TORSION FUNCTORS OF COMINIMAX MODULES

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Let R be a commutative Noetherian ring with identity and $\mathfrak{a}$ be an ideal of R. Also, suppose that M and N are two non-zero R-modules such that M is $\mathfrak{a}$-cominimax and N is finitely generated. We show that $\mathrm{Tor}_i^R(N, M)$ is $\mathfrak{a}$-cominimax for all $i \geq 0$, whenever $\dim N \leq 2$ or $\dim M \leq 1$. As an immediate consequence, we obtain that if M is a non-zero minimax R-module such that $\dim H_{\mathfrak{a}}^i(M) \leq 1$, then for each finitely generated R-module N, $\mathrm{Tor}_j^R\left(N, H_{\mathfrak{a}}^i(M)\right)$ is $\mathfrak{a}$-cominimax for all $i \geq 0$ and $j \geq 0$. Moreover, we prove that if R is local, M is $\mathfrak{a}$-cominimax and N is finitely generated, then the R-module $\mathrm{Tor}_i^R(N, M)$ is $\mathfrak{a}$-weakly cofinite for all $i \geq 0$, when $\dim N = 3$ or $\dim M \leq 2$.

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1. INTRODUCTION

Let R be a commutative Noetherian ring with identity and $\mathfrak{a}$ be an ideal of R. For an R-module M, the ith local cohomology module of M with respect to $\mathfrak{a}$ is defined as

$$H_{\mathfrak{a}}^i(M) \cong \varinjlim_{n \in \mathbb{N}} \mathrm{Ext}_R^i(R/\mathfrak{a}^n, M).$$

For more details about the local cohomology, we refer the reader to [6].

In 1968, Grothendieck [12] conjectured that for any ideal $\mathfrak{a}$ of R and any finitely generated R-module M, $\mathrm{Hom}_R\left(R/\mathfrak{a}, H_{\mathfrak{a}}^i(M)\right)$ is a finitely generated R-module for all i. One year later, Hartshorne [13] provided a counterexample to Grothendieck's conjecture and introduced the class of cofinite modules with respect to an ideal. He defined an R-module M to be $\mathfrak{a}$-*cofinite* if $\mathrm{Supp}_R(M) \subseteq V(\mathfrak{a})$ and $\mathrm{Ext}_R^j(R/\mathfrak{a}, M)$ is finitely generated for all j and he asked:

For which rings R and ideals $\mathfrak{a}$ are the modules $H_{\mathfrak{a}}^i(M)$ $\mathfrak{a}$-cofinite for all i and all finitely generated R-modules M?

There are several papers devoted to this question (see [5,7–9,13,15,17,27]).

In [28], Zöschinger introduced the class of minimax modules and in [28] and [29] gave equivalent conditions for a module to be minimax. An R-module

M is called *minimax* if there is a finitely generated submodule N of M such that M/N is Artinian. The concept of $\mathfrak{a}$-cominimax modules was introduced in [3] as a generalization of $\mathfrak{a}$-cofinite modules. According to the definition, an R-module M is said to be $\mathfrak{a}$-*cominimax* if $\operatorname{Supp}_R(M) \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^j_R(R/\mathfrak{a}, M)$ is minimax for all j. Since the concept of minimax modules is a natural generalization of the concept of finitely generated modules, many authors studied the minimaxness of local cohomology modules and answered the Hartshorn's question in the class of minimax modules (see [2, 3, 18]).

In this paper, we continue the study of cominimax modules with respect to an ideal of a commutative Noetherian ring. Melkersson in [23, Theorem 2.1] showed the striking result that $\mathfrak{a}$-cofiniteness of a module M over a Noetherian ring R is actually equivalent to the finiteness of the R-modules $\operatorname{Tor}^R_i(R/\mathfrak{a}, M)$ for all i. In this paper, we extend this result for any Serre subcategory of the category of all R-modules. This extension plays an important role in the proof of the results of this paper.

Recall that an R-module X is said to be *weakly Laskerian* if the set of associated primes of any quotient module of X is finite. Also, an R-module X is said to be $\mathfrak{a}$-*weakly cofinite* if $\operatorname{Supp}_R(X) \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ is weakly Laskerian for all $i \geq 0$ (see [10] and [11]).

