MODULES OF FINITE LENGTH

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In this paper, we first present necessary and sufficient conditions for various modules to be of finite length. We then use our results to give an alternative proof of the well-known result that if R is a Noetherian local ring with maximal ideal $\mathfrak{M}$ and M is a finitely generated R-module of dimension d, then $\mathrm{H}^d_{\mathfrak{M}}(M)$ is finitely generated if and only if $d = 0$.

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1. INTRODUCTION

This note is motivated by the question of what are the most elementary properties that are required for an R-module M to be of finite length? That is M being both Noetherian and Artinian. We first provide conditions equivalent to M and all its Koszul cohomology modules $\mathrm{H}^i(\mathbf{x}^\infty M)$ to be of length at most two, where $\mathbf{x} = \{x_1, x_2, \cdots, x_n\}$ is any sequence of elements in R. We then consider the case for divisible modules and show that over a reduced Noetherian ring, finitely generated divisible modules are of finite length and that a reduced local ring R with finitely many prime ideals possesses a nonzero finitely generated divisible module implies that R is of Krull dimension zero. We use these results to show that if R is a Noetherian local ring with maximal ideal $\mathfrak{M}$ and M a finitely generated R-module with dimension d, then $\mathrm{H}^d_{\mathfrak{M}}(M)$ is finitely generated if and only if $d = 0$. We then prove that for any ideal I of R, $\mathrm{H}^d_I(M)$ is a nonzero finitely generated module over a reduced Noetherian local ring R if and only if R is a field provided that the set of zero divisors of M is contained in the set of zero divisors of R.

Throughout, R denotes a commutative ring with identity and M denotes a non-zero R-module, and that $Z(M)$ will denote the set of zero divisors of M.

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2. MODULES OF LENGTH AT MOST TWO

We begin by recalling the following:

Let R be a Noetherian ring and I be an ideal of R. For an R-module M, set

$$\Gamma_I(M) = \{x \in M : \textit{there exists an } n \in \mathbb{N} \textit{ such that } I^n x = 0\}.$$

Then the i^{th} local cohomology of M with respect to the ideal I is the i^{th} cohomology module of the sequence obtained by applying $\Gamma_I(-)$ to an injective resolution of M and this module is denoted by $\mathrm{H}^i_I(M)$. Equivalently, the i^{th} local cohomology of M with respect to I is defined to be

$$\mathrm{H}^i_I(M) = \lim_{\rightarrow} \mathrm{Ext}^i_R(R/I^n, M).$$

Let now R be a ring (not necessarily Noetherian) and $\mathbf{x} = \{x_1, x_2, \cdots, x_n\}$ be a sequence of elements of R and let $\mathbf{x}^t$ denote the sequence $\{x_1^t, x_2^t, \cdots, x_n^t\}$. Then the i^{th} Koszul cohomology of M is defined to be:

$$\mathrm{H}^i(\mathbf{x}^\infty; M) = \mathrm{H}^i(\underset{\longrightarrow_t}{\lim}\ \mathbf{K}^\bullet(\mathbf{x}^t; M))$$

where $\mathbf{K}^\bullet(\mathbf{x}^t; M)$ denote the cohomological Koszul complex. In particular, if R is Noetherian, then for all i

$$\mathrm{H}^i(\mathbf{x}^\infty; M) \cong \mathrm{H}^i_I(M)$$

where $I = (x_1, x_2, \cdots, x_n)$. For more details, we refer the reader to Section 7 of [3].

In this section, we make some simple but somehow interesting observations, the first of which is the following:

Proposition 2.1. *For an R-module M the following statements are equivalent.*

(i) *For any two distinct proper submodules K, L of M,* $\mathrm{Ann}(K) + \mathrm{Ann}(L) = R$.

(ii) *For any two distinct proper submodules K, L of M,* $\mathrm{Hom}_R(K, L) = 0$.

