AVKHADIEV-BECKER TYPE P-VALENT CONDITIONS FOR HARMONIC MAPPINGS OF THE UNIT DISK AND ITS EXTERIOR

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We obtain Avkhadiev–Becker type p-valent conditions for harmonic mappings of the unit disk and its exterior, and prove a generalization of John's p-valent condition.

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1. INTRODUCTION

Let f be a complex-valued function defined on a simply connected subdomain E of the complex plane $\mathbb{C}$. It has been shown that for a function f harmonic in E, there exist two functions g and h analytic in E, such that

$$f(z) = h(z) + \overline{g(z)}, \quad z \in E.$$

H. Lewy proved that f is locally univalent and sense-preserving in E if and only if $|g'(z)| < |h'(z)|$ in E (see [16, 21] for more information). In the recent paper [10], F.G. Avkhadiev *et al.* used the methods of L. Ahlfors and G. Weill [1] to establish the univalency condition for harmonic mappings of the unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

and its exterior

$$\mathbb{D}^{-} = \{z \in \mathbb{C} : |z| > 1\}.$$

Namely, the authors proved the following

THEOREM A. *Let h and g be holomorphic functions in the unit disk $\mathbb{D}$, and for all $z \in \mathbb{D}$,*

$$h'(z) \neq 0, \; |\omega(z)| < 1,$$

where $\omega(z) = g'(z)/h'(z)$. Let for all $z \in \mathbb{D}$,

$$|\omega(z)| + (1 - |z|^2) \left| z \frac{h''(z)}{h'(z)} \right| \leq 1.$$

Then $f = h + \overline{g}$ *is univalent in* $\mathbb{D}$.

Note that Theorem A implies the following J. Becker's theorem for analytic functions (see more details in [5, 6, 8, 9, 12–14, 22]).

THEOREM B. *Let* f *be an analytic mapping from* $\mathbb{D}$ *into* $\mathbb{C}$, *such that* $f'(z) \neq 0$ *for all* $z \in \mathbb{D}$. *Let for all* $z \in \mathbb{D}$,

$$(1 - |z|^2)|zf''(z)/f'(z)| \leq 1.$$

Then f *is univalent in* $\mathbb{D}$.

Note that there are analogues of Theorem B for analytic functions proved by F.G. Avkhadiev (see [5, 6, 9, 14, 22]), and for harmonic functions proved by R. Hernandez and M.J. Martin in [20] and by Sh.L. Chen, S. Ponnusamy, A. Rasila and X.T. Wang in [15].

Let p be a natural number. We say that a function f is p-valent in a domain, if

a) for all $w \in \mathbb{C}$, the equation $f(z) = w$ has m roots, where $0 \leq m \leq p$;

b) there exists $w_0 \in \mathbb{C}$ such that the equation $f(z) = w_0$ has exactly p roots.

In [7], F.G. Avkhadiev obtained p-valent conditions for analytic functions. Namely, the author proved the following

THEOREM A2. *Let* f *be an analytic function in* $\mathbb{D} \setminus \{0\}$, $n \neq 0$ *be an integer, and*

$$\lim_{z \to 0} z^{-n} f(z) = a_1 \in \mathbb{C} \setminus \{0\}.$$

Let for all $z \in \mathbb{D}$, $|z| < 1$,

$$\sup_{z \in \mathbb{D}} \left| \left(1 - |z|^{2n}\right) \left(n - 1 - z \frac{f''(z)}{f'(z)}\right) \right| \leq |n|.$$

Then f *is* $|n|$*-valent in* $\mathbb{D}$.

The aim of this paper is to obtain Avkhadiev–Becker type p-valent conditions for harmonic mappings of the unit disk and its exterior. We will use the methods from [4–6]. The main result for harmonic mappings in $\mathbb{D}$ is the following assertion.

THEOREM 1. *Let* $n \neq 0$ *be an integer,* $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, h *and* g *be analytic in* $\mathbb{D} \setminus \{0\}$, $h'(z) \neq 0$, $|\omega(z)| < 1$ *for all* $z \in \mathbb{D} \setminus \{0\}$, *where*

$$\omega(z) = g'(z)/h'(z),$$

moreover,

$$\lim_{z\to 0} z^{-n}h(z) = 1. \tag{1}$$

Let $f(z) = h(z) + \overline{g(z)}$ *satisfy the condition*

$$|n|\,|\omega(z)| + \left(1 - |z|^{2n}\right)\left|n - 1 - z\frac{h''(z)}{h'(z)}\right| \le |n|$$

for all $z \in \mathbb{D}$. *Then* $f(z)$ *is* $|n|$*-valent in* $\mathbb{D}$.

