ON SATURATIONS OF SUBMODULES

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Some properties of saturations of submodules of modules is investigated. In particular, saturations of the zero submodule of injective modules and the associated prime ideals of saturations of submodules of Noetherian modules are studied.

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1. INTRODUCTION

Throughout the paper, R is a commutative ring with nonzero identity, all modules are unitary and $\mathfrak{p}$ denotes a prime ideal of R. Moreover, the set of all ideals of R not contained in $\mathfrak{p}$ is designated by $S(\mathfrak{p})$. It is clear that $S(\mathfrak{p})$ is a multiplicatively closed set of ideals of R.

The notion of saturation of ideals (or submodules) has been studied in many literature, such as [1, 5, 7]. Let N be a submodule of an R-module M. Then the *saturation of N with respect to* $\mathfrak{p}$ is the contraction of $N_{\mathfrak{p}}$ in M and designated by $S_{\mathfrak{p}}(N)$. It is known that $S_{\mathfrak{p}}(N) = \{e \in M \mid se \in N \text{ for some } s \in R \setminus \mathfrak{p}\}$. For more detail, we refer the reader to [5]. The submodule N is said to be $\mathfrak{p}$-*saturated* if $S_{\mathfrak{p}}(N) = N$.

A *prime* submodule (or a $\mathfrak{p}$-*prime* submodule) of M is a proper submodule P with $(P :_R M) = \mathfrak{p}$ such that whenever $re \in P$ for $r \in R$ and $e \in M$, either $e \in P$ or $r \in \mathfrak{p}$. If P is a prime submodule, then $\mathfrak{p} = (P :_R M)$ is necessarily a prime ideal of R (see [4]). C.P. Lu in [5] investigated the relationship between prime submodules and saturations of submodules.

In this short paper, some properties of saturations of submodules of modules are investigated. In Section 2, we show that if $\operatorname{Spec}(R) \setminus V(R \setminus \mathfrak{p})$ is a Noetherian scheme and for any finitely generated module, the saturation of the zero submodule with respect to $\mathfrak{p}$ is finitely generated, then for any finitely generated R-module M and any submodule N of M, $S_{\mathfrak{p}}(N)$ is finitely generated (Theorem 2.6). In Theorem 2.7, we prove that if R is a Noetherian ring and M is an injective R-module, then $S_{\mathfrak{p}}(0)$ is an injective R-module. Also, the associated prime ideals of saturations of submodules of Noetherian modules is studied (see Proposition 2.9 and Theorem 2.11).

2. MAIN RESULTS

We begin by some properties of saturations of submodules.

PROPOSITION 2.1. *Let M be an R-module and N a submodule of M.*

1. $S_{\mathfrak{p}}(N :_M J) = (S_{\mathfrak{p}}(N) :_M J)$ *for each finitely generated ideal J of R.*
2. *Let S be a multiplicatively closed subset of R. If M is Noetherian and N is $\mathfrak{p}$-saturated, then $S^{-1}N \cap M$ is $\mathfrak{p}$-saturated.*
3. *Let $A = \{N \leq M \mid S_{\mathfrak{p}}(N) = N \neq M\}$. Then the maximal elements of A are prime submodules.*
4. *Let $\{N_i\}_{i=1}^n$ be a finite family of submodules of M. Then $S_{\mathfrak{p}}(\bigcap_{i=1}^n N_i) = \bigcap_{i=1}^n S_{\mathfrak{p}}(N_i)$.*

Proof. (1) Let $m \in S_{\mathfrak{p}}(N :_M J)$. Then $sm \in (N :_M J)$ for some $s \in R\setminus \mathfrak{p}$. Thus $sJm \subseteq N$. So, $Jm \subseteq S_{\mathfrak{p}}(N)$. This yields that $m \in (S_{\mathfrak{p}}(N) :_M J)$.