(iii) *M is a direct sum of at most two non-isomorphic simple submodules.*

(iv) *M has length at most two with non-isomorphic simple quotient modules.*

(v) *For all i and for all sequences $\mathbf{x} = \{x_1, x_2, \cdots, x_n\}$ of elements in R, $\mathrm{H}^i(\mathbf{x}^\infty; M)$ is of length at most two with non-isomorphic simple quotient modules. (In particular if R is Noetherian, then for all i and for all ideals I of R, $\mathrm{H}^i_I(R)$ is of length at most two with non-isomorphic simple quotient modules.)*

Proof. (i)⇒(iii). Let K be any non-zero proper submodule of M and x a non-zero element of K. We claim that $K = Rx$. Suppose not, then Rx is a proper submodule of K and hence by the assumption $\operatorname{Ann}(Rx)+\operatorname{Ann}(K) = R$. But this is the same thing as saying that $\operatorname{Ann}(Rx) = R$, a contradiction to the fact that x is non-zero in K. Therefore it follows that every proper submodule of M is simple. If K is different than M, then there is a proper submodule L of M different than K, and for the same reason as above $L = Ry$, for some y in M. Thus $K \cong R/\operatorname{Ann}(x)$ and $L \cong R/\operatorname{Ann}(y)$, and $\operatorname{Ann}(x)$, $\operatorname{Ann}(y)$ are two distinct maximal ideals of R. Next we show that $K \cap L = 0$. But this follows from the fact that if $K \cap L$ has a non-zero element z, then $\operatorname{Ann}(z)+\operatorname{Ann}(K) = \operatorname{Ann}(z) = R$, which is not possible. Therefore $M = K \oplus L$. Since otherwise $K \oplus L$ would be a proper submodule of M which would then contradict the fact that K is a non-zero and yet $\operatorname{Ann}(K) = \operatorname{Ann}(K) + \operatorname{Ann}(K \oplus L) = R$.

(iii)⇒(i). Is clear.

(iii) ⇒ (ii). Is clear.

(ii)⇒(iii). Let K be again a proper submodule of M and x a non-zero element of K. If Rx is different than K, then $\operatorname{Hom}_R(Rx, K) \neq 0$, a contradiction. Therefore each proper submodule of M is simple. If L is another proper submodule of M different from K, then $M = K \oplus L$. Because otherwise $K \oplus L$ would be a proper submodule of M, and that would give $\operatorname{Hom}_R(K, K \oplus L) \neq 0$, contradicting the assumption.

(iii) ⇒ (iv). Follows from the fact that $length(M_1 \oplus M_2) = length(M_1) + length(M_2)$.

(iv)⇒(iii). Is clear.

(iii) ⇒ (v). Let K be one of the simple submodules of M and $\mathbf{x} = \{x_1, x_2, \cdots, x_n\}$ be a sequence of elements in R. Let $I = (x_1, x_2, \cdots, x_n)$. Then by Lemma 7.7 of [3], $\mathrm{H}^i(\mathbf{x}^\infty; K) = 0$ for all $i > 0$. Therefore we only consider the remaining case $\mathrm{H}^0(\mathbf{x}^\infty; K) = \{x \in K : I^t x = 0 \; for\ some\ positive\ integer\ t\}$. Since K is simple, $K \cong R/\mathfrak{M}$ for some maximal ideal $\mathfrak{M}$ of R. If now $I \nsubseteq \mathfrak{M}$ then $I + \mathfrak{M} = I + \operatorname{Ann} K = R$ and since $I + \operatorname{Ann} K$ annihilates $\mathrm{H}^0(\mathbf{x}^\infty; K)$, $\mathrm{H}^0(\mathbf{x}^\infty; K) = 0$. If however $I \subseteq \mathfrak{M}$, then it follows from the definition that $\mathrm{H}^0(\mathbf{x}^\infty; K) = K$. Since Koszul cohomology commutes with direct sum, it follows that for any sequence $\mathbf{x} = \{x_1, x_2, \cdots, x_n\}$ of elements of R and any i, $\mathrm{H}^i(\mathbf{x}^\infty; M)$ is either zero or one of the factors of M or is M itself. Therefore for all i and for all sequences $\mathbf{x} = \{x_1, x_2, \cdots, x_n\}$ of elements in R, $\mathrm{H}^i(\mathbf{x}^\infty; M)$ has length at most two with non-isomorphic simple quotient submodules.