Recently, Naghipour et al. in [24] proved that if $(R, \mathfrak{m})$ is a local ring, M is a non-zero $\mathfrak{a}$-cofinite R-module and N be a finitely generated R-module such that $\dim N = 3$ or $\dim M \leq 2$, then the R-module $\operatorname{Tor}^R_i(N, M)$ is $\mathfrak{a}$-weakly cofinite for all $i \geq 0$. As a main goal of this paper, we will show that the assertion in this result holds when we replace "$\mathfrak{a}$-cofinite" by the more general condition "$\mathfrak{a}$-cominimax". In this direction, we prove the following result:

THEOREM 1.1. *Let M be a non-zero $\mathfrak{a}$-cominimax R-module and N be a finitely generated R-module. Then the R-module $\operatorname{Tor}^R_i(N, M)$ is $\mathfrak{a}$-cominimax for all $i \geq 0$, when one of the following statements holds:*

1. $\dim N \leq 2$; *or*
2. $\dim M \leq 1$.

In particular, the Bass numbers and Betti numbers of R-module $\operatorname{Tor}^R_i(N, M)$ is finite for all $i \geq 0$

As the consequences, we derive the following corollaries of the above theorem.

COROLLARY 1.2. *Let M be a non-zero minimax R-module such that $\dim H^i_{\mathfrak{a}}(M) \leq 1$ (e.g. $\dim M \leq 1$ or $\dim R/\mathfrak{a} \leq 1$). Then for each finitely generated R-module N, the R-module $\operatorname{Tor}^R_j\left(N, H^i_{\mathfrak{a}}(M)\right)$ is $\mathfrak{a}$-cominimax for all $i \geq 0$ and $j \geq 0$.*

COROLLARY 1.3. *Let $(R, \mathfrak{m})$ be a local ring and M be a minimax R-module such that $\dim H^i_{\mathfrak{a}}(M) \leq 2$ (e.g. $\dim R/\mathfrak{a} \leq 2$). Then for each finitely generated*

R-module N, the R-module $\mathrm{Tor}_j^R\left(N, H_{\mathfrak{a}}^i(M)\right)$ *is $\mathfrak{a}$-weakly cofinite for all $i \geq 0$ and $j \geq 0$.*

Throughout this paper, we assume that R is a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ is an ideal of R, $V(\mathfrak{a})$ is the set of all prime ideals of R containing $\mathfrak{a}$ and $\mathrm{Max}(R)$ is the set of all maximal ideals of R. For any unexplained notation and terminology we refer the reader to [6] and [21].

2. PRELIMINARIES

Recall that a class of R-modules is a *Serre subcategory* of the category of R-modules when it is closed under taking submodules, quotients and extensions. For example, the classes of Noetherian modules, Artinian modules or minimax modules are Serre subcategories. As in standard notation, we let $\mathcal{S}$ stand for a Serre subcategory of the category of R-modules. The following lemma which is needed in the next section, immediately follows from the definition of Ext and Tor modules.

Lemma 2.1. *Suppose that M is a finitely generated R-module and $N \in \mathcal{S}$. Then* $\mathrm{Ext}_R^i(M, N) \in \mathcal{S}$ *and* $\mathrm{Tor}_i^R(M, N) \in \mathcal{S}$ *for all $i \geq 0$.*

Proof. See [1, Lemma 2.2]. □

Lemma 2.2. *Let N be an arbitrary R-module. Then the following conditions are equivalent:*

1. $\mathrm{Tor}_i^R(R/\mathfrak{a}, N) \in \mathcal{S}$ *for all $i \geq 0$;*
2. $\mathrm{Ext}_R^i(R/\mathfrak{a}, N) \in \mathcal{S}$ *for all $i \geq 0$;*
3. $\mathrm{Ext}_R^i(L, N) \in \mathcal{S}$ *for all $i \geq 0$ and for any finitely generated R-module L with* $\mathrm{Supp}_R(L) \subseteq V(\mathfrak{a})$*;*
4. *$Tor_i^R(L, N) \in \mathcal{S}$ for all $i \geq 0$ and for any finitely generated R-module L with* $\mathrm{Supp}_R(L) \subseteq V(\mathfrak{a})$*.*

Proof. (1.) $\Leftrightarrow$ (2.) : This follows by the same method as in [26, Theorem 2.7]. (2.) $\Leftrightarrow$ (3.) : This follows from Lemma 2.1.