Now, suppose $x \in (S_{\mathfrak{p}}(N) :_M J)$ and J is generated by $c_1, \dots, c_t$. Then there are $b_1, \dots, b_t \in R\setminus\mathfrak{p}$ such that $b_i c_i x \in N$. Set $b := b_1 \cdots b_t$. Then $b \in R\setminus\mathfrak{p}$ and $Jbx \subseteq N$. Therefore, $bx \in (N :_M J)$ and so, $x \in S_{\mathfrak{p}}(N :_M J)$.

(2) By assumption, there are elements $g_1, \dots, g_t \in M$ such that $S^{-1}N \cap M = \sum_{i=1}^t Rg_i$. By [7, p. 137, Proposition 1], there are $s_1, \dots, s_t \in S$ such that $s_i g_i \in N$ for each $1 \leq i \leq t$. Set $s = s_1 \cdots s_t$. Then $s \in S$ and $s(S^{-1}N \cap M) \subseteq N$. This implies that $S^{-1}N \cap M = (N :_M s)$. Hence,

$$\begin{aligned} S_{\mathfrak{p}}(S^{-1}N \cap M) &= S_{\mathfrak{p}}(N :_M s) \\ &= (S_{\mathfrak{p}}(N) :_M s) \qquad \text{(by part ((1)))} \\ &= (N :_M s) = S^{-1}N \cap M. \end{aligned}$$

(3) Let P be a maximal element of A and $rm \in P$, where $r \in R$ and $m \in M \setminus P$. We are going to show that $r \in (P :_R M)$. According to part ((1)) we have

$$(P :_M Rr) = (S_{\mathfrak{p}}(P) :_M Rr) = S_{\mathfrak{p}}(P :_M Rr).$$

Thus $(P :_M Rr)$ is $\mathfrak{p}$-saturated. By our choice of m we have $P \subsetneqq (P :_M Rr)$, since $m \in (P :_M Rr) \setminus P$. By maximality of P, we infer that $(P :_M Rr) = M$. Thus $rM \subseteq P$, as desired.

(4) Obviously, $S_{\mathfrak{p}}(\bigcap_{i=1}^n N_i) \subseteq \bigcap_{i=1}^n S_{\mathfrak{p}}(N_i)$. Conversely, if $m \in \bigcap_{i=1}^n S_{\mathfrak{p}}(N_i)$, then there exist $b_1, \dots, b_n \in R \setminus \mathfrak{p}$ such that $b_i m \in N_i$ for each $1 \leq i \leq n$. Then $b := b_1 \cdots b_n \in R \setminus \mathfrak{p}$. This shows that $bm \in \bigcap_{i=1}^n N_i$. Therefore, $m \in S_{\mathfrak{p}}(\bigcap_{i=1}^n N_i)$. $\square$

Proposition 2.2. *Let M be an R-module. Then for each submodule N of M and ideal I of R we have $IS_{\mathfrak{p}}(N) \subseteq S_{\mathfrak{p}}(IN)$. In particular, $S_{\mathfrak{p}}(IS_{\mathfrak{p}}(N)) = S_{\mathfrak{p}}(IN)$.*

Proof. Let $m \in IS_{\mathfrak{p}}(N)$. Then there are $t \in \mathbb{N}$, $a_1, \ldots, a_t \in I$ and $m_1, \ldots, m_t \in S_{\mathfrak{p}}(N)$ such that $m = \sum_{i=1}^{t} a_i m_i$. So, there are $b_1, \ldots, b_t \in R \setminus \mathfrak{p}$ such that $b_i m_i \in N$ for each $1 \leq i \leq t$. Set $b = \prod_{i=1}^{t} b_i$. Then $bm_i \in N$ for each $1 \leq i \leq t$. This implies that $bm \in IN$. Therefore, $m \in S_{\mathfrak{p}}(IN)$. This completes the proof. $\square$

Proposition 2.3. *Let M be an R-module, N a submodule of M and $I \in S(\mathfrak{p})$ be a finitely generated ideal. Then we have $S_{\mathfrak{p}}(N) = (IS_{\mathfrak{p}}(N) :_M I) = (S_{\mathfrak{p}}(IN) :_M I)$.*