(v)⇒(i). Let K and L be any two distinct proper submodules of M and $J = \operatorname{Ann}(K + L)$ which is both contained in $\operatorname{Ann}(K)$ and $\operatorname{Ann}(L)$. Let now $\mathbf{y} = \{y\}$ where $y \in J$. Then it follows from the definition of Koszul cohomology that $K = \mathrm{H}^0(\mathbf{y}^\infty; K) \subseteq \mathrm{H}^0(\mathbf{y}^\infty; M)$ and $L = \mathrm{H}^0(\mathbf{y}^\infty; L) \subseteq \mathrm{H}^0(\mathbf{y}^\infty; M)$ and so K and L are also distinct proper submodules of $\mathrm{H}^0(\mathbf{y}^\infty; M)$. Hence by assumption $\mathrm{Ann}(K) + \mathrm{Ann}(L) = R$. □

We note that if M and N are any two R-modules, then it is not hard to see that $\mathrm{Ann}(\mathrm{Hom}_R(M, N))$ and $\mathrm{Ann}(M \otimes_R N)$ contains both $\mathrm{Ann}(M)$ and $\mathrm{Ann}(N)$ and so, if $\mathrm{Ann}(M) + \mathrm{Ann}(N) = R$, then we necessarily have $\mathrm{Hom}_R(M, N) = 0$ and $M \otimes_R N = 0$. Of course, in general, neither $M \otimes_R N = 0$ nor $\mathrm{Hom}_R(M, N) = 0$ implies that $\mathrm{Ann}(M) + \mathrm{Ann}(N) = R$.

PROPOSITION 2.2. *Let $\{M_i\}_{i\in\mathfrak{I}}$ be a family of R-modules such that for all pairs $i \neq j$ in $\mathfrak{I}$, $\mathrm{Ann}(M_i) + \mathrm{Ann}(M_j) = R$. Then*

(i) $\sum_{i\in\mathfrak{I}} M_i = \bigoplus_{i\in\mathfrak{I}} M_i$.

(ii) $\mathrm{Hom}_R(\bigoplus_{i\in\mathfrak{I}_1} M_i, \bigoplus_{j\in\mathfrak{I}_2} M_j) = 0$, *for any two finite disjoint subsets $\mathfrak{I}_1$ and $\mathfrak{I}_2$ of $\mathfrak{I}$.*

(iii) $(\bigoplus_{i\in\mathfrak{I}_1} M_i) \otimes_R (\bigoplus_{j\in\mathfrak{I}_2} M_j) = 0$, *for any two disjoint subsets $\mathfrak{I}_1$ and $\mathfrak{I}_2$ of $\mathfrak{I}$.*

(iv) $\mathrm{Ext}^k_R(\bigoplus_{i\in\mathfrak{I}_1} M_i, \bigoplus_{j\in\mathfrak{I}_2} M_j) = 0$, *for any two finite disjoint subsets $\mathfrak{I}_1$ and $\mathfrak{I}_2$ of $\mathfrak{I}$ and all $k \geq 1$.*

(v) *If moreover for each $i \in \mathfrak{I}$, M_i is simple, then* $\mathrm{Tor}^R_1(\bigoplus_{j\in\mathfrak{I}_1} M_i, \bigoplus_{j\in\mathfrak{I}_2} M_j) = 0$, *for any two disjoint subsets $\mathfrak{I}_1$ and $\mathfrak{I}_2$ of $\mathfrak{I}$.*

Proof. (i) Since every element of $\sum_{i\in\mathfrak{I}} M_i$ is contained in a submodule generated by a finite number of the M_i and since for any finite subset $\mathfrak{J}$ of $\mathfrak{I}$ not containing i, $\mathrm{Ann}(M_i) + \bigcap_{j\in\mathfrak{J}} \mathrm{Ann}(M_j) = R$, it follows that $M_i \cap \sum_{j\in\mathfrak{J}} M_j = 0$, and so $\sum_{i\in\mathfrak{I}} M_i$ is a direct sum.