COROLLARY 1. *Let* $n \neq 0$ *be an integer,* $\mathbb{D}^- = \{\zeta \in \mathbb{C} : |\zeta| > 1\}$, h *and* g *be analytic in* $\mathbb{D}^- \setminus \{\infty\}$, $h'(\zeta) \neq 0$, $|\omega(\zeta)| < 1$ *for all* $\zeta \in \mathbb{D}^- \setminus \{\infty\}$, *where*

$$\omega(\zeta) = g'(\zeta)/h'(\zeta),$$

moreover,

$$\lim_{\zeta\to\infty} \zeta^{n}h(\zeta) = 1. \tag{2}$$

Let $f(\zeta) = h(\zeta) + \overline{g(\zeta)}$ *satisfy the condition*

$$|n|\,|\omega(\zeta)| + \left(|\zeta|^{2n} - 1\right)\left|n + 1 + \zeta\frac{h''(\zeta)}{h'(\zeta)}\right| \le |n|$$

for all $\zeta \in \mathbb{D}^-$. *Then* $f(\zeta)$ *is* $|n|$*-valent in* $\mathbb{D}^-$.

To obtain Corollary 1, we apply Theorem 1 to the function f_1 defined by $f_1(\zeta) = f(z)$, $z = 1/\zeta$.

F. John, F.G. Avkhadiev and J. Gevirtz obtained sufficient univalence conditions of the type

$$m < |f'(z)| < M$$

for analytic functions (see [8, 18]). Sh.L. Chen, S. Ponnusamy, A. Rasila and X.T. Wang in [15] got univalence condition of this type for harmonic mappings. We obtain a p-valent condition of this type for harmonic mappings.

THEOREM 2. *Let* $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $n \neq 0$ *be an integer,* $q \in [0, 1)$, h *and* g *be holomorphic mappings in* $\mathbb{D} \setminus \{0\}$, $h'(z) \neq 0$, *moreover,*

$$\lim_{z\to 0} z^{-n}h(z) = 1,\ \ h(z) - z^n = O\left(|z|^{|n|}\right). \tag{3}$$

Then a harmonic mapping $f(z) = h(z) + \overline{g(z)}$ *is* $|n|$*-valent in* $\mathbb{D}$, *provided that for all* $z \in \mathbb{D}$,

$$m \le |h'(z)z^{1-n}| \le M, \quad |g'(z)/h'(z)| \le q,$$

where the positive constants m *and* M *are such that*

$$1 < \frac{M}{m} \le \exp\left(\frac{\pi(1-q)}{2}\right) \ \text{for } n \ge 1,$$

$$1 < \frac{M}{m} \leq \exp\left(\frac{\pi(1-q)}{4}\right) \quad for\ n \leq -1.$$

Note that Theorem A follows from Theorem 1 for $n = 1$. In [12, 13], using Levner–Kufarev's equation, J. Becker proved the statement of Theorem 1 for analytic functions in the case of $n = \pm 1$. P.L. Duren, M.S. Shapiro, A.L. Shields in the paper [17] and F.G. Avkhadiev in [2, 3] obtained the analogues of Theorem 1 using other methods.

In [10], the authors also obtained univalence conditions for harmonic mappings from the exterior of the unit disc $\mathbb{D}^-$ into $\mathbb{C}$.

Let

$$F(\zeta) = H(\zeta) + \overline{G(\zeta)},\ \zeta \in \mathbb{D}^-,$$

where H and G are analytic functions in $\mathbb{D}^- \setminus \{\infty\}$.