(4.) $\Leftrightarrow$ (1.) : The forward direction is clear, because $\mathrm{Supp}_R(R/\mathfrak{a}) = V(\mathfrak{a})$. For the other direction, using [21, Theorem 6.4], there exist prime ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_t$ of R and a chain $0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_t = L$ of submodules of L such that $L_j/L_{j-1} \cong R/\mathfrak{p}_j$ for all $j = 1, \cdots, t$. Since $\mathfrak{p}_j \in \mathrm{Supp}_R(L) \subseteq V(\mathfrak{a})$, by induction on the length of this filtration of L, it is enough to show that $\mathrm{Tor}_i^R(R/\mathfrak{p}, N) \in \mathcal{S}$ for all $i \geq 0$ and all $\mathfrak{p} \in \mathrm{Supp}_R(L)$. This follows by the equivalence of (1.) – (3.) and the fact that $\mathrm{Supp}_R(R/\mathfrak{p}) \subseteq V(\mathfrak{a})$. □

Remark 2.3. The following statements hold:

1. The class of minimax modules contains all finitely generated and all Artinian modules.

2. Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules. Then M is minimax if and only if L and N are both minimax (see [4, Lemma 2.1]). Thus any submodule and quotient of a minimax module is minimax.
3. The set of associated primes of any minimax R-module is finite.
4. If M is a minimax R-module and $\mathfrak{p}$ is a non-maximal prime ideal of R, then $M_{\mathfrak{p}}$ is a finitely generated $R_{\mathfrak{p}}$-module.

3. MAIN RESULTS

Theorem 3.1. *Let M be a non-zero $\mathfrak{a}$-cominimax R-module. Then for each non-zero R-module N of finite length, the R-module $\operatorname{Tor}_i^R(N, M)$ has finite length for all $i \geq 0$.*

Proof. Since N is a non-zero R-module of finite length, the set $\operatorname{Supp}_R(N)$ is a finite non-empty subset of $\operatorname{Max}(R)$ by the definition. Let $\operatorname{Supp}_R(N) := \{\mathfrak{m}_1, \cdots, \mathfrak{m}_n\}$ and $\mathfrak{b} := \mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n$. As $\operatorname{Supp}_R(N) = V(\mathfrak{b})$, we need only to show that the R-module $\operatorname{Tor}_i^R(R/\mathfrak{b}, M)$ is of finite length for all $i \geq 0$, by Lemma 2.2. Since $\operatorname{Tor}_i^R(R/\mathfrak{b}, M) \cong \bigoplus_{j=1}^n \operatorname{Tor}_i^R(R/\mathfrak{m}_j, M)$, without loss of generality, we may assume that $n = 1$ and $\mathfrak{b} = \mathfrak{m}_1$. Finally, let $i \geq 0$ be an integer such that $\operatorname{Tor}_i^R(R/\mathfrak{m}_1, M) \neq 0$. It is clear that $\mathfrak{m}_1 \in \operatorname{Supp}_R(M) \subseteq V(\mathfrak{a})$. Therefore, in view of Lemma 2.2, the R-module $\operatorname{Tor}_i^R(R/\mathfrak{m}_1, M)$ is minimax and hence is of finite length for all $i \geq 0$ by [14, Lemma 2.5], as desired. □

Theorem 3.2. *Let M be a non-zero $\mathfrak{a}$-cominimax R-module and N be a finitely generated R-module such that $\dim N = 1$. Then the R-module $\operatorname{Tor}_i^R(N, M)$ is minimax for all $i \geq 0$.*

Proof. In the case where $N = \Gamma_{\mathfrak{a}}(N)$, we have $\operatorname{Supp}_R(N) \subseteq V(\mathfrak{a})$ and so the assertion follows from Lemma 2.2. Otherwise, if $\overline{N} = N/\Gamma_{\mathfrak{a}}(N)$, then in the light of the exact sequence

$$0 \to \Gamma_I(N) \to N \to \overline{N} \to 0$$

and Lemma 2.2, we need only to show that $\operatorname{Tor}_i^R(\overline{N}, M)$ is minimax for all $i \geq 0$. Therefore, without loss of generality, we may assume that $\Gamma_I(N) = 0$ and $\dim N = 1$. Hence, there exists an element $x \in \mathfrak{a} \setminus \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(N)} \mathfrak{p}$, by [6, Lemma 2.1.1]. Now, the exact sequence

$$0 \to N \xrightarrow{\cdot x} N \to N/xN \to 0$$

induces the following exact sequence for all $i \geq 0$:

$$\cdots \to \operatorname{Tor}_{i+1}^R(N/xN, M) \to \operatorname{Tor}_i^R(N, M) \xrightarrow{\cdot x} \operatorname{Tor}_i^R(N, M) \to \cdots .$$