Proof. According to Proposition 2.2, we have

$$S_{\mathfrak{p}}(N) \subseteq (IS_{\mathfrak{p}}(N) :_M I) \subseteq (S_{\mathfrak{p}}(IN) :_M I).$$

Now, suppose that $m \in (S_{\mathfrak{p}}(IN) :_M I)$ and I is generated by $x_1, \ldots, x_t \in R$. Then $Im \subseteq S_{\mathfrak{p}}(IN)$ and there are elements $b_1, \ldots, b_t \in R \setminus \mathfrak{p}$ such that $b_i x_i m \in IN$. Set $b = b_1 \cdots b_t$. Then $bx_i m \in IN \subseteq N$. This implies that $bIm \subseteq N$ and so $m \in S_{\mathfrak{p}}(N)$, since $bI \in S(\mathfrak{p})$. This completes the proof. $\square$

Proposition 2.4. *Let M be an R-module, N, L be two submodules of M and $I \in S(\mathfrak{p})$ be a finitely generated ideal. If $IN \subseteq S_{\mathfrak{p}}(IL)$, then $N \subseteq S_{\mathfrak{p}}(L)$. In particular, if $S_{\mathfrak{p}}(IN) = S_{\mathfrak{p}}(IL)$, then $S_{\mathfrak{p}}(N) = S_{\mathfrak{p}}(L)$.*

Proof. Assume that $IN \subseteq S_{\mathfrak{p}}(IL)$. Then $N \subseteq (S_{\mathfrak{p}}(IL) :_M I) = S_{\mathfrak{p}}(L)$ by Proposition 2.3.

For the second assertion, if $S_{\mathfrak{p}}(IN) = S_{\mathfrak{p}}(IL)$, then $IN \subseteq S_{\mathfrak{p}}(IL)$. So, $N \subseteq S_{\mathfrak{p}}(L)$ by the first part of the proposition. This yields that

$$S_{\mathfrak{p}}(N) \subseteq S_{\mathfrak{p}}(S_{\mathfrak{p}}(L)) = S_{\mathfrak{p}}(L).$$

Thus, by similarity, we infer that $S_{\mathfrak{p}}(N) = S_{\mathfrak{p}}(L)$. $\square$

The converse of Proposition 2.4 also holds. More precisely, if $S_{\mathfrak{p}}(N) = S_{\mathfrak{p}}(L)$, then $KS_{\mathfrak{p}}(N) = KS_{\mathfrak{p}}(L)$ for all ideals K of R. Thus,

$$S_{\mathfrak{p}}(KN) = S_{\mathfrak{p}}(KS_{\mathfrak{p}}(N)) = S_{\mathfrak{p}}(KS_{\mathfrak{p}}(L)) = S_{\mathfrak{p}}(KL),$$

by Proposition 2.2.

Proposition 2.5. *Let M be a Noetherian R-module and N a submodule of M. Then there exists an ideal $J \in S(\mathfrak{p})$ such that*

(1) *$S_{\mathfrak{p}}(N) = (N :_M J)$;*

(2) *$S_{\mathfrak{p}}(N) = (S_{\mathfrak{p}}(N) :_M J) = (S_{\mathfrak{p}}(JN) :_M J) = (JS_{\mathfrak{p}}(N) :_M J)$;*

(3) $S_{\mathfrak{p}}(N) = (KN :_M KJ)$ *for all* $K \in S(\mathfrak{p})$.

Proof. (1) It is straightforward.

(2) According to Proposition 2.2 and part (1), we have

$$S_{\mathfrak{p}}(N) \subseteq (JS_{\mathfrak{p}}(N) :_M J) \subseteq (S_{\mathfrak{p}}(JN) :_M J) \subseteq (S_{\mathfrak{p}}(N) :_M J) = S_{\mathfrak{p}}(N).$$

(3) Let $K \in S(\mathfrak{p})$. Then $KJ \in S(\mathfrak{p})$ and by part(1) we have

$$\begin{aligned} S_{\mathfrak{p}}(N) &= (N :_M J) \subseteq (K(N :_M J) :_M K) \subseteq (KN :_M KJ) \\ &\subseteq (N :_M KJ) \subseteq \bigcup_{a \in R \setminus \mathfrak{p}} (N :_M a) = S_{\mathfrak{p}}(N). \end{aligned}$$