THEOREM 3. *Let G and F be holomorphic functions in $\mathbb{D}^- \setminus \{\infty\}$, such that*

$$G(\zeta) = \sum_{k=n}^{\infty} g_k/\zeta^k$$

and

$$H(\zeta) = \zeta^n + \sum_{k=n}^{\infty} h_k/\zeta^k.$$

Moreover, for each positive integer $n \neq 0$, let the function H have a pole of order n at the point $\zeta = \infty$, and for all $\zeta \in \mathbb{D}^-$,

$$\lim_{\zeta\to\infty} \zeta^{-n} H(\zeta) = 1, \quad H'(\zeta) \neq 0.$$

Let for all $|\zeta| > 1$,

$$n\left|\zeta^2 \frac{H'(\zeta)}{G'(\zeta)}\right| + \left(|\zeta|^{2n} - 1\right)\left|n - 1 - \zeta\frac{H''(\zeta)}{H'(\zeta)}\right| \leq n.$$

Then $F = H + \overline{G}$ is n-valent in $\mathbb{D}^-$.

THEOREM 4. *Let*

$$G(\zeta) = \zeta^n + \sum_{k=n}^{\infty} g_k/\zeta^k$$

and

$$H(\zeta) = \zeta^n + \sum_{k=n}^{\infty} h_k/\zeta^k$$

be holomorphic in $\mathbb{D}^- \setminus \{\infty\}$, and have a pole of order n at $\zeta = \infty$, moreover, $|H'(\zeta)| - |G'(\zeta)| > 0$ for all $\zeta \in \mathbb{D}^-$. Let for all $|\zeta| \geq 1$,

$$|H''(\zeta)|+|G''(\zeta)|\le n\frac{|H'(\zeta)|-|G'(z)|}{|\zeta|^{2n+1}-|\zeta|}-(n-1)\frac{|H'(\zeta)|+|G'(\zeta)|}{|\zeta|}.$$

Then $F=H+\overline{G}$ is n-valent in $\mathbb{D}^-=\{\zeta\in\overline{\mathbb{C}}:|\zeta|>1\}$.

We note that the sufficient conditions in the unit disk were announced without proof in the short communication [23].

2. PROOF OF THE SUFFICIENT CONDITIONS IN THE UNIT DISK

Proof of Theorem 1. Fix $r\in(0,1)$, and denote

$$\mathbb{D}_r=\{z\in\mathbb{C}:|z|\le r\},\ \mathbb{D}_r^-=\{z\in\mathbb{C}:|z|\ge r\}.$$

Consider the mapping $\widehat{f}:\mathbb{C}\to\mathbb{C}$ given by

$$\widehat{f}(z)=\begin{cases} f(z), & |z|\le r,\\ f\left(r^2/\overline{z}\right)+(z^n-r^{2n}/\overline{z}^n)f'\left(r^2/\overline{z}\right)\overline{z}^{n-1}/(nr^{2(n-1)}), & |z|\ge r.\end{cases}$$

Using the decomposition $f(z)=h(z)+\overline{g(z)}$, we obtain

$$\widehat{f}(z)=\begin{cases} f(z), & |z|\le r\\ h\left(r^2/\overline{z}\right)+\overline{g\left(r^2/\overline{z}\right)}+\left(z^n-r^{2n}/\overline{z}^n\right)\frac{h'\left(r^2/\overline{z}\right)\overline{z}^{n-1}}{nr^{2(n-1)}}, & |z|\ge r.\end{cases}$$

The function $\widehat{f}$ is obviously continuous. We will prove that $\widehat{f}(z)\to\infty$ as $z\to\infty$, when $n\ne 0$ is an integer. It is easily shown that

$$\lim_{|z|\to\infty} h\left(r^2/\overline{z}\right)+\overline{g\left(r^2/\overline{z}\right)}=h(0)+\overline{g(0)}=f(0).$$

Due to condition (2),

$$h(z)=z^n+a_{n+1}z^{n+1}+\cdots=z^n+\sum_{k=n+1}^{\infty}a_kz^k,$$

where a_k are complex numbers.

Straightforward computations give the following equalities.

$$h'\left(\frac{r^2}{\overline{z}}\right)=n\left(\frac{r^2}{\overline{z}}\right)^{n-1}+(n+1)a_{n+1}\left(\frac{r^2}{\overline{z}}\right)^{n}+\ldots=$$

$$=n\left(\frac{r^2}{\overline{z}}\right)^{n-1}+\sum_{k=n+1}^{\infty}ka_k\left(\frac{r^2}{\overline{z}}\right)^{k-1}.$$

Consequently, for $n>0$ we have

$$\frac{z^n\overline{z}^{n-1}}{nr^{2(n-1)}}h'\left(\frac{r^2}{\overline{z}}\right)=z^n\left(1+\sum_{k=n+1}^{\infty}ka_k\left(\frac{r^2}{\overline{z}}\right)^{k-n}\right)=O(z^n),$$

and for $n<0$ we obtain

$$\frac{r^2}{n\overline{z}}h'\left(\frac{r^2}{\overline{z}}\right)=\frac{r^{2n}}{\overline{z}^n}+\sum_{k=n+1}^{\infty}ka_k\left(\frac{r^2}{\overline{z}}\right)^k=O(z^{|n|}).$$

Therefore, $\lim_{z\to\infty}\widehat{f}(z)=\infty$ for any integer $n\neq 0$.

