ON THE LOCAL-GLOBAL PRINCIPLE FOR THE LOCAL COHOMOLOGY MODULES

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Communicated by Marian Aprodu

Let R be a commutative Noetherian ring. We wish to investigate the minimaxness and weakly Laskerianness of local cohomology modules. We prove that the local-global principle for the minimaxness of local cohomology modules, introduced in [15], holds at level $n \in \mathbb{N}$ over R if $H_I^i(M)$ is I-cominimax for all $i < n$. Also, we introduce the concept of local-global principle for the weakly Laskerianness of local cohomology modules over a commutative Noetherian ring R and we show that if R is semi-local, then this principle holds at level 2. Moreover, if $\dim R \leq 3$, we establish this principle at all levels over R.

AMS 2020 Subject Classification: 13D45, 13C13, 13C99.

Key words: local cohomology modules, local-global principle, minimax modules, weakly Laskerian modules.

1. INTRODUCTION

Let R be a commutative Noetherian ring with identity and I be an ideal of R. We will denote by $\mathbb{N}$ (resp. $\mathbb{N}_0$) the set of all positive (non-negative) integers. For an R-module M, the ith local cohomology module of M with respect to I is defined as

$$H_I^i(M) \cong \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}_R^i(R/I^n, M).$$

We refer the reader to [11] for more details about the local cohomology. An important theorem in local cohomology is Faltings' local-global principal for the finiteness of local cohomology modules [16, Satz 1], which states that for a finitely generated R-module M and a positive integer n, the R-module $H_{IR_\mathfrak{p}}^i(M_\mathfrak{p})$ is finitely generated for all $i \leq n$ and for all $\mathfrak{p} \in \operatorname{Spec}(R)$ if and only if the R-module $H_I^i(M)$ is finitely generated for all $i \leq n$.

Another formulation of Faltings' local-global principle, particularly relevant for this paper, is in terms of the generalization of the finiteness dimension $f_I(M)$ of M relative to I, where

$$f_I(M) := \inf\{i \in \mathbb{N}_0 \mid H_I^i(M) \text{ is not finitely generated}\},$$

with the usual convention that the infimum of the empty set of integers is interpreted as ∞. In view of [11, 9.6.2], it follows that for a finitely generated R-module M,

$$\begin{aligned} f_I(M) &:= \inf\{i \in \mathbb{N}_0 \mid I^t H_I^i(M) \neq 0 \text{ for all } t \in \mathbb{N}_0\} \\ &= \inf\{f_{IR_\mathfrak{p}}(M_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\} \\ &= \inf\{f_{IR_\mathfrak{p}}(M_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Supp}_R(M/IM) \text{ and } \dim R/\mathfrak{p} \geq 0\}. \end{aligned}$$

Let M be a finitely generated R-module and J be a second ideal of R such that $J \subseteq I$. Based on [11, Definition 9.1.5], the J-finiteness dimension $f_I^J(M)$ of M relative to I is defined by

$$f_I^J(M) := \inf\{i \in \mathbb{N}_0 \mid J^t H_I^i(M) \neq 0 \text{ for all } t \in \mathbb{N}_0\}.$$

Let n be a non-negative integer. As another generalization of the notion of $f_I(M)$, Bahmanpour et al. introduced in [8] the notion of the nth finiteness dimension $f_I^n(M)$ of M relative to I by

$$f_I^n(M) := \inf\{f_{IR_\mathfrak{p}}(M_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Supp}_R(M/IM) \text{ and } \dim R/\mathfrak{p} \geq n\}.$$

Note that $f_I^J(M)$ and $f_I^n(M)$ are either a positive integer or ∞ and $f_I^0(M) = f_I(M) = f_I^I(M)$. They showed that

$$f_I^1(M) = \inf\{i \in \mathbb{N}_0 \mid H_I^i(M) \text{ is not minimax}\}$$

and

$$f_I^2(M) = \inf\{i \in \mathbb{N}_0 \mid H_I^i(M) \text{ is not weakly Laskerian}\},$$

whenever R is semi-local. Recently, Doustimehr and Naghipour proved in [15] that

$$f_I^1(M) = \inf\{i \in \mathbb{N}_0 \mid I^t H_I^i(M) \text{ is not minimax for all } t \in \mathbb{N}\}$$

and introduced the J-minimaxness dimension $\mu_I^J(M)$ of M relative to I (as a generalization of J-finiteness dimension $f_I^J(M)$) by

$$\mu_I^J(M) := \inf\{i \in \mathbb{N}_0 \mid J^t H_I^i(M) \text{ is not minimax for all } t \in \mathbb{N}\}.$$

In view of the above definition, it is natural to ask: Is there a greatest integer i such that $I^t H_I^i(M)$ is weakly Laskerian for some integer $t \in \mathbb{N}$, and if so, what is it?

In this paper, we prove that

$$f_I^2(M) = \inf\{i \in \mathbb{N}_0 \mid I^t H_I^i(M) \text{ is not weakly Laskerian for all } t \in \mathbb{N}\},$$

whenever R is a semi-local ring. So, the above definition motivates to introduce the notion of the J-weakly Laskerianness dimension $\omega_I^J(M)$ of M relative to I by

$$\omega_I^J(M) := \inf\{i \in \mathbb{N}_0 \mid J^t H_I^i(M) \text{ is not weakly Laskerian for all } t \in \mathbb{N}\},$$

which we abbreviate by J-wL dimension of M relative to I.