Finally, we obtain the exact sequence

$$\operatorname{Tor}_{i+1}^R(N/xN, M) \to (0 :_{\operatorname{Tor}_i^R(N,M)} x) \to 0 \qquad (1)$$

for all $i \geq 0$. Since $\dim N/xN = 0$ and so N/xN has finite length, it follows from the exact sequence (1) and Lemma 3.1 that the R-module $\left(0 :_{\mathrm{Tor}_i^R(N,M)} x\right)$ is of finite length for all $i \geq 0$. Therefore, the R-module $\left(0 :_{\mathrm{Tor}_i^R(N,M)} \mathfrak{a}\right)$ is also of finite length for all $i \geq 0$, because $x \in \mathfrak{a}$. But, as $\mathrm{Supp}_R\left(\mathrm{Tor}_i^R(N,M)\right) \subseteq V(\mathfrak{a})$, we infer that the R-module $\mathrm{Tor}_i^R(N,M)$ is $\mathfrak{a}$-torsion. Thus, by Melkersson's result [22, Theorem 1.3], $\mathrm{Tor}_i^R(N,M)$ is Artinian and so is minimax for all $i \geq 0$, as required. □

THEOREM 3.3. *Let M be a non-zero $\mathfrak{a}$-cominimax module and N be a finitely generated R-module such that* $\dim N = 2$. *Then the R-module* $\mathrm{Tor}_i^R(N, M)$ *is $\mathfrak{a}$-cominimax for all $i \geq 0$.*

Proof. As in the proof of Theorem 3.2, we may assume that $\Gamma_{\mathfrak{a}}(N) = 0$ and $\dim N = 2$. So, by [6, Lemma 2.1.1], there exists an element $x \in \mathfrak{a} \setminus \bigcup_{\mathfrak{p}\in \mathrm{Ass}_R(N)} \mathfrak{p}$. Now, the short exact sequence

$$0 \to N \xrightarrow{.x} N \to N/xN \to 0$$

induces the following exact sequence for all $i \geq 0$:

$$\cdots \to \mathrm{Tor}_{i+1}^R(N/xN, M) \to \mathrm{Tor}_i^R(N,M) \xrightarrow{.x} \mathrm{Tor}_i^R(N,M) \to \mathrm{Tor}_i^R(N/xN, M) \to \cdots .$$

Since $\dim N/xN \leq 1$, by Theorems 3.1 and 3.2, the R-modules $(0 :_{\mathrm{Tor}_i^R(N,M)} x)$ and $\mathrm{Tor}_i^R(N,M)/x\mathrm{Tor}_i^R(N,M)$ are minimax and so are $\mathfrak{a}$-cominimax for all $i \geq 0$. Therefore, the R-module $\mathrm{Tor}_i^R(N,M)$ is $\mathfrak{a}$-cominimax for all $i \geq 0$, by [14, Lemma 2.6]. □

THEOREM 3.4. *Let M be a non-zero $\mathfrak{a}$-cominimax and* $\dim M \leq 1$. *Then for each non-zero finitely generated R-module N, the R-module* $\mathrm{Tor}_i^R(N,M)$ *is $\mathfrak{a}$-cominimax for all $i \geq 0$.*

Proof. Since N is a finitely generated module over the Noetherian ring R, it has a free resolution

$$\mathbf{F}_\bullet : \cdots \to F_n \to \cdots \to F_2 \to F_1 \to F_0 \to 0,$$

where all the free R-modules F_i have finite ranks. Hence, $\mathrm{Tor}_i^R(N,M) = H_i(\mathbf{F}_\bullet \otimes_R M)$ is a subquotient of a direct sum of finitely many copies of M. Now, the assertion follows easily from [16, Theorem 2.5]. □

Recall that for an R-module M, the ith Bass number and Betti number of M with respect to a prime ideal $\mathfrak{p}$ is defined as $\mu_i(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} \mathrm{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p})$ and $\beta_i(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} \mathrm{Tor}_i^{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p})$, respectively. Applying the same technique in the proof of [2, Corollary 2.3] using Lemma 2.2, one can see the

Bass numbers and Betti numbers of any $\mathfrak{a}$-cominimax module are finite. Hence, the next result immediately follows from Theorems 3.1-3.4 and Lemma 2.2.

COROLLARY 3.5. *Suppose that M and N are two non-zero R-modules such that M is $\mathfrak{a}$-cominimax and N is finitely generated. Then the Bass numbers and Betti numbers of R-module $\operatorname{Tor}_i^R(N, M)$ is finite for all $i \geq 0$, when one of the following statements holds:*

1. $\dim N \leq 2$; *or*
2. $\dim M \leq 1$.