Denote by $J_{\widehat{f}}$ the Jacobian of $\widehat{f}$. Since

$$J_{\widehat{f}}=|\widehat{f}_z|^2-|\widehat{f}_{\overline{z}}|^2,$$

the Jacobian $J_{\widehat{f}}$ is positive for $|z|\le r$, provided that

$$|\widehat{f}_z|-|\widehat{f}_{\overline{z}}|=|h_z|-|g_z|>0.$$

The last statement follows from the condition $|\omega(z)|<1$ for all $z\in D$.

Now we will show that the Jacobian $J_{\widehat{f}}$ is positive for $|z|\ge r$. By straightforward calculations we get

$$\widehat{f}_z=\frac{(z\overline{z})^{n-1}}{r^{2(n-1)}}h'\left(\frac{r^2}{\overline{z}}\right)-\frac{r^2}{z^2}\overline{g'\left(\frac{r^2}{\overline{z}}\right)}$$

and

$$\begin{aligned}\widehat{f}_{\overline{z}}&=-\frac{r^2}{\overline{z}^2}\left[h'\left(\frac{r^2}{\overline{z}}\right)-\frac{(n-1)|z|^{2n}+r^{2n}}{nr^{2n}}h'\left(\frac{r^2}{\overline{z}}\right)+\frac{|z|^{2n}-r^{2n}}{nr^{2(n-1)}\overline{z}}h''\left(\frac{r^2}{\overline{z}}\right)\right]=\\&=-\frac{r^2}{\overline{z}^2}\left[-h'\left(\frac{r^2}{\overline{z}}\right)\frac{n-1}{n}\left(\frac{|z|^{2n}}{r^{2n}}-1\right)+\frac{r^2}{n\overline{z}}\left(\frac{|z|^{2n}}{r^{2n}}-1\right)h''\left(\frac{r^2}{\overline{z}}\right)\right]=\\&=\frac{r^2}{\overline{z}^2}\frac{1}{n}h'\left(\frac{r^2}{\overline{z}}\right)\left(\frac{|z|^{2n}}{r^{2n}}-1\right)\left(n-1-\frac{r^2}{\overline{z}}\frac{h''\left(\frac{r^2}{\overline{z}}\right)}{h'\left(\frac{r^2}{\overline{z}}\right)}\right).\end{aligned}$$

We replace $r^2/\overline{z}$ by ζ in the last statements, and obtain

$$\widehat{f}_z=\frac{r^{2(n-1)}}{\zeta^{n-1}\overline{\zeta}^{n-1}}h'(\zeta)-\frac{\overline{\zeta}^2}{r^2}\overline{g'(\zeta)}=\frac{r^{2(n-1)}}{|\zeta|^{2(n-1)}}h'(\zeta)-\frac{\overline{\zeta}^2}{r^2}\overline{g'(\zeta)}$$

and

$$\widehat{f}_{\overline{z}}=\frac{1}{n}\frac{\zeta^2}{r^2}h'(\zeta)\left(\frac{r^{2n}}{\zeta^n\overline{\zeta}^n}-1\right)\left(n-1-\zeta\frac{h''(\zeta)}{h'(\zeta)}\right).$$