This completes the proof. □

The next theorem is one of the main results of the paper. Recall that a scheme X is *locally Noetherian* if it can be covered by open affine subsets $\mathrm{Spec}(A_i)$, where each A_i is a Noetherian ring. X is *Noetherian* if it is locally Noetherian and quasi-compact. Equivalently, X is Noetherian if it can be covered by a finite number of open affine subsets $\mathrm{Spec}(A_i)$, with each A_i a Noetherian ring (see [3]). In the sequel, $\mathbb{N}_0$ denotes the set of all non-negative integers.

Theorem 2.6. *Let I be a finitely generated ideal of R such that $I \cap \mathfrak{p} = \{0\}$. Then the following statements are equivalent:*

(1) $\mathrm{Spec}(R) \setminus V(I)$ *is a Noetherian scheme and for any finitely generated R-module M, $S_{\mathfrak{p}}(0_M)$ is finitely generated;*

(2) *for any finitely generated R-module M and any submodule N of M, $S_{\mathfrak{p}}(N)$ is finitely generated;*

(3) *for any finitely generated R-module M, $M/S_{\mathfrak{p}}(0_M)$ is finitely presented.*

Proof. (1) ⇒ (2) Let M be a finitely generated R-module and N a submodule of M. Note that $\mathrm{Spec}(R) \setminus V(I)$ is quasi-compact, since I is finitely generated. By [3, Proposition 3.2], there are non-zero elements $a_1, \ldots, a_n \in I$ such that $\mathrm{Spec}(R) \setminus V(I)$ is covered by affine open subsets $\mathrm{Spec}(R_{a_i})$ and R_{a_i} is a Noetherian ring, where $R_{a_i} = S_i^{-1}R$ and $S_i = \{a_i^t \,|\, t \in \mathbb{N}_0\}$. Thus N_{a_i} is a finitely generated submodule of the Noetherian R_{a_i}-module M_{a_i}. So, as it is known, there exists a finitely generated submodule L of N such that $N_{a_i} = L_{a_i}$. Obviously, $S_{\mathfrak{p}}(L) \subseteq S_{\mathfrak{p}}(N)$. Let $m \in S_{\mathfrak{p}}(N)$. Then there exists $t \in R \setminus \mathfrak{p}$ such that $tm \in N$. Therefore, $\frac{t}{1}\frac{m}{1} = S_i^{-1}(tm) \in N_{a_i} = L_{a_i}$. There exists $r \in \mathbb{N}_0$ such that $a_i^r(tm) \in L$. Thus $m \in S_{\mathfrak{p}}(L)$, since $a_i^r t \in R \setminus \mathfrak{p}$. This yields that $S_{\mathfrak{p}}(N) = S_{\mathfrak{p}}(L)$. So it is enough for us to show that $S_{\mathfrak{p}}(L)$ is finitely generated. We deduce from the following short exact sequence

$$0 \to L \to S_{\mathfrak{p}}(L) \to \frac{S_{\mathfrak{p}}(L)}{L} \to 0$$

that $S_{\mathfrak{p}}(L)$ is finitely generated, since L is finitely generated (as we mentioned) and by assumption $\frac{S_{\mathfrak{p}}(L)}{L} = S_{\mathfrak{p}}(0_{\frac{M}{L}})$ is finitely generated too.

(2) ⇒ (1) We must show that $\mathrm{Spec}(R) \setminus V(I)$ is a Noetherian scheme. By definition, it is enough to show that R_a is a Noetherian ring for each $a \in I$. To do this, we will show that every ideal of R_a is finitely generated. Let H be an ideal of R_a and $H' = H \cap R$ be the contraction of H by the natural map $R \to R_a$. It is easy to verify that $H' = S_{\mathfrak{p}}(H')$ and so H' is a finitely generated ideal of R, by (2). Therefore, $H = H'R_a$ is finitely generated.