For the proofs of (ii) and (iii) use the fact that Hom is distributive over finite direct sum and the fact that $\mathrm{Hom}_R(M_i, M_j) = 0$, and Tensor product is distributive over arbitrary direct sum and the fact that $M_i \otimes_R M_j = 0$.

(iv) Since for each pair $i \neq j$ in $\mathfrak{I}$, $\mathrm{Ann}(M_i) + \mathrm{Ann}(M_j) \subseteq \mathrm{Ann}(\mathrm{Ext}^k_R(M_i, M_j))$, it follows that $\mathrm{Ext}^k_R(M_i, M_j) = 0$ for all $k \geq 1$. Hence $\mathrm{Ext}^k_R(\bigoplus_{i\in\mathfrak{I}_1} M_i, \bigoplus_{j\in\mathfrak{I}_2} M_j) = 0$ follows from the fact that Ext is distributive over finite direct sum.

(v) If for each $i \in \mathfrak{I}$, M_i is simple, then $M_i \cong R/\mathrm{Ann}(M_i)$. But then from $\mathrm{Ann}(M_i) + \mathrm{Ann}(M_j) = R$ we have $\mathrm{Ann}(M_i)\,\mathrm{Ann}(M_j) = \mathrm{Ann}(M_i) \cap \mathrm{Ann}(M_j)$. Therefore $\mathrm{Tor}^R_1(M_i, M_j) \cong \mathrm{Ann}(M_i) \cap \mathrm{Ann}(M_j)/\mathrm{Ann}(M_i)\,\mathrm{Ann}(M_j) = 0$. Now, $\mathrm{Tor}^R_1(\bigoplus_{j\in\mathfrak{I}_1} M_i, \bigoplus_{j\in\mathfrak{I}_2} M_j) = 0$ is a consequence of the fact that Tor is distributive over arbitrary direct sum. □

It may be worth mentioning that if R is a Noetherian ring and M and N are two finitely generated R-modules with $\mathrm{Ann}(M) + \mathrm{Ann}(N) = R$, then the i^{th} local cohomology of M with respect to the ideal $\mathrm{Ann}\, N$ is zero. That is, $\mathrm{H}^i_{\mathrm{Ann}\,N}(M) = \varinjlim \mathrm{Ext}^i_R(R/(\mathrm{Ann}\,N)^n, M) = 0$, which easily follows from the proof of part (iv) of Proposition 2.2 above.

Proposition 2.3. *Let R be a ring and M and N be R-modules. Suppose that* $\mathrm{Ann}(M) \neq 0$ *and that* $\mathrm{Ann}(M)$ *is not contained in $Z(N)$, the set of zero divisors of N. Then* $\mathrm{Hom}_R(M, N) = 0$.

Proof. Suppose that $\mathrm{Hom}_R(M, N) \neq 0$, and let f be a non-zero element of $\mathrm{Hom}_R(M, N)$. Then there is a non-zero element m in M such that $f(m) \neq 0$ in N. Let now r be any non-zero element of $\mathrm{Ann}(M)$ which is not contained in $Z(N)$. Then $rf(m) = f(rm) = 0$. But this is a contradiction to the fact that r is not in $Z(N)$. Therefore $\mathrm{Hom}_R(M, N) = 0$. □

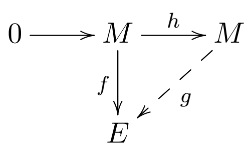
We note that if R is an integral domain and K is the field of fractions of R, then for any non-zero ideal I of R, $\mathrm{Hom}_R(R/I, K) = 0$. Now applying $\mathrm{Hom}(-, K)$ to the short exact sequence $0 \to I \to R \to R/I \to 0$ we obtain $\mathrm{Hom}_R(R, K) \cong \mathrm{Hom}_R(I, K)$, as K is an injective R-module. On the other hand, applying $\mathrm{Hom}(R/I, -)$ to the same short exact sequence we obtain $\mathrm{Hom}_R(R/I, R/I) \cong \mathrm{H}^R_1(R/I, R/I)$, the first homology of the Hom functor applied to the given sequence. Also the conditions that $\mathrm{Ann}(M) \neq 0$ and $\mathrm{Ann}(M) \nsubseteq Z(N)$ in the statement of the above proposition has to be retained for its conclusion. For let $M = R$ and $N = K$, then clearly $\mathrm{Hom}_R(M, N) \neq 0$.