Hence,

$$\begin{aligned}\frac{|f_{\overline{z}}|}{|f_z|}&=\frac{\left|\frac{1}{n}\frac{\zeta^2}{r^2}h'(\zeta)\left(\frac{r^{2n}}{|\zeta|^{2n}}-1\right)\left(n-1-\zeta\frac{h''(\zeta)}{h'(\zeta)}\right)\right|}{\left|\frac{r^{2(n-1)}}{|\zeta|^{2(n-1)}}h'(\zeta)-\frac{\overline{\zeta}^2}{r^2}\overline{g'(\zeta)}\right|}\le\\&\le\frac{\left|\left(\frac{r^{2n}}{|\zeta|^{2n}}-1\right)\left(n-1-\zeta\frac{h''(\zeta)}{h'(\zeta)}\right)\right|}{|n|\left(\frac{r^{2n}}{|\zeta|^{2n}}-\left|\frac{\overline{g'(\zeta)}}{h'(\zeta)}\right|\right)}.\end{aligned}$$

Thus, the positivity of the Jacobian for $|\zeta| \leq r$ is implied by the following inequalities:

$$n\frac{|\zeta|^{2n}}{r^{2n}}|\omega(\zeta)| + \left|\left(1-\frac{|\zeta|^{2n}}{r^{2n}}\right)\left(n-1-\zeta\frac{h''(\zeta)}{h'(\zeta)}\right)\right| < n, \qquad n > 0,$$

and

$$|n|\frac{|\zeta|^{2|n|}}{r^{2|n|}}|\omega(\zeta)| + \left|\left(1-\frac{|\zeta|^{2|n|}}{r^{2|n|}}\right)\left(n-1-\zeta\frac{h''(\zeta)}{h'(\zeta)}\right)\right| < |n|, \qquad n < 0.$$

Using the last inequality, condition $|\omega(\zeta)| < 1, \forall\zeta \in \mathbb{D}$, and inequalities

$$\frac{|\zeta|^{2n}}{r^{2n}} \leq 1, \quad 1-\frac{|\zeta|^{2n}}{r^{2n}} \leq 1-|\zeta|^{2n},$$

we get the positivity of the Jacobian for $|\zeta| \leq r$.

Thus, we obtain that the function $\widehat{f}$ is continuous in $\mathbb{C}\backslash\{0\}$, and it follows from the positivity of the Jacobian that $\widehat{f}$ is a local homeomorphism in $0 < |z| \leq r$ and in $r \leq |z| < \infty$ separately. Using the following lemma from [5], we get that f is a local homeomorphism in $\mathbb{C}\backslash\{0\}$.

Lemma A. *Let D_1 and D_2 be nonintersecting domains which have common part of their boundaries including an open Jordan arc L, where $D_1 \cup D_2 \cup L$ is a domain. Let $f_i(z)$, $i = 1, 2$, be similarly oriented interior mappings of D_i in the sense of Stoilow [24] which are continuous except at finite number of poles and locally univalent in $D_i \cup L$, $i = 1, 2$, and $f_1(z) = f_2(z)$ for all $z \in L$. Then the function $f(z) = \{f_1(z), z \in D_1 \cup L; f_2(z), z \in D_2 \cup L\}$ is an interior locally univalent mapping of the domain $D = D_1 \cup D_2 \cup L$.*

Hence, using Stoilow's theorem [5,24], we get that $\widehat{f}$ is topologically equivalent to z^n.

Theorem of Stoilow. *Let $f\colon \mathbb{C} \to \mathbb{C}$ be an interior mapping in the sense of Stoilow, moreover, $f(z) \to \infty$ as $z \to \infty$. Then there exists a homeomorphic mapping ψ of the Euclidean plane onto itself, $\psi(0) = 0$, and a holomorphic in $\mathbb{C}$ mapping g such that $f(z) = g(\psi(z))$.*

This proves that f is $|n|$-valent harmonic mapping.

Proof of Theorem 2. Since the unit disk is a simply connected domain, and

$$\lim_{z\to 0} z^{-n}h(z) = 1, \quad h(z) - z^n = O\left(|z|^{|n|}\right),$$

it follows that there exists a holomorphic function defined as a single-valued branch of $\ln h'(z)/z^{n-1}$. Under the condition of Theorem 2, namely,

$$m \leq |h'(z)z^{1-n}| \leq M,$$

we see that the values of the function $s(z) = \ln h'(z)/z^{n-1}$ lie in the strip

$$S(m, M) = \{w \in \mathbb{C} : \ln m < Re\ w < \ln M\}.$$

It means that $s(z)$ is subordinated to the function

$$\frac{2\ln(M/m)}{\pi i} \ln \frac{1+z}{1-z},$$

which maps $\mathbb{D}$ onto the strip.