Based on [10] (resp. [15]) we say that the local-global principle for the annihilation (resp. minimaxness) of local cohomology modules holds at level n if for every choice of ideals I and J of R with $J \subseteq I$ and every choice of finitely generated R-module M, it is the case that

$$f_I^J(M) > n \Longleftrightarrow f_{IR_{\mathfrak{p}}}^{JR_{\mathfrak{p}}}(M_{\mathfrak{p}}) > n \text{ for all } \mathfrak{p} \in \operatorname{Spec}(R)$$
$$(\mu_I^J(M) > n \Longleftrightarrow \mu_{IR_{\mathfrak{p}}}^{JR_{\mathfrak{p}}}(M_{\mathfrak{p}}) > n \text{ for all } \mathfrak{p} \in \operatorname{Spec}(R)).$$

It is shown in [10] that the local-global principle for the annihilation of local cohomology modules holds at levels $1, 2$, over an arbitrary commutative Noetherian ring R and at all levels whenever $\dim R \leq 4$. Recently, in [15], these results are extended to the class of minimax modules and it is shown that the local-global principle for the minimaxness of local cohomology modules holds at levels $1, 2$ over a commutative Noetherian ring R and at all levels whenever $\dim R \leq 3$. One of the main tools for proving these results is [15, Theorem 2.6] which shows that for ideals I, J of the commutative Noetherian ring R with $J \subseteq I$ and finitely generated R-module M, the local-global principle for the minimaxness of local cohomology modules holds at level $n \in \mathbb{N}$ if the local cohomology module $H_I^i(M)$ is I-cofinite for all $i < n$.

In Section 2, we prove that the assertion in [15, Theorem 2.6] holds under a more general assumption on local cohomology module $H_I^i(M)$. More precisely, we prove the following theorem

Theorem 1.1. *Let R be a Noetherian ring, I, J be two ideals of R such that $J \subseteq I$, M be a finitely generated R-module and n be a positive integer such that the local cohomology module $H_I^i(M)$ is I-cominimax for all $i < n$. Then*

$$\mu_I^J(M) > n \Longleftrightarrow \mu_{IR_{\mathfrak{p}}}^{JR_{\mathfrak{p}}}(M_{\mathfrak{p}}) > n \textit{ for all } \mathfrak{p} \in \operatorname{Spec}(R).$$

In Section 3, we introduce the concept of local-global principle for the weakly Laskerianness of local cohomology modules over a commutative Noetherian ring R and we show that this principle holds at levels $1, 2$ over a commutative Noetherian semi-local rings. We also establish the same principle at all levels over an arbitrary commutative Noetherian semi-local ring of dimension not exceeding 3. Our tool for proving the main result is the following theorem:

Theorem 1.2. *Let R be a semi-local ring, I, J be two ideals of R such that $J \subseteq I$, M be a finitely generated R-module and n be a positive integer such that the local cohomology module $H_I^i(M)$ is I-weakly cofinite for all $i < n$.*

Then

$$\omega_I^J(M) > n \Longleftrightarrow \omega_{IR_\mathfrak{p}}^{JR_\mathfrak{p}}(M_\mathfrak{p}) > n \text{ for all } \mathfrak{p} \in \operatorname{Spec}(R).$$

As a consequence of this theorem, we establish the following result which is a generalization of [10, Corollary 2.3], [15, Corollaries 2.9, 2.10, 2.11] and a result of Raghavan [20] for a commutative Noetherian semi-local ring.

COROLLARY 1.3. *Let R be a semi-local ring, I be an ideal of R and M be a finitely generated R-module such that $IM \neq M$. Let $n \in \{1, \operatorname{grade}_I(M), f_I(M), f_I^1(M), f_I^2(M)\}$. Then*

$$\omega_I^J(M) > n \Longleftrightarrow \omega_{IR_\mathfrak{p}}^{JR_\mathfrak{p}}(M) > n \text{ for all } \mathfrak{p} \in \operatorname{Spec}(R).$$

We also establish an interrelation between these principles over a commutative Noetherian semi-local ring in Proposition 3.21 and Corollary 3.22.

Throughout this paper, we assume that R is a commutative Noetherian ring with non-zero identity, I is an ideal of R, $V(I)$ is the set of all prime ideals of R containing I, $\operatorname{Spec}(R)$ is the set of all prime ideals and $\operatorname{Max}(R)$ is the set of all maximal ideals of R.

2. LOCAL-GLOBAL PRINCIPLE FOR MINIMAXNESS OF LOCAL COHOMOLOGY MODULES

The concept of I-cofinite module was introduced by Hartshorne [17]. An R-module M is *I-cofinite* if $\operatorname{Supp}_R(M) \subseteq V(I)$ and $\operatorname{Ext}_R^j(R/I, M)$ is finitely generated for all j. Zöschinger introduced in [22] the interesting class of minimax modules and in [22] and [23] gave equivalent conditions for a module to be minimax. An R-module M is called *minimax* if there is a finitely generated submodule N of M such that M/N is Artinian. The concept of I-cominimax modules was introduced in [4] as a generalization of I-cofinite modules. According to the definition, an R-module M is said to be *I-cominimax* if $\operatorname{Supp}_R(M) \subseteq V(I)$ and $\operatorname{Ext}_R^j(R/I, M)$ is minimax for all j. Here are some elementary properties of this concepts that will be useful in the sequel.