3. DIVISIBLE MODULES OF FINITE LENGTH

Recall that an $R-$module M is divisible if for any nonzero divisor r in R, $M = rM$. In this section, we examine conditions under which a divisible module is of finite length.

Proposition 3.1. *Let M be an R-module with $Z(M) \subseteq Z(R)$ and E be an injective R-module. Then* $\mathrm{Hom}_R(M, E)$ *is a divisible R-module.*

Proof. Let E be an injective R-module and M be any R-module with $Z(M) \subseteq Z(R)$, and let f be a non-zero element of $\mathrm{Hom}_R(M, E)$ and r be a non-zero divisor in R. We want to show that there exists a $g \in \mathrm{Hom}_R(M, E)$ such that $f = rg$. For this, we define $h : M \to M$ by $h(m) = rm$. Then it is clear that h is well-defined and one-to-one. Now using the injectivity of E, one obtains the following commutative diagram:



Thus for any $m \in M$, we have

$$f(m) = gh(m) = g(rm) = rg(m)$$

that is $f = rg$. Therefore $\mathrm{Hom}_R(M, E)$ is divisible. $\square$

Corollary 3.2. *Let E be a torsion free injective module over an integral domain R. Then for any torsion free R-module M, $\mathrm{Hom}_R(M, E)$ is an injective R-module and in particular, $\mathrm{End}_R(E)$ is injective as an R-module.*

Proof. It is easy to see that $\mathrm{Hom}_R(M, E)$ is torsion free. Hence by Proposition 3.1, it is also divisible. Since over an integral domain a torsion free divisible module is injective, $\mathrm{Hom}_R(M, E)$ is an injective R-module. $\square$

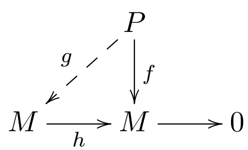
Corollary 3.3. *Let M be an R-module and E be an injective R-module. Then $\mathrm{Hom}_R(\mathrm{Hom}_R(M, R), E)$ is a divisible R-module.*

Proof. Let $r \in Z(\mathrm{Hom}_R(M, R))$. Then there exists a nonzero element $f \in \mathrm{Hom}_R(M, R)$ such that $rf = 0$. Since f is nonzero, $0 \neq f(m) \in R$ for some $m \in M$. But then $rf(m) = 0$ and so $r \in Z(R)$. Therefore $Z(\mathrm{Hom}_R(M, R)) \subseteq Z(R)$ and the result follows from Proposition 3.1. $\square$

The following statement may be considered as the dual of Proposition 3.1:

Proposition 3.4. *Let M be a divisible R-module. Then for any projective R-module P, $\mathrm{Hom}_R(P, M)$ is a divisible R-module.*

Proof. Let M be a divisible and P be a projective R-module, and let f be a non-zero element of $\mathrm{Hom}_R(P, M)$ and r be a non-zero divisor in R. We want to show that there exists a $g \in \mathrm{Hom}_R(P, M)$ such that $f = rg$. For this, we define $h : M \to M$ by $h(m) = rm$. Then from the divisibility of M, h is onto. Now using the projectivity of P, one obtains the following commutative diagram:



Thus for any $p \in P$, we have

$$f(p) = h(g(p)) = rg(p)$$

that is $f = rg$. Therefore $\mathrm{Hom}_R(P, M)$ is divisible. $\square$

Proposition 3.5. *Let M be an Artinian R-module with $Z(M) \subseteq Z(R)$. Then M is divisible.*

Proof. Let $r \in R - Z(R) \subseteq R - Z(M)$. Then because M is Artinian, the chain

$$rM \supseteq r^2M \supseteq \cdots$$

must stabilize *i.e.* $r^nM = r^{n+1}M$ for some positive integer n. Let now $x \in M$ then $r^nx = r^{n+1}y$ for some $y \in M$. Hence $r^n(x - ry) = 0$ and since $r^n \in R - Z(M)$, $x - ry = 0$ implies $x = ry$. Therefore $M = rM$ for all $r \in R - Z(R)$ and so M is divisible. □