Consequently, there exists an analytic in the disk $\mathbb{D}$ function φ, such that

$$s(z) = \frac{2\ln(M/m)}{\pi i} \ln \frac{1+\varphi(z)}{1-\varphi(z)} + const,$$

where $|\varphi(z)| < 1$, $\varphi(0) \in D$, and for $n \geq 1$,

$$\varphi'(0) = \varphi''(0) = \ldots = \varphi^{(2|n|-1)}(0) = 0.$$

Further, using the Schwarz lemma and the inequality of Goluzin [11, 19], we have

$$\frac{|\varphi'(z)|}{1-|\varphi(z)|^2} \leq \frac{1}{1-|z|^2} \text{ for } n \geq 1,$$

and

$$\frac{|\varphi'(z)|}{1-|\varphi(z)|^2} \leq \frac{2|n||z|^{2|n|-1}}{1-|z|^{4|n|}} \text{ for } n \leq -1.$$

It is clear that

$$s'(z) = \frac{4\ln(M/m)}{\pi i} \frac{\varphi'(z)}{1-\varphi^2(z)}.$$

Denote by $R(w, S(M, m))$ the conformal radius of the domain $S(M, m)$ at the point w. We obtain

$$|s'(w)| = \frac{R(w, S(M, m))}{1-|w|^2}, \quad w = \varphi(z),$$

and

$$|s'(z)| = |s'(w)||\varphi'(z)| = R(s(z), S(M, m)) \frac{|\varphi'(z)|}{1-|\varphi(z)|^2}.$$

Since

$$s(z) = \ln z^{n-1} h'(z), \quad s'(z) = h''(z)/h'(z) - (n-1)/z,$$

one can show that for all $z \in \mathbb{D}$,

$$\left| z\frac{h''(z)}{h'(z)} - n + 1 \right| \leq \begin{cases} 2/\pi \ln(M/m)|z|(1-|z|^2)^{-1} & \text{for } n \geq 1, \\ 4/\pi \ln(M/m)|n||z|^{2|n|}(1-|z|^{4|n|})^{-1} & \text{for } n \leq -1. \end{cases}$$

Using the inequalities

$$|\omega(z)| = \frac{|g'(z)|}{|h'(z)|} \leq q, \quad 1 < \frac{M}{m} \leq \exp\left(\frac{\pi(1-q)}{2}\right) \text{ for } n \geq 1,$$

and

$$|\omega(z)| = \frac{|g'(z)|}{|h'(z)|} \leq q, \quad 1 < \frac{M}{m} \leq \exp\left(\frac{\pi(1-q)}{4}\right) \text{ for } n \leq -1,$$

we obtain

$$n\,|\omega(z)| + \left|\left(1-|z|^{2n}\right)\left(n-1-z\frac{h''(z)}{h'(z)}\right)\right| \leq$$
$$\leq nq + \left|\frac{2}{\pi}\ln\left(\frac{M}{m}\right)\frac{|z|\left(1-|z|^{2n}\right)}{1-|z|^2}\right| \leq n,$$

and

$$|n|\,|\omega(z)| + \left|\left(1-|z|^{2n}\right)\left(n-1-z\frac{h''(z)}{h'(z)}\right)\right| \leq$$
$$\leq |n|q + \left|\frac{4}{\pi}\ln(M/m)\frac{|n||z|^{2n}\left(1-|z|^{2|n|}\right)}{1-|z|^{4|n|}}\right| \leq |n|.$$

Hence, due to Theorem 1, $f = h + \overline{g}$ is $|n|$-valent in $\mathbb{D}$.

3. PROOF OF THE SUFFICIENT CONDITIONS IN THE EXTERIOR OF THE UNIT DISK

Proof of Theorem 3. Let

$$\widehat{G}(\zeta) = \begin{cases} F(\zeta), & |\zeta| \geq r, \\ F\left(r^2/\overline{\zeta}\right) + \left(\zeta^n - r^{2n}/\overline{\zeta}^n\right) H'\left(r^2/\overline{\zeta}\right)\overline{\zeta}^{n-1}/(nr^{2(n-1)}), & |\zeta| \leq r, \end{cases}$$

where $r \in (1, \infty)$.