LEMMA 2.1. *The following statements hold:*

(i) *The class of minimax modules contains all finitely generated and all Artinian modules.*

(ii) *Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules. Then M is minimax if and only if L and N are both minimax (see [6, Lemma 2.1]). Thus any submodule and quotient of a minimax module is minimax.*

(iii) *The set of associated primes of any minimax R-module is finite.*

(iv) *If M is a minimax R-module and $\mathfrak{p}$ is a non-maximal prime ideal of R, then $M_{\mathfrak{p}}$ is a finitely generated $R_{\mathfrak{p}}$-module.*

(v) *If M is a minimax R-module with* $\dim M = 0$*, then M is Artinian.*

LEMMA 2.2. *Let M be a finitely generated R-module and t be a non-negative integer such that $H^i_I(M)$ is I-cominimax for all $i < t$. Then* $\mathrm{Hom}_R(R/I, H^t_I(M))$ *is minimax. In particular, the set* $\mathrm{Ass}_R(H^t_I(M))$ *is finite.*

Proof. The assertion follows from [1, Theorem 2.8], Lemma 2.1(iii) and the fact that

$$\mathrm{Ass}_R(\mathrm{Hom}_R(R/I, H^t_I(M))) = V(I) \cap \mathrm{Ass}_R(H^t_I(M)) = \mathrm{Ass}_R(H^t_I(M)). \quad \square$$

In the following theorem, we show that [15, Theorem 2.6] which plays a key role in the proof of the main results of [15], remains valid with "I-cofinite" replaced by "I-cominimax". The proof involves little modification from that given in [15, Theorem 2.6], but we include it here for the convenience of the reader.

THEOREM 2.3. *Let M be a finitely generated R-module and n be a positive integer such that the local cohomology module $H^i_I(M)$ is I-cominimax for all $i < n$. Suppose that J is a second ideal of R such that $J \subseteq I$. Then*

$$\mu^J_I(M) > n \Longleftrightarrow \mu^{JR_{\mathfrak{p}}}_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}) > n \text{ for all } \mathfrak{p} \in \mathrm{Spec}(R).$$

Proof. Let i be an arbitrary non-negative integer such that $i \leq n$. It is enough to show that there is a non-negative integer k_0 such that $J^{k_0} H^i_I(M)$ is minimax. To do this, note that by Lemma 2.2, the set $\mathrm{Ass}_R(J^k H^i_I(M))$ is finite for all $k \in \mathbb{N}_0$. Thus for all $k \in \mathbb{N}_0$, the set $\mathrm{Supp}_R(J^k H^i_I(M))$ is a closed subset of $\mathrm{Spec}(R)$ (in the Zariski topology) and

$$\cdots \supseteq \mathrm{Supp}_R(J^k H^i_I(M)) \supseteq \mathrm{Supp}_R(J^{k+1} H^i_I(M)) \supseteq \cdots$$

is a descending chain of the Noetherian space $\mathrm{Spec}(R)$ and so is eventually stationary. Thus, there exists a non-negative integer k_0 such that for each $k \geq k_0$,

$$\mathrm{Supp}_R(J^{k_0} H^i_I(M)) = \mathrm{Supp}_R(J^k H^i_I(M)).$$

If $\mathrm{Spec}(R) = \mathrm{Max}(R)$, then $\dim R = 0$ and so there is nothing to prove by assumption and Grothendieck's Vanishing Theorem [11, Theorem 6.1.2]. Otherwise, let $\mathfrak{p}$ be a non-maximal prime ideal of R and $\mathfrak{m} \in \mathrm{Max}(R)$ such that $\mathfrak{p} \subsetneq \mathfrak{m}$. Since $\mu^{JR_{\mathfrak{p}}}_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}) > n$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$, it follows that there is an integer $t \geq k_0$ such that $(JR_{\mathfrak{m}})^t H^i_{IR_{\mathfrak{m}}}(M_{\mathfrak{m}})$ is minimax. Thus, by the isomorphism

$$((JR_{\mathfrak{m}})^t H^i_{IR_{\mathfrak{m}}}(M_{\mathfrak{m}}))_{\mathfrak{p}R_{\mathfrak{m}}} \cong (J^t H^i_I(M))_{\mathfrak{p}}$$