(3) ⇒ (2) Let M be a finitely generated R-module and N a submodule of M. Then, the R-module

$$\frac{M}{S_{\mathfrak{p}}(N)} \cong \frac{M/N}{S_{\mathfrak{p}}(N)/N} = \frac{M/N}{S_{\mathfrak{p}}(0_{M/N})}$$

is finitely presented, by (3). Consider the following short exact sequence

$$0 \to S_{\mathfrak{p}}(N) \to M \to \frac{M}{S_{\mathfrak{p}}(N)} \to 0.$$

Then, by [2, Chap. I, §2.8, Lemma 9], we infer that $S_{\mathfrak{p}}(N)$ is finitely generated.

(2) ⇒ (3) Let M be a finitely generated R-module. Then there exists a finitely generated free R-module F_0 such that the sequence $F_0 \xrightarrow{f} \frac{M}{S_{\mathfrak{p}}(0)} \to 0$ is exact. We claim that $\ker(f)$ is finitely generated. Let $m \in S_{\mathfrak{p}}(\ker(f))$. Then there exists $t \in R \setminus \mathfrak{p}$ such that $tm \in \ker(f)$. Thus $tf(m) = f(tm) = 0_{M/S_{\mathfrak{p}}(0)}$. This shows that $m \in \ker(f)$. Therefore, $\ker(f) = S_{\mathfrak{p}}(\ker(f))$ and so is finitely generated by (2). Again, there exists a finitely generated free R-module F_1 such that the sequence $F_1 \to \ker(f) \to 0$ is exact. A standard technique shows that the combined sequence $F_1 \to F_0 \to \frac{M}{S_{\mathfrak{p}}(0)} \to 0$ is exact. Consequently, $M/S_{\mathfrak{p}}(0)$ is finitely presented. □

In the next theorem, we consider the saturations of the zero submodule of injective modules.

THEOREM 2.7. *Let R be a Noetherian ring. If M is an injective R-module, then so is $S_{\mathfrak{p}}(0_M)$.*

Proof. We recall the following well-known facts:

(1) If R is a commutative Noetherian ring and M is an injective R-module, then it is a direct sum of modules of the form $E(R/\mathfrak{q})$ for $\mathfrak{q} \in \mathrm{Spec}(R)$.

(2) Fix $\mathfrak{p} \in \mathrm{Spec}(R)$, then for any $\mathfrak{q} \in \mathrm{Spec}(R)$,

$$E(R/\mathfrak{q})_{\mathfrak{p}} = \begin{cases} E(R/\mathfrak{q}) & \mathfrak{q} \subseteq \mathfrak{p} \\ 0 & \text{otherwise} \end{cases}$$

(3) If $\mathfrak{p} \in \operatorname{Spec}(R)$ and M is an R-module, then $S_{\mathfrak{p}}(0)$ is exactly the kernel of the canonical localization map $M \to M_{\mathfrak{p}}$.

If we combine the above facts it follows that when M is an injective module, then $S_{\mathfrak{p}}(0)$ is exactly the direct sum of all direct summands of M that are isomorphic to $E(R/\mathfrak{q})$ for $\mathfrak{q} \nsubseteq \mathfrak{p}$ so it is a direct summand of M, hence injective (see [6, Theorem 18.5]). □

Corollary 2.8. *Let R be a Noetherian ring and M be an injective R-module. Then the canonical exact sequence $0 \to S_{\mathfrak{p}}(0) \to M \to \frac{M}{S_{\mathfrak{p}}(0)} \to 0$ splits.*

Proof. Use Theorem 2.7. □

In the sequel, among other things, we provide some results on the associated prime ideals of $M/S_{\mathfrak{p}}(N)$.