It is obvious that the function $\widehat{G}(\zeta)$ is continuous and has a pole of order n at the point $\zeta = \infty$. Under the condition of Theorem 3, we have

$$\left|\zeta^2 G'(\zeta)/H'(\zeta)\right| < 1$$

for all $\zeta \in \mathbb{D}^-$. Since $|\zeta| \geq 1$, it follows that

$$\left|G'(\zeta)\right| < \left|H'(\zeta)\right|$$

for all $\zeta \in D^-$. Hence, $\widehat{G}(\zeta)$ is locally univalent in $|\zeta| \geq r$.

Now we will prove that the Jacobian is positive in $|\zeta| \leq r$. By straightforward computations, we obtain that

$$\widehat{G}_\zeta = \frac{(\zeta\overline{\zeta})^{n-1}}{r^{2(n-1)}} H'\left(\frac{r^2}{\overline{\zeta}}\right) - \frac{r^2}{\zeta^2}\overline{G'\left(\frac{r^2}{\overline{\zeta}}\right)},$$

and

$$\widehat{G}_{\overline{\zeta}} = \frac{r^2}{\overline{\zeta}^2}\frac{1}{n}H'\left(\frac{r^2}{\overline{\zeta}}\right)\left(\frac{\zeta^n\overline{\zeta}^n}{r^{2n}}-1\right)\left(n-1-\frac{r^2}{\overline{\zeta}}\frac{H''\left(\frac{r^2}{\overline{\zeta}}\right)}{H'\left(\frac{r^2}{\overline{\zeta}}\right)}\right).$$

Let $z = r^2/\overline{\zeta}$, *i.e.* $|z| \geq r$. Hence,

$$\widehat{G}_\zeta = \frac{r^{2(n-1)}}{z^{n-1}\overline{z}^{n-1}}H'(z) - \frac{\overline{z}^2}{r^2}\overline{G'(z)} = \frac{r^{2(n-1)}}{|z|^{2(n-1)}}H'(z) - \frac{\overline{z}^2}{r^2}\overline{G'(z)},$$

and

$$\widehat{G}_{\overline{\zeta}} = \frac{1}{n}\frac{z^2}{r^2}H'(z)\left(\frac{r^{2n}}{z^n\overline{z}^n}-1\right)\left(n-1-\zeta\frac{H''(z)}{H'(z)}\right).$$

It is straightforward that

$$\frac{|\widehat{G}_{\overline{\zeta}}|}{|G_\zeta|} = \frac{\left|z^2/r^2H'(z)\left(r^{2n}/|z|^{2n}-1\right)\left(n-1-zH''(z)/H'(z)\right)\right|}{n\left|r^{2(n-1)}/|z|^{2(n-1)}H'(z)-\overline{z}^2/r^2\overline{G'(z)}\right|} \leq$$

$$\leq \frac{\left|\left(r^{2n}/|z|^{2n}-1\right)\left(n-1-zH''(z)/H'(z)\right)\right|}{n\left(r^{2n}/|z|^{2n}-\left|\overline{G'(z)}/H'(z)\right|\right)}.$$

We need to check that

$$n\frac{|z|^{2n}}{r^{2n}}\left|G'(z)/H'(z)\right| + \left|\left(\frac{|z|^{2n}}{r^{2n}}-1\right)\left(n-1-z\frac{H''(z)}{H'(z)}\right)\right| \leq n, \qquad |z| \geq r.$$

Since

$$\frac{|z|^{2n}}{r^{2n}} \geq |z|^{2n}, \qquad \frac{|z|^{2n}}{r^{2n}} - 1 \geq |z|^{2n} - 1,$$

positivity of the Jacobian follows from the condition of Theorem 3. Using Lemma A and Stoilow's theorem, we obtain that F is n-valent in $\mathbb{D}^-$.

Proof of Theorem 4. Consider the mapping $\widehat{F} : \mathbb{C} \to \mathbb{C}$ given by

$$\widehat{F}(\zeta) = \begin{cases} F(\zeta), & |\zeta| \geq r, \\ F\left(\frac{r^2}{\overline{\zeta}}\right) + A(\zeta), & |\zeta| \leq r, \end{cases}$$

where $r \in (1, \infty)$ and

$$A(\zeta) = \left(\zeta^n - \frac{r^{2n}}{\overline{\zeta}^n}\right)H'\left(\frac{r^2}{\overline{\zeta}}\right)\frac{\overline{\zeta}^{n-1}}{nr^{2(n-1)}} + \left(\overline{\zeta}^n - \frac{r^{2n}}{\zeta^n}\right)G'\left(\frac{r^2}{\zeta}\right)\frac{\zeta^{n-1}}{nr^{2(n-1)}}.$$