COROLLARY 3.6. *Let M be a non-zero minimax R-module such that $\dim H_{\mathfrak{a}}^i(M) \leq 1$ for all $i \geq 0$ (e.g. $\dim M \leq 1$ or $\dim R/\mathfrak{a} \leq 1$). Then for each finitely generated R-module N, the R-module $\operatorname{Tor}_j^R\left(N, H_{\mathfrak{a}}^i(M)\right)$ is $\mathfrak{a}$-cominimax for all $i \geq 0$ and $j \geq 0$. In particular, the R-module $\operatorname{Tor}_j^R\left(N, H_{\mathfrak{a}}^i(M)\right)$ is $\mathfrak{a}$-cofinite for all $i \geq 1$ and $j \geq 0$.*

Proof. In the light of [2, Theorem 2.2], $H_{\mathfrak{a}}^i(M)$ is an $\mathfrak{a}$-cominimax R-module for all $i \geq 0$. Hence, Theorem 3.4 yields the result. The last assertion follows from [2, Theorem 2.2] and [24, Lemma 3.3]. □

The following corollary which is a generalization of [24, Corollary 3.5], gives an extension of Corollary 3.6 for local (Noetherian) rings.

COROLLARY 3.7. *Let $(R, \mathfrak{m})$ be a local ring and M be a minimax R-module such that $\dim H_{\mathfrak{a}}^i(M) \leq 2$ (e.g. $\dim R/\mathfrak{a} \leq 2$). Then for each finitely generated R-module N, the R-module $\operatorname{Tor}_j^R\left(N, H_{\mathfrak{a}}^i(M)\right)$ is $\mathfrak{a}$-weakly cofinite for all $i \geq 0$ and $j \geq 0$.*

Proof. By Lemma 2.2, we need only to show that $\operatorname{Tor}_k^R\left(R/\mathfrak{a}, \operatorname{Tor}_j^R\left(N, H_{\mathfrak{a}}^i(M)\right)\right)$ is weakly Laskerian for all $i, j, k \geq 0$. Let

$$\Omega = \left\{\operatorname{Tor}_k^R\left(R/\mathfrak{a}, \operatorname{Tor}_j^R\left(N, H_{\mathfrak{a}}^i(M)\right)\right) \mid i \geq 0, j \geq 0, k \geq 0\right\}.$$

Suppose that $K \in \Omega$ and K' is a submodule of K. By definition, it suffices to show that $\operatorname{Ass}_R(K/K')$ is a finite set. For do this, in view of the Flat Base Change Theorem [6, Theorem 4.3.2], [21, Ex. 7.7] and [19, Lemma 2.1], without loss of generality, we can assume that R is complete.

Now, on the contrary, suppose that the set $\operatorname{Ass}_R(K/K')$ is infinite. So, there is a countably infinite subset $\{\mathfrak{p}_r\}_{r=1}^{\infty}$ of non-maximal elements of $\operatorname{Ass}_R(K/K')$. Thus, $\mathfrak{m} \nsubseteq \bigcup_{r=1}^{\infty} \mathfrak{p}_r$ by [20, Lemma 3.2]. Let S be the multiplicatively closed subset $R \setminus \bigcup_{r=1}^{\infty} \mathfrak{p}_r$. By hypothesis, [6, Corollary 4.3.3] and [25, Lemma 3.4], it easily follows that $S^{-1}M$ is a minimax $S^{-1}R$-module such that $\dim H_{S^{-1}\mathfrak{a}}^i(S^{-1}M) \leq 1$. Therefore, Corollary 3.6 implies that $S^{-1}K/S^{-1}K'$ is a minimax $S^{-1}R$-module and so $\operatorname{Ass}_{S^{-1}R}\left(S^{-1}K/S^{-1}K'\right)$ is a finite set by Remark 2.3. But, $S^{-1}\mathfrak{p}_r \in \operatorname{Ass}_{S^{-1}R}\left(S^{-1}K/S^{-1}K'\right)$ for all $r = 1, 2, \cdots$, a contradiction. □

Finally, as an application, we state the following theorem which is a generalization of [24, Theorem 3.2 and Corollary 3.6].

Theorem 3.8. *Let $(R, \mathfrak{m})$ be a local ring, M be a non-zero $\mathfrak{a}$-cominimax R-module and N be a finitely generated R-module. Then the R-module* $\operatorname{Tor}_i^R(N, M)$ *is $\mathfrak{a}$-weakly cofinite for all $i \geq 0$, when one of the following statements holds:*

1. $\dim N = 3$*; or*
2. $\dim M \leq 2$.

Proof. In the light of Theorems 3.1-3.4, the assertion follows by the same method as in Corollary 3.7. □

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