Proposition 2.9. *Let R be a Noetherian ring and M be an R-module. Then the following statements hold:*

(1) $\mathfrak{q} \in \operatorname{Ass}(S_{\mathfrak{p}}(0))$ *if and only if* $\mathfrak{q} \in \operatorname{Ass}(M)$ *and* $\mathfrak{q} \nsubseteq \mathfrak{p}$*;*

(2) $\mathfrak{q} \in \operatorname{Ass}(M/S_{\mathfrak{p}}(0))$ *if and only if* $\mathfrak{q} \in \operatorname{Ass}(M)$ *and* $\mathfrak{q} \subseteq \mathfrak{p}$*.*

Proof. (1) Let $\mathfrak{q} \in \operatorname{Ass}(S_{\mathfrak{p}}(0))$. Then there exists a nonzero element $m \in S_{\mathfrak{p}}(0)$ such that $\mathfrak{q} = (0 :_R m)$. So, there is an element $t \in R \setminus \mathfrak{p}$ such that $tm = 0$. Thus, $t \in \mathfrak{q}$. This shows that $\mathfrak{q} \in \operatorname{Ass}(M)$ and $\mathfrak{q} \nsubseteq \mathfrak{p}$.

Conversely, suppose that $\mathfrak{q} \in \operatorname{Ass}(M)$ and $\mathfrak{q} \nsubseteq \mathfrak{p}$. Then, there is a nonzero element $m \in M$ and $t \in (R \setminus \mathfrak{p}) \cap \mathfrak{q}$ such that $\mathfrak{q} = (0 :_R m) \supseteq t$. Hence, $tm = 0$. Therefore, $m \in S_{\mathfrak{p}}(0)$. This implies that $\mathfrak{q} \in \operatorname{Ass}(S_{\mathfrak{p}}(0))$.

(2) Let $\mathfrak{q} \in \operatorname{Ass}(M)$ and $\mathfrak{q} \subseteq \mathfrak{p}$. According to the canonical exact sequence

$$0 \to S_{\mathfrak{p}}(0) \to M \to \frac{M}{S_{\mathfrak{p}}(0)} \to 0,$$

we have $\operatorname{Ass}(M) \subseteq \operatorname{Ass}(S_{\mathfrak{p}}(0)) \cup \operatorname{Ass}(M/S_{\mathfrak{p}}(0))$. Since $\mathfrak{q} \subseteq \mathfrak{p}$, we infer from (1) that $\mathfrak{q} \in \operatorname{Ass}(M/S_{\mathfrak{p}}(0))$.

Conversely, suppose that $\mathfrak{q} \in \operatorname{Ass}(M/S_{\mathfrak{p}}(0))$. Then, there exists $y \in M \setminus S_{\mathfrak{p}}(0)$ such that $\mathfrak{q} = (S_{\mathfrak{p}}(0) :_R Ry)$. Note that $x \in (S_{\mathfrak{p}}(0) :_R Ry)$ if and only if $xy \in S_{\mathfrak{p}}(0)$ if and only if $txy = 0$ for some $t \in R \setminus \mathfrak{p}$ if and only if $x \in S_{\mathfrak{p}}(0 :_R Ry)$. Therefore, $\mathfrak{q} = S_{\mathfrak{p}}(0 :_R Ry)$. Since R is Noetherian, by Proposition 2.5, there exists $J \in S(\mathfrak{p})$ such that $\mathfrak{q} = ((0 :_R Ry) :_R J)$ and $J = \sum_{i=1}^{l} Rr_i$ where $r_i \in R$ for each $i \in \{1, \dots l\}$. Thus,

$$\mathfrak{q} = \bigcap_{i=1}^{l} ((0 :_R Ry) :_R Rr_i).$$

So, $\mathfrak{q} = ((0 :_R Ry) :_R Rr_j) = (0 :_R r_j y)$ for some $j \in \{1, \dots l\}$. Consequently, $\mathfrak{q} \in \mathrm{Ass}(M)$. We claim that $\mathfrak{q} \subseteq \mathfrak{p}$. For, if $t \in (R \setminus \mathfrak{p}) \cap \mathfrak{q}$, then $t \in (S_{\mathfrak{p}}(0) :_R Ry)$. Thus $ty \in S_{\mathfrak{p}}(0)$ and so there is $s \in R \setminus \mathfrak{p}$ such that $sty = 0$. This implies that $y \in S_{\mathfrak{p}}(0)$, since $st \in R \setminus \mathfrak{p}$. This is a contradiction. $\square$