and Lemma 2.1(iv) we infer that $(J^tH^i_I(M))_{\mathfrak{p}}$ is a finitely generated $R_{\mathfrak{p}}$-module. Since $(J^tH^i_I(M))_{\mathfrak{p}}$ is $IR_{\mathfrak{p}}$-torsion and $J \subseteq I$, there is an integer $s \in \mathbb{N}$ such that $(J^{t+s}H^i_I(M))_{\mathfrak{p}} = 0$. Hence, $\mathfrak{p} \notin \mathrm{Supp}_R(J^{k_0}H^i_I(M))$ and so, $\mathrm{Supp}_R(J^{k_0}H^i_I(M)) \subseteq \mathrm{Max}(R)$. Therefore, $\mathrm{Supp}_R(\mathrm{Hom}_R(R/I, J^{k_0}H^i_I(M))) \subseteq \mathrm{Max}(R)$. Furthermore, by assumption and Lemma 2.2, the R-module $\mathrm{Hom}_R(R/I, J^{k_0}H^i_I(M))$ is minimax. Thus, in the light of Lemma 2.1(v), we conclude that $\mathrm{Hom}_R(R/I, J^{k_0}H^i_I(M))$ is an Artinian R-module. Hence, as $J^{k_0}H^i_I(M)$ is I-torsion, it follows from [18, Theorem 1.3] that $J^{k_0}H^i_I(M)$ is Artinian and so is minimax. This is the desired conclusion. □

COROLLARY 2.4. *Let I be an ideal of R and M be a finitely generated R-module such that one of the following conditions is satisfied:*

(i) $\dim H^i_I(M) \leq 1$ *for all* $i \geq 0$ *(e.g.* $\dim R/I \leq 1$*);*

(ii) $\mathrm{cd}(I) = 1$*;*

(iii) $\dim R \leq 2$*.*

Let J be a second ideal of R such that $J \subseteq I$. Then for any integer n,

$$\mu^J_I(M) > n \Longleftrightarrow \mu^{JR_{\mathfrak{p}}}_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}) > n \text{ for all } \mathfrak{p} \in \mathrm{Spec}(R).$$

Proof. The assertion follows from [21, Theorem 3.1] and Theorem 2.3. □

3. LOCAL-GLOBAL PRINCIPLE FOR THE WEAKLY LASKERIANNESS OF LOCAL COHOMOLOGY MODULES

In this section, our main results are stated and proved. Based on [14] and [13], an R-module X is called *weakly Laskerian* if the set of associated primes of any quotient module of X is finite. Also, an R-module X is said to be *I-weakly cofinite* if $\mathrm{Supp}_R(X) \subseteq V(I)$ and $\mathrm{Ext}^i_R(R/I, X)$ is weakly Laskerian for all $i \geq 0$. In [19], Quy has introduced the class of FSF modules, modules containing some finitely generated submodules such that the support of the quotient module is finite. It has been shown in [5, Theorem 3.3] that over a Noetherian ring R, an R-module M is weakly Laskerian if and only if it is FSF. We first mention some basic properties of these concepts that are needed in what follows.

LEMMA 3.1. *If M is a weakly Laskerian R-module and $\mathfrak{p}$ is a prime ideal of R with $\dim R/\mathfrak{p} \geq 2$, then $M_{\mathfrak{p}}$ is a finitely generated $R_{\mathfrak{p}}$-module.*

Proof. In view of [5, Theorem 3.3], there is a finitely generated submodule N of M such that $\mathrm{Supp}_R(M/N)$ is a finite set. Hence, $\dim M/N \leq 1$ and so, $M_{\mathfrak{p}} \cong N_{\mathfrak{p}}$. This completes the proof. □

Lemma 3.2. *Let M be a weakly Laskerian R-module and n be a positive integer such that $H^i_I(M)$ is I-weakly cofinite for all $i < n$. Then the set of associated primes of $H^i_I(M)$ is finite for all $i \leq n$.*

Proof. The assertion follows from [13, Theorem 2.5] and the fact that

$$\mathrm{Ass}_R(\mathrm{Hom}_R(R/I, H^i_I(M))) = \mathrm{Ass}_R(H^i_I(M)) \cap V(I) = \mathrm{Ass}_R(H^i_I(M))$$

for all $i \geq 0$. □

Recall that a class of R-modules is a *Serre subcategory* of the category of R-modules when it is closed under taking submodules, quotients and extensions. For example, the classes of Noetherian modules, Artinian modules, minimax modules or weakly Laskerian modules are Serre subcategories. As in standard notation, we let $\mathcal{S}$ stand for a Serre subcategory of the category of R-modules.

Lemma 3.3. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules and M be an arbitrary R-module. Then IM belongs to $\mathcal{S}$ if and only if $M/(0 :_M I)$ belongs to $\mathcal{S}$. In particular, IM is weakly Laskerian if and only if $M/(0 :_M I)$ is weakly Laskerian.*

Proof. This follows easily by induction on the number of generators of I and the definition of Serre subcategory. □

Proposition 3.4. *Let M be a finitely generated R-module and n be a positive integer. Then the following statements are equivalent:*

(i) *$H^i_I(M)$ is weakly Laskerian for all $i < n$;*

(ii) *There exists a positive integer k such that $I^k H^i_I(M)$ is weakly Laskerian for all $i < n$.*