Obviously, the function $\widehat{F}$ have a pole of order n at the point $\zeta = \infty$ and

$$\lim_{\zeta\to 0}\zeta^n\widehat{F}(\zeta) = const \in \mathbb{C}\setminus\{0\}.$$

Since $|H'(\zeta)| - |G'(\zeta)| > 0$ it follows that $\widehat{F}$ is locally univalent in $|\zeta| \geq r$. Now we will prove that the Jacobian is positive in $|\zeta| \leq r$. By straightforward computations, we obtain that

$$\widehat{F}_\zeta = \frac{(\zeta\overline{\zeta})^{n-1}}{r^{2(n-1)}} H'\left(\frac{r^2}{\overline{\zeta}}\right) + \frac{n-1}{n}\left(\frac{\overline{\zeta}^n \zeta^{n-2}}{r^{2(n-1)}} - \frac{r^2}{\zeta^2}\right) G'\left(\frac{r^2}{\zeta}\right) -$$
$$-\frac{r^2}{n\zeta^2}\left(\frac{\overline{\zeta}^n \zeta^{n-1}}{r^{2(n-1)}} - \frac{r^2}{\zeta}\right) G''\left(\frac{r^2}{\zeta}\right)$$

and

$$\widehat{F}_{\overline{\zeta}} = \frac{(\zeta\overline{\zeta})^{n-1}}{r^{2(n-1)}} \overline{G'\left(\frac{r^2}{\overline{\zeta}}\right)} + \frac{n-1}{n}\left(\frac{\zeta^n \overline{\zeta}^{n-2}}{r^{2(n-1)}} - \frac{r^2}{\overline{\zeta}^2}\right) H'\left(\frac{r^2}{\overline{\zeta}}\right) -$$
$$-\frac{r^2}{n\overline{\zeta}^2}\left(\frac{\overline{\zeta}^n \zeta^{n-1}}{r^{2(n-1)}} - \frac{r^2}{\overline{\zeta}}\right) H''\left(\frac{r^2}{\overline{\zeta}}\right).$$

Let $z = r^2/\overline{\zeta}$, *i.e.* $|z| \geq r$. Hence,

$$|\widehat{F}_\zeta| \geq \frac{r^{2(n-1)}}{|z|^{2(n-1)}}|H'(z)| - \left|\frac{r^{2n}}{|z|^{2n}} - 1\right| \left(\frac{n-1}{n}\frac{|z|^2}{r^2}|G'(z)| + \frac{|z|^3}{nr^2}|G''(z)|\right)$$

and

$$\left|\widehat{F}_{\overline{\zeta}}\right| \leq \frac{r^{2(n-1)}}{|z|^{n-1}}|G'(z)| + \left|\frac{r^{2n}}{|z|^{2n}} - 1\right| \left(\frac{n-1}{n}\frac{|z|^2}{r^2}|H'(z)| + \frac{|z|^3}{nr^2}|H''(z)|\right).$$

Consequently, we need to check that

$$\frac{r^{2(n-1)}}{|z|^{2(n-1)}}|G'(z)| + \left|\frac{r^{2n}}{|z|^{2n}} - 1\right| \left(\frac{n-1}{n}\frac{|z|^2}{r^2}|H'(z)| + \frac{|z|^3}{nr^2}|H''(z)|\right) \leq$$
$$\leq \frac{r^{2(n-1)}}{|z|^{2(n-1)}}|H'(z)| - \left|\frac{r^{2n}}{|z|^{2n}} - 1\right| \left(\frac{n-1}{n}\frac{|z|^2}{r^2}|G'(z)| + \frac{|z|^3}{nr^2}|G''(z)|\right).$$

Since

$$|H''(z)| + |G''(z)| \leq n\frac{|H'(z)| - |G'(z)|}{|z|^{2n+1} - |z|} - (n-1)\frac{|H'(z)| + |G'(z)|}{|z|}, \quad |z| \geq r,$$

we get the positivity of the Jacobian for $|\zeta| \leq r$. Therefore, using Lemma A and Stoilow's theorem, we obtain that F is n-valent in $\mathbb{D}^-$.

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