COROLLARY 2.10. *Let R be a Noetherian ring and M be an R-module. Then* $\mathrm{Ass}(S_{\mathfrak{p}}(0))$ *and* $\mathrm{Ass}(M/S_{\mathfrak{p}}(0))$ *are disjoint, and*

$$\mathrm{Ass}(M) = \mathrm{Ass}(S_{\mathfrak{p}}(0)) \cup \mathrm{Ass}(M/S_{\mathfrak{p}}(0)).$$

Proof. Use Proposition 2.9. $\square$

THEOREM 2.11. *Let M be a Noetherian R-module and N be a submodule of M. Then the following hold:*

(1) *If $\mathfrak{q} \in \mathrm{Ass}(\frac{M}{S_{\mathfrak{p}}(N)})$, then $\mathfrak{q} \in \mathrm{Ass}(\frac{M}{JS_{\mathfrak{p}}(N)}) \cap \mathrm{Ass}(\frac{M}{S_{\mathfrak{p}}(JN)})$ for all finitely generated ideals $J \in S(\mathfrak{p})$;*

(2) *If $\mathfrak{q} \in \mathrm{Ass}(\frac{M}{S_{\mathfrak{p}}(N)})$, then $\mathfrak{q} \in \mathrm{Ass}(\frac{M}{JN})$ for all $J \in S(\mathfrak{p})$.*

Proof. (1) Let $\mathfrak{q} \in \mathrm{Ass}(\frac{M}{S_{\mathfrak{p}}(N)})$ and let $J \in S(\mathfrak{p})$ be a finitely generated ideal. Then there exists $m \in M$ such that $\mathfrak{q} = (S_{\mathfrak{p}}(N) :_R m)$. This implies that

$$\begin{aligned} \mathfrak{q} = (S_{\mathfrak{p}}(N) :_R m) &\subseteq (JS_{\mathfrak{p}}(N) :_R Jm) \\ &\subseteq (S_{\mathfrak{p}}(JN) :_R Jm) \qquad \text{(by Proposition 2.2)} \\ &= ((S_{\mathfrak{p}}(JN) :_M J) :_R m) \\ &= (S_{\mathfrak{p}}(N) :_R m) \qquad \text{(by Proposition 2.3)} \\ &= \mathfrak{q}. \end{aligned}$$

Since M is Noetherian, Jm is a finitely generated submodule of M. Suppose $Jm = \sum_{i=1}^{t} Rg_i$, where $g_i \in M$ for each $i \in \{1, \dots, t\}$. Then

$$\mathfrak{q} = (JS_{\mathfrak{p}}(N) :_R Jm) = \cap_{i=1}^{t} (JS_{\mathfrak{p}}(N) :_R Rg_i).$$

Thus, there exists $j \in \{1, \dots, t\}$ such that $\mathfrak{q} = (JS_{\mathfrak{p}}(N) :_R g_j)$. Therefore, $\mathfrak{q} \in \mathrm{Ass}(\frac{M}{JS_{\mathfrak{p}}(N)})$. Similarly, $\mathfrak{q} \in \mathrm{Ass}(\frac{M}{S_{\mathfrak{p}}(JN)})$.

(2) By Proposition 2.5, there exists $K \in S(\mathfrak{p})$ such that $S_{\mathfrak{p}}(N) = (JN :_M JK)$ for all $J \in S(\mathfrak{p})$. Suppose that $\mathfrak{q} \in \mathrm{Ass}(\frac{M}{S_{\mathfrak{p}}(N)})$ and $J \in S(\mathfrak{p})$. Then there exists $m \in M$ such that

$$\mathfrak{q} = (S_{\mathfrak{p}}(N) :_R m) = ((JN :_M JK) :_R m) = (JN :_M JKm).$$

Hence $\mathfrak{q} \in \mathrm{Ass}(\frac{M}{JN})$, since JKm is finitely generated. $\square$

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