Proof. The forward direction is clear. For the other direction, we proceed by induction on n. If $n = 1$, then there is nothing to prove. Now, assume that $n > 1$ and the result has been proved for all $i < n$. By inductive hypothesis the R-module $H^i_I(M)$ is weakly Laskerian for all $i < n - 1$, and so it suffices to prove that the R-module $H^{n-1}_I(M)$ is weakly Laskerian. By assumption and Lemma 3.3, the R-module $H^{n-1}_I(M)/(0 :_{H^{n-1}_I(M)} I^k)$ is weakly Laskerian. Hence, in view of [5, Theorem 3.3] there exists a finitely generated submodule H of $H^{n-1}_I(M)$ such that the set $\mathrm{Supp}_R(H^{n-1}_I(M)/H + (0 :_{H^{n-1}_I(M)} I^k))$ is finite. On the other hand, in the light of [2, Corollary 3.6], the R-module $(0 :_{H^{n-1}_I(M)} I^k)$ is finitely generated and hence $H + (0 :_{H^{n-1}_I(M)} I^k)$ is a finitely generated submodule of $H^{n-1}_I(M)$. Therefore, $H^{n-1}_I(M)$ is weakly Laskerian. This completes the proof. □

Corollary 3.5. *Let M be a finitely generated R-module and $f_I^2(M)$ denote the 2nd finiteness dimension of M relative to I. Then*

$$\inf\{i \in \mathbb{N}_0 \mid I^k H_I^i(M) \text{ is not weakly Laskerian for all } k \in \mathbb{N}\} \leq f_I^2(M).$$

Equality holds if R is semi-local.

Proof. The result follows immediately from [8, Proposition 3.7] and Proposition 3.4. □

Based on [2], an R-module X is called $FD_{\leq n}$ if there is a finitely generated submodule L of X such that $\dim X/L \leq n$. It is clear that a minimax module is $FD_{\leq 0}$ and a weakly Laskerian module is $FD_{\leq 1}$.

Remark 3.6. The reader can easily see that Proposition 3.4 remains valid with "weakly Laskerian" replaced by $FD_{\leq 1}$, provided that $H_I^i(M)$ is $FD_{\leq 1}$ for all $i < n$ if and only if there is $t \in \mathbb{N}$ such that $I^t H_I^i(M)$ is $FD_{\leq 1}$ for all $i < n$. Hence, in the light of [2, Theorem 3.4] we establish the following result which is a generalization of [15, Theorem 3.6] and the main results of [9] and [19].

Corollary 3.7. *Let M be a finitely generated R-module and n be a non-negative integer such that $I^k H_I^i(M)$ is $FD_{\leq 1}$ for some $k \in \mathbb{N}_0$ and all $i < n$. Then the following assertions hold:*

(i) *The R-module $H_I^i(M)$ is I-cofinite for all $i < n$.*

(ii) *For all $FD_{\leq 0}$ (or minimax) submodules N of $H_I^n(M)$, the R-modules*

$$\mathrm{Hom}_R(R/I, H_I^n(M)/N) \quad \text{and} \quad \mathrm{Ext}_R^1(R/I, H_I^n(M)/N)$$

are finitely generated. In particular, the set $\mathrm{Ass}_R(H_I^n(M)/N)$ is finite.

In view of Corollary 3.5 it is natural to ask: For a given ideal J of R such that $J \subseteq I$ and a finitely generated R-module M, what can we say about the greatest integer i such that $J^t H_I^i(M)$ is weakly Laskerian for some integer $t \in \mathbb{N}$? This suggests that we introduce the notion of J-weakly Laskerianness dimension $\omega_J^I(M)$ (as a generalization of J-finiteness dimension $f_I^J(M)$ [11, Definition 9.1.5] and J-minimaxness dimension $\mu_I^J(M)$ [15]) of M relative to I.

Definition 3.8. Let M be a finitely generated R-module and let I, J be two ideals of R such that $J \subseteq I$. We define the J-weakly Laskerianness dimension $\omega_J^I(M)$ of M relative to I by

$$\omega_I^J(M) := \inf\{i \in \mathbb{N}_0 \mid J^k H_I^i(M) \text{ is not weakly Laskerian for all } k \in \mathbb{N}\}.$$

Note that $\omega_I^J(M) > 0$ since $\Gamma_I(M)$ is weakly Laskerian, that is $\omega_I^J(M)$ is either a positive integer or ∞. It is clear that $f_I^J(M) \leq \mu_I^J(M) \leq \omega_I^J(M)$. Also, by Corollary 3.5, $\omega_I^I(M) = f_I^2(M)$ when R is semi-local.

We also introduce the local-global principle for the weakly Laskerianness of local cohomology modules as a generalization of the Faltings' local-global principle for the annihilation and for the minimaxness of local cohomology modules.

Definition 3.9. We say that the local-global principle for the weakly Laskerianness of local cohomology modules holds at level $n \in \mathbb{N}$ (over the commutative Noetherian ring R) if for every choice of ideals I, J of R with $J \subseteq I$ and for every choice of finitely generated R-module M, it is the case that

$$\omega_I^J(M) > n \Longleftrightarrow \omega_{IR_\mathfrak{p}}^{JR_\mathfrak{p}}(M_\mathfrak{p}) > n \text{ for all } \mathfrak{p} \in \operatorname{Spec}(R).$$

From now on, except as indicated, R will be assumed to be a commutative Noetherian semi-local ring and I, J be two ideals of R such that $J \subseteq I$.

