members of C_1 and two members of $C_1|S$. If $S_0 = C_1uv \subset S$ with $u, v \in C_1|S$ and

$$x_i + x_j + u + v = 0. \tag{1}$$

Then

$$\sigma(S_0) = u + v = x_k + x_t. \tag{2}$$

Since S is a set, by (1), $x_\alpha + x_\beta + u + v \neq 0$ for every

$$(\alpha, \beta) \in \{(i, k), (i, t), (j, k), (j, t)\}.$$

Also, since S is $3^{\leq}$-zero sum free, by (2), we have $x_k + x_t + u + v \neq 0$.

Consider $S_i = x_jx_kx_tuvw$, with $w \in S_0|S$. We have the cases $\sigma(C') = 0$ where $C' \subset S_i$ with $|C'| = 4$ and we discard x_kx_tuw and x_kx_tvw since, by (2), if $0 = x_k+x_t+u+w = 2u+v+w \Rightarrow u = v+w$ and $0 = x_k+x_t+v+w = 2v+u+w \Rightarrow v = u+w$. But in both cases, $C_1|S = uvwz$ is zero sum implies that $u = z$ or $v = z$ (respectively). And this is not possible since S does not have repeated elements. Thus, we analyze the cases $C' \in \{x_jx_kuw, x_jx_tuw, x_jx_kvw, x_jx_tvw\}$. But only one of them is possible since in any case the fact that S is a set is contradicted. In fact:

- If $x_j + x_k + u + w = 0$ $(*)$, then none of the other is zero sum. In fact:
  - For $x_j + x_t + u + w = 0$ or $x_j + x_k + v + w = 0$, by $(*)$, it follows $x_t = x_k$ or $u = v$ (respectively).
  - From $x_j + x_t + v + w = 0$, by $(*)$, we have $x_k + u = x_t + v$ and by (2), it follows $2u + v + x_k = 2x_t + v + x_k \Rightarrow u = x_t$.
- If $x_j + x_t + u + w = 0$ $(**)$ then for:.
  - $x_j + x_k + v + w = 0$, by $(**)$ we have $x_t + u = x_k + v$ and by (2) it follows $2u + v + x_t = 2x_k + v + x_t \Rightarrow u = x_k$.
  - $x_j + x_t + v + w = 0$, by $(**)$ we have $u = v$.

Thus, we may assume that

$$x_j + x_k + u + w = 0 \tag{3}$$

Now, consider $S_j = x_ix_kx_tuvw$. Since S does not have repeated elements, we discard x_ix_kuw and x_kx_tuw. This is because from (3), it follows $x_i = x_j$

and $x_j = x_t$. On the other hand, since S is $3^{\leq}$-zero sum free, it is not possible that $x_i + x_t + u + w = 0$, because, by (3), it would be $x_j + x_k = x_i + x_t$ and since $x_1 + x_2 + x_3 + x_4 = 0$, then $x_j + x_k = 0 = x_i + x_t$.

Hence, if S_j contains a zero sum set, it has to be in $\{x_i x_k vw, x_i x_t vw, x_k x_t vw\}$.

- If $x_i + x_k + v + w = 0$, from (3), it follows $x_i + v = x_j + u$ and since $x_i + x_j + u + v = 0$, then $x_i + v = x_j + u = 0$, which contradicts that S is $3^{\leq}$-zero sum free.
- For $x_i + x_t + v + w = 0$, from (1), we get $x_j + u = x_t + w$ and from (1) and (3) it follows $x_i + v = x_k + w$. Adding these results and using (1), it follows $2w + x_k + x_t = 0$. Now, by (2), we have $w = u + v$. But this is not possible since $\sigma(C_1|S) = u + v + w + z = 0 \Rightarrow w = z$ which contradicts that S is a set.
- If $x_k + x_t + v + w = 0$, from (2), we obtain $u + 2v + w = 0 \Rightarrow v = u + w$ and since $u + v + w + z = 0$, then $v = z$, which contradicts that S is a set.

Therefore S_j zero sum free subset of S with $|S_j| = 6$. □

Proposition 4. *Every zero sum set T in $G = \mathbb{Z}_3^3$ with $|T| = 8$ is $3^{\leq}$-zero sum free if and only if it has the following structure*

$$a_1 a_2 a_3 \underbrace{(a_1 + a_2)}_{a_4} \underbrace{(a_1 + a_3)}_{a_5} \underbrace{(2a_1 + a_2 + a_3)}_{a_6} \underbrace{(2a_1 + 2a_2 + a_3)}_{a_7} \underbrace{(2a_1 + a_2 + 2a_3)}_{a_8}$$

where $\{a_1, a_2, a_3\}$ is a basis.