PROPOSITION 3.6. *Over an integral domain R which is not a field the only finitely generated divisible module is the zero module.*

Proof. Let M be a finitely generated divisible module over the integral domain R. Then for any nonzero prime ideal P of R and any nonzero element r in P we have $rM = M$ and hence $\frac{r}{1}M_P = M_P$ as R_P-modules. But then by Nakayama's Lemma $M_P = 0$. Thus $M_P = 0$ for all prime ideals P of R and therefore $M = 0$. □

When R is not an integral domain there are cases where R possesses a nonzero finitely generated divisible module and we now establish these facts.

PROPOSITION 3.7. *Let M be a nonzero finitely generated divisible R-module. Then any maximal ideal in the support of M consists of zero divisors.*

Proof. Let $\mathfrak{M}$ be a maximal ideal of R such that $M_{\mathfrak{M}} \neq 0$ and r be a nonzero divisor in $\mathfrak{M}$. Then as M is divisible, $rM = M$ and hence $\frac{r}{1}M_{\mathfrak{M}} = M_{\mathfrak{M}}$ as $R_{\mathfrak{M}}$-modules. But then again by Nakayama's Lemma, $M_{\mathfrak{M}} = 0$. This contradiction shows that $\mathfrak{M}$ consists of zero divisors. □

COROLLARY 3.8. *Let R be a ring and M a finitely generated non-zero R-module. Suppose that the Jacobson radical, $J(R)$, of R is non-zero and that M is divisible. Then $J(R)$ consists of only zero divisors.*

COROLLARY 3.9. *Let R be a local ring with maximal ideal $\mathfrak{M}$ and M be a finitely generated non-zero R-module. Then M is divisible implies that $\mathfrak{M}$ consists of zero divisors.*

PROPOSITION 3.10. *A reduced local ring with finitely many minimal prime ideals which possesses a non-zero finitely generated divisible module is of Krull dimension zero.*

Proof. Let R be a reduced local ring with finitely many minimal prime ideals which possesses a nonzero finitely generated divisible module and $\mathfrak{M}$ be a maximal ideal of R. Then it follows from the above corollary that $\mathfrak{M}$ consists of zero divisors. But then since R is reduced, we have

$$\mathfrak{M} = Z(R) = \cup_{i=1}^{n}\{P_i \mid P_i \textit{ is a minimal prime ideal of } R\}$$

which implies that $\mathfrak{M} = P_i$ for some i and so the height of $\mathfrak{M}$ is zero. Therefore R is of Krull dimension zero. □

It follows from Corollary 3.9 and Proposition 3.10 that if A is a Noetherian local ring which is not a field and possesses a nonzero finitely generated divisible module, then the maximal ideal of A consists of zero divisors and contains at least one nonzero nilpotent element. Therefore the reduced Noetherian local ring $A = k[[x, y]]/(xy)$ does not have a nonzero finitely generated divisible module.

On the other hand, let $R = k[[x, y]]/(x^2, xy)$. Then since every element of R is either a unit or a zero divisor, every R-module is divisible. Note also that R is of Krull dimension one and therefore non-Artinian. Thus there are Noetherian divisible modules that are not Artinian. One also knows that the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty) = \mathbb{Z}[1/p]/\mathbb{Z}$ is an Artinian divisible module which is not Noetherian.