THEOREM 3.10. *Let M be a finitely generated R-module and n be a positive integer such that the local cohomology module $H_I^i(M)$ is I-weakly cofinite for all $i < n$. Then*

$$\omega_I^J(M) > n \Longleftrightarrow \omega_{IR_\mathfrak{p}}^{JR_\mathfrak{p}}(M_\mathfrak{p}) > n \textit{ for all } \mathfrak{p} \in \operatorname{Spec}(R).$$

Proof. The implication ($\Rightarrow$) is obviously true. In order to show ($\Leftarrow$), let i be an arbitrary non-negative integer such that $i \leq n$. It is enough to show that there is a positive integer k_0 such that $J^{k_0} H_I^i(M)$ is weakly Laskerian. By assumption and Lemma 3.2, the set $\operatorname{Ass}_R(J^k H_I^i(M))$ is finite for all $k \in \mathbb{N}_0$. Hence, $\operatorname{Supp}_R(J^k H_I^i(M))$ is a closed subset of $\operatorname{Spec}(R)$ in the Zariski topology for all $k \in \mathbb{N}_0$. Therefore,

$$\cdots \supseteq J^k H_I^i(M) \supseteq J^{k+1} H_I^i(M) \supseteq \cdots$$

is a descending chain of the Noetherian space $\operatorname{Spec}(R)$ and so is eventually stationary. Thus, there exists $k_0 \in \mathbb{N}_0$ such that for each $k \geq k_0$,

$$\operatorname{Supp}_R(J^{k_0} H_I^i(M)) = \operatorname{Supp}_R(J^k H_I^i(M)).$$

Now, let $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\dim R/\mathfrak{p} \geq 2$. Then there exist $\mathfrak{q} \in \operatorname{Spec}(R)$ and $\mathfrak{m} \in \operatorname{Max}(R)$ such that $\mathfrak{p} \subsetneq \mathfrak{q} \subsetneq \mathfrak{m}$. Since, $\omega_{IR_\mathfrak{p}}^{JR_\mathfrak{p}}(M_\mathfrak{p}) > n$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$, it follows that there is an integer $u \geq k_0$ such that $(JR_\mathfrak{m})^u H_{IR_\mathfrak{m}}^i(M_\mathfrak{m})$ is weakly Laskerian. Hence, in the light of Lemma 3.1 and the isomorphism

$$((JR_\mathfrak{m})^u H_{IR_\mathfrak{m}}^i(M_\mathfrak{m}))_{\mathfrak{p}R_\mathfrak{m}} \cong (J^u H_I^i(M))_\mathfrak{p},$$

we infer that $(J^u H_I^i(M))_\mathfrak{p}$ is finitely generated. Since, $(J^u H_I^i(M))_\mathfrak{p}$ is $IR_\mathfrak{p}$-torsion and $J \subseteq I$, there is a positive integer v such that $(J^{u+v} H_I^i(M))_\mathfrak{p} = 0$ and so $\mathfrak{p} \notin \operatorname{Supp}_R(J^{u+v} H_I^i(M)) = \operatorname{Supp}_R(J^{k_0} H_I^i(M))$. Therefore,

$$\operatorname{Supp}_R(J^{k_0} H_I^i(M)) \subseteq \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \dim R/\mathfrak{p} \leq 1\}.$$

Let $T := \{\mathfrak{p} \in \operatorname{Supp}_R(J^{k_0} H_I^i(M)) \mid \dim R/\mathfrak{p} = 1\}$. Obviously, $T \subseteq \operatorname{Ass}_R(J^{k_0} H_I^i(M))$ is a finite set. Consequently, as the set $\operatorname{Max}(R)$ is finite, it follows that the set $\operatorname{Supp}_R(J^{k_0} H_I^i(M))$ is finite and so, the R-module $J^{k_0} H_I^i(M)$ is weakly Laskerian, as required. □

In the sequel, we mention some important consequences of Theorem 3.10.

COROLLARY 3.11. *The local-global principle (for the weakly Laskerianness of local cohomology modules) holds over R at level 1.*

COROLLARY 3.12. *If $\dim R \leq 2$, then the local-global principle (for the weakly Laskerianness of local cohomology modules) holds over R at all levels $n \in \mathbb{N}$.*

Proof. The result follows from [12, Theorem 1.3] and Theorem 3.10. □

COROLLARY 3.13. *Let M be a finitely generated R-module and n be a positive integer such that $H_I^i(M)$ is $FD_{\leq 1}$ (or weakly Laskerian) R-module for all $i < n$. Then*

$$\omega_I^J(M) > n \Longleftrightarrow \omega_{IR_\mathfrak{p}}^{JR_\mathfrak{p}}(M_\mathfrak{p}) > n \text{ for all } \mathfrak{p} \in \operatorname{Spec}(R).$$

Proof. The result follows from [2, Theorem 3.4] and Theorem 3.10. □

COROLLARY 3.14. *Let M be a finitely generated R-module such that $\dim M/IM \leq 1$. Then for any integer n,*

$$\omega_I^J(M) > n \Longleftrightarrow \omega_{IR_\mathfrak{p}}^{JR_\mathfrak{p}}(M_\mathfrak{p}) > n \text{ for all } \mathfrak{p} \in \operatorname{Spec}(R).$$

Proof. The assertion follows from [7, Corollary 2.7] and Theorem 3.10. □

COROLLARY 3.15. *Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module such that $\dim M/IM \leq 2$. Then for any integer n,*

$$\omega_I^J(M) > n \Longleftrightarrow \omega_{IR_\mathfrak{p}}^{JR_\mathfrak{p}}(M_\mathfrak{p}) > n \text{ for all } \mathfrak{p} \in \operatorname{Spec}(R).$$