Proof. ($\Leftarrow$) Consider the zero sum set in $\mathbb{Z}_3^3$, where $\{a_1, a_2, a_3\}$ is a basis.

$$T = a_1 a_2 a_3 \underbrace{(a_1 + a_2)}_{a_4} \underbrace{(a_1 + a_3)}_{a_5} \underbrace{(2a_1 + a_2 + a_3)}_{a_6} \underbrace{(2a_1 + 2a_2 + a_3)}_{a_7} \underbrace{(2a_1 + a_2 + 2a_3)}_{a_8}$$

it is immediate that T is $2^{\leq}$-zero sum free, since zero is not in T and it does not contain any member together with its inverse. Furthermore, T does not contain a zero sum set with cardinality 3 since there are no distinct a_i, a_j, a_k with $i, j, k \in [1, 8]$ such that $a_i = 2a_j + 2a_k$.

($\Rightarrow$) Let $T \in \mathcal{F}(G)$ be a zero sum set with $|T| = 8$ and $3^{\leq}$-zero sum free. By Proposition 3, T contains a zero sum free set with cardinality 6, and by Theorem 2, the set is $S = a_1 a_2 a_3 (a_1 + a_2)(a_1 + a_3)(2a_1 + a_2 + a_3)$. We need to extend such set to a $3^{\leq}$-zero sum free set. From these 6 elements we discard their inverses. Remaining 14 from $|G - \{0\}| = 26$. Since $exp(G) = 3$, in order to discard those whose sum is zero with 6 of the initial elements, we use that $x + y + z = 0$ if and only if any of them is the inverse of the sum of the two others, that is, $z = 2(x + y) = 2x + 2y$. These give $\binom{6}{2} = 15$, from which, we exclude those of the form $2x+2y$ for $x+y \in S$, since they have been counted already. Also, $2a_2+2a_3$ and $2a_1+2a_2+2a_3$, which are considered twice. Substracting: $a_2+a_3, a_1+a_2+a_3, 2a_1+2a_2+a_3$ y $2a_1+a_2+2a_3$. From which, the required elements are $2a_1+2a_2+a_3$ y $2a_1+a_2+2a_3$ since otherwise their sum is the inverse of a member of S. Finally, we have the zero sum set

$$a_1a_2a_3(a_1+a_2)(a_1+a_3)(2a_1+a_2+a_3)(2a_1+2a_2+a_3)(2a_1+a_2+2a_3). \quad \square$$

Definition 1. Let G be a finite abelian group. Given a positive integer k, the constant $O^k(G)$ is the least positive integer l such that every set S in G with $|S| \geq l$ contains a $k^{\leq}$-zero sum set.

As an application of the obtained results we calculate exact values of $O^k(\mathbb{Z}_3^3)$ for $k \in [3,7]$.

Proposition 5. *Let $G = \mathbb{Z}_3^3$, then $O^3(G) = 9$ and $O^k(G) = 7$ for every $k \in [4,7]$.*

Proof. ($\geq$) The sets $S = a_1a_2a_3(a_1+a_2)(a_1+a_3)(2a_1+a_2+a_3)$ and $S(2a_1+2a_2+a_3)(2a_1+a_2+2a_3)$ given by Theorem 2 and Proposition 4 respectively, prove that $O^k(G) \geq 7$ for $k \in [4,7]$ and $O^3(G) \geq 9$.

($\leq$) Let $T \in \mathcal{F}(G)$ be a set. If $|T| = 7$ and T is $3^{\leq}$-zero sum free, then it contains the zero sum free set $S = a_1a_2a_3(a_1+a_2)(a_1+a_3)(2a_1+a_2+a_3)$ given by Theorem 2, where $\{a_1,a_2,a_3\}$ is a basis. Thus, $T = Sg$ with $g \in \{a_2+a_3, a_1+a_2+a_3, 2a_1+2a_2+a_3, 2a_1+a_2+2a_3\}$, hence in every case, the set given by g and three members of S is zero sum. Therefore $O^k(G) \leq 7$ for $k \in [4,7]$. On the other hand, if $|T| = 9$, by Proposition 4 there exists a $3^{\leq}$-zero sum subset of T. This proves that $O^3(G) \leq 9$. $\square$

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