With this in mind, we have the following result:

Theorem 3.11. *Over a reduced Noetherian ring R, a finitely generated divisible module M is Artinian and $Z(M) \subseteq Z(R)$.*

Proof. Let $\mathfrak{M}$ be a maximal ideal of R containing $\operatorname{Ann} M$. Then by Proposition 3.7 and the fact that R is reduced, we have

$$\mathfrak{M} \subseteq Z(R) = \cup_{i=1}^{n}\{P_i \mid P_i \textit{ is a minimal prime ideal of } R\}$$

which implies that $\mathfrak{M} = P_i$ for some i and so height of $\mathfrak{M}$ is zero. Therefore $R/\operatorname{Ann} M$ is of Krull dimension zero and hence is Artinian. Since a finitely generated module over an Artinian ring is Artinian, M is Artinian as an $R/\operatorname{Ann} M$-module. But then since M as an R-module and as an $R/\operatorname{Ann} M$- module is one and the same it follows that M is an Artinian R-module. Let now $r \in R$ be a nonzero divisor in R and define $f : M \to M$ by $f(m) = rm$. It is clear from the divisibility of M that f is onto and also since M is Noetherian, f must be an isomorphism. Therefore $\operatorname{Ker} f = 0$ and so $rm = 0$ implies $m = 0$ which implies that $r \in R$ is a nonzero divisor of M. Thus $R - Z(R) \subseteq R - Z(M)$ and hence we have $Z(M) \subseteq Z(R)$. □

Proposition 3.12. *Over a Noetherian integral domain R of Krull dimension 1, a finitely generated module M with $\operatorname{Ann} M \neq 0$ is Artinian and so is of finite length.*

Proof. Since R is of Krull dimension 1, $rad(\operatorname{Ann} M)$, the radical of $\operatorname{Ann} M$, is a finite product of maximal ideals of R, and so $R/\operatorname{Ann} M$ is Artinian. Hence M is Artinian both as an R-module and an $R/\operatorname{Ann} M$-module. □

We also would like to mention that if R is any ring with $J(R) \neq 0$ and M is an Artinian R-module with $Z(M) \subseteq Z(R)$, then $J(R)$ is contained in the set of zero divisors of R. This easily follows from the proof of the following proposition.

PROPOSITION 3.13. *Let R be a ring with nonzero Jacobson radical $J(R)$ and M be an Artinian R-module. Then $J(R) \subseteq Z(M)$.*

Proof. Suppose $J(R) \nsubseteq Z(M)$ and let $r \in J(R) - Z(M)$. Then for any nonzero $x \in M$, the Nakayama's Lemma would give a non-stationary descending chain of submodules of M

$$Rx \supsetneqq rRx \supsetneqq r^2Rx \supsetneqq \cdots$$

But then this yields a contradiction. Therefore $J(R) \subseteq Z(M)$. □

4. LOCAL COHOMOLOGY MODULES OF FINITE LENGTH

Let $(R, \mathfrak{M})$ be a Noetherian local ring and I be an ideal of R. Then for any finitely generated R-module M with dimension d, one knows that for all i, $\mathrm{H}^i_{\mathfrak{M}}(M)$ and $\mathrm{H}^d_I(M)$ are Artinian modules. Here in this section, we use the information of Section 3 to give necessary and sufficient conditions for $\mathrm{H}^d_I(M)$ to be of finite length. The following is yet another proof (that uses Proposition 3.6) of the so-called Grothendieck's non-vanishing theorem, see for example Section 6.1.4 of [1] and [7].

THEOREM 4.1. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{M}$ and M be a finitely generated R- module of dimension d. Then $\mathrm{H}^d_{\mathfrak{M}}(M)$ is finitely generated if and only if $d = 0$.*

Proof. By the Independence of Base [4, Proposition 2.14], we may place R by $R/\operatorname{Ann} M$. Therefore we may assume that $\operatorname{Ann} M = 0$ and so $d = \dim_R M = \dim_R R$. On the other hand, as is well-known that $\mathrm{H}^i_{\hat{\mathfrak{M}}}(\hat{M}) \cong \mathrm{H}^i_{\mathfrak{M}}(M)$, we may also assume that R is complete, here $\hat{M}$ denotes the $\mathfrak{M}$-adic completion of R. Then by Cohen's structure theorem, R is the homomorphic image of a complete regular Noetherian local ring T with dimension $n \geq d$ and thus there is a surjective homomorphism $\phi : T \to R$ and clearly $I = Ker\phi$ is an ideal of T with height $n - d$. Since every regular local ring is Cohen-Macaulay, I contains a regular sequence $(x