Proof. The assertion follows from [7, Corollary 3.2] and Theorem 3.10. □

COROLLARY 3.16. *Let M be a finitely generated R-module such that $IM \neq M$. Then*

$$\omega_I^J(M) > \operatorname{grade}_M I \Longleftrightarrow \omega_{IR_\mathfrak{p}}^{JR_\mathfrak{p}}(M_\mathfrak{p}) > \operatorname{grade}_M I \text{ for all } \mathfrak{p} \in \operatorname{Spec}(R).$$

Proof. The assertion follows from [11, Theorem 6.2.7] and Theorem 3.10. □

COROLLARY 3.17. *Let M be a finitely generated R-module and $n \in \{f_I(M), f_I^1(M), f_I^2(M)\}$. Then*

$$\omega_I^J(M) > n \Longleftrightarrow \omega_{IR_\mathfrak{p}}^{JR_\mathfrak{p}}(M) > n \text{ for all } \mathfrak{p} \in \operatorname{Spec}(R).$$

Proof. The assertion follows from [8, Theorem 2.3 and Proposition 3.7] and Theorem 3.10. □

COROLLARY 3.18. *Let $(R, \mathfrak{m})$ be a complete local ring and M be a finitely generated R-module. If $n = f_I^3(M)$ is finite, then*

$$\omega_I^J(M) > n \Longleftrightarrow \omega_{IR_\mathfrak{p}}^{JR_\mathfrak{p}}(M) > n \text{ for all } \mathfrak{p} \in \operatorname{Spec}(R).$$

Proof. The assertion follows from [8, Theorem 3.9] and Theorem 3.10. □

We are now ready to state and prove the main theorem of this section.

THEOREM 3.19. *The local-global principle (for the weakly Laskerianness of local cohomology modules) holds over R at level 2.*

Proof. Let M be a finitely generated R-module, such that $\omega_{IR_\mathfrak{p}}^{JR_\mathfrak{p}}(M) > 2$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$. If we prove that $\omega_I^J(M) > 2$, the assertion follows. To this end, by Corollary 3.11, we only need to show that there exists a non-negative integer k such that the R-module $J^k H_I^2(M)$ is weakly Laskerian. For this purpose, let $\overline{M} = M/\Gamma_J(M)$. We begin by proving that the set $\operatorname{Ass}_R(H_I^2(\overline{M}))$ is finite. Consider the long exact sequence

$$(1) \qquad H_I^1(M) \to H_I^1(\overline{M}) \to H_I^2(\Gamma_J(M)) \to H_I^2(M) \to H_I^2(\overline{M}),$$

induced by the short exact sequence

$$0 \to \Gamma_J(M) \to M \to \overline{M} \to 0.$$

Let $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R/\mathfrak{p} > 1$. Since M is a finitely generated R-module, the set $\operatorname{Ass}_R(H_I^1(M))$ is finite by [9, Theorem 2.2]. Hence, as in the proof of Theorem 3.10, there exists $u \in \mathbb{N}_0$ such that $(J^u H_I^1(M))_\mathfrak{p} = 0$. Moreover, there exists $t \in \mathbb{N}_0$ such that $J^t H_I^i(\Gamma_J(M)) = 0$ for all $i \geq 0$. So, it follows from the exact sequence

$$H_{IR_\mathfrak{p}}^1(M_\mathfrak{p}) \to H_{IR_\mathfrak{p}}^1(\overline{M}_\mathfrak{p}) \to H_{IR_\mathfrak{p}}^2(\Gamma_{JR_\mathfrak{p}}(M_\mathfrak{p}))$$

and [11, Lemma 9.1.1] that there is $v \in \mathbb{N}_0$ such that $(JR_\mathfrak{p})^v H_{IR_\mathfrak{p}}^1(\overline{M}_\mathfrak{p}) = 0$. Furthermore, by [11, Lemma 2.1.1], there is $r \in J$ which is a non-zerodivisor on $\overline{M}$. So, $r^v H_{IR_\mathfrak{p}}^1(\overline{M}_\mathfrak{p}) = 0$. Now, considering the exact sequence

$$H_{IR_\mathfrak{p}}^0(\overline{M}_\mathfrak{p}/r^v\overline{M}_\mathfrak{p}) \to H_{IR_\mathfrak{p}}^1(\overline{M}_\mathfrak{p}) \to 0$$

induced by the exact sequence

$$0 \to \overline{M}_\mathfrak{p} \xrightarrow{r^v} \overline{M}_\mathfrak{p} \to \overline{M}_\mathfrak{p}/r^v\overline{M}_\mathfrak{p} \to 0,$$

we conclude that $H_{IR_\mathfrak{p}}^1(\overline{M}_\mathfrak{p})$ is a finitely generated $R_\mathfrak{p}$-module. Therefore, in the light of [8, Proposition 3.1], $\mathrm{Hom}_R(R/I, H_I^2(\overline{M}))$ is a finitely generated R-module and so, the set $\mathrm{Ass}_R(H_I^2(\overline{M}))$ is finite.

Since $\omega_{IR_\mathfrak{p}}^{JR_\mathfrak{p}}(M) > 2$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$, by the same method as in the proof of Theorem 3.10, there is $l \in \mathbb{N}_0$ such that $(JR_\mathfrak{p})^l H_{IR_\mathfrak{p}}^2(M_\mathfrak{p}) = 0$. Thus, by the exact sequence

$$H_{IR_\mathfrak{p}}^2(M_\mathfrak{p}) \to H_{IR_\mathfrak{p}}^2(\overline{M}_\mathfrak{p}) \to H_{IR_\mathfrak{p}}^3(\Gamma_J(M_\mathfrak{p}))$$

and [11, Lemma 9.1.1], we infer that there exists $n \in \mathbb{N}_0$ such that $(JR_\mathfrak{p})^n H_{IR_\mathfrak{p}}^2 (\overline{M}_\mathfrak{p}) = 0$ and so, $\mathfrak{p} \notin \mathrm{Supp}_R(J^n H_I^2(\overline{M}))$. Thus,

$$\mathrm{Supp}_R(J^n H_I^2(\overline{M})) \subseteq \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \dim R/\mathfrak{p} \leq 1\}.$$

Since $\mathrm{Spec}(R)$ is Noetherian, it follows from the proof of Theorem 3.10 that there is a non-negative integer s such that

$$\mathrm{Supp}_R(J^s H_I^2(\overline{M})) \setminus \mathrm{Max}(R) \subseteq \mathrm{Ass}_R(H_I^2(\overline{M})).$$

Hence, $\mathrm{Supp}_R(J^s H_I^2(\overline{M}))$ is a finite set. Let $k = s + t$ and $\mathfrak{p} \notin \mathrm{Supp}_R(J^s H_I^2(\overline{M}))$. Then $(JR_\mathfrak{p})^s(H_I^2(\overline{M}))_\mathfrak{p} = 0$ and so in view of the exact sequence (1) and [11, Lemma 9.1.1], $(JR_\mathfrak{p})^k(H_I^2(M))_\mathfrak{p} = 0$. Hence, $\mathfrak{p} \notin \mathrm{Supp}_R(J^k H_I^2(M))$. This implies that

$$\mathrm{Supp}_R(J^k H_I^2(M)) \subseteq \mathrm{Supp}_R(J^s H_I^2(\overline{M})).$$

Therefore, the set $\mathrm{Ass}_R(J^k H_I^2(M))$ is finite and so, $J^k H_I^2(M)$ is weakly Laskerian, as desired. □

COROLLARY 3.20. *If* $\dim R \leq 3$, *then the local-global principle (for the weakly Laskerianness of local cohomology modules) holds over* R *at all levels* $n \in \mathbb{N}$.

Proof. Apply Corollary 3.11, Theorem 3.19 and [11, Corollary 7.1.7]. □

In the following results, we state a relationship between the local-global principle for the weakly Laskerianness, minimaxness and annihilation (of local cohomology modules) over a commutative Noetherian semi-local ring R.

PROPOSITION 3.21. *The local-global principle for the weakly Laskerianness (of local cohomology modules) implies the local-global principle for the minimaxness (of local cohomology modules) over* R.

Proof. Let n be a non-negative integer and suppose that the local-global principle for the weakly Laskerianness of local cohomology modules holds at level n. Let M be a finitely generated R-module such that $\mu_{IR_\mathfrak{p}}^{JR_\mathfrak{p}}(M_\mathfrak{p}) > n$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$. We must show that $\mu_I^J(M) > n$. Let $i \leq n$. Then for all $\mathfrak{p} \in \mathrm{Spec}(R)$ there exists $k_\mathfrak{p} \in \mathbb{N}$ such that the $R_\mathfrak{p}$-module $(JR_\mathfrak{p})^{k_\mathfrak{p}} H_{IR_\mathfrak{p}}^i(M_\mathfrak{p})$ is minimax and so is weakly Laskerian. Thus $\omega_{IR_{\mathfrak{p}}}^{JR_{\mathfrak{p}}}(M) > n$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Therefore, by assumption $\omega_I^J(M) > n$ and so there is $s \in \mathbb{N}$ such that $J^s H_I^i(M)$ is weakly Laskerian. Now, as R is semi-local and $mu_{IR_{\mathfrak{m}}}^{JR_{\mathfrak{m}}}(M_{\mathfrak{m}}) > n$ for all $\mathfrak{m} \in \operatorname{Max}(R)$, there exists $t \in \mathbb{N}$ such that $(J^t H_I^i(M))_{\mathfrak{m}}$ is minimax for all $\mathfrak{m} \in \operatorname{Max}(R)$ and $J^t H_I^i(M)$ is weakly Laskerian. Hence, it follows from [3, Proposition 2.2] that $J^t H_I^i(M)$ is minimax, as required. □

Corollary 3.22. *The local-global principle for the weakly Laskerianness (of local cohomology modules) implies the local-global principle for the annihilation (of local cohomology modules) over R.*

Proof. The assertion follows from Proposition 3.21 and [15, Proposition 3.1]. □

Acknowledgements. The author is deeply grateful to the referee for his or her valuable suggestions on the paper.

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Received 6 April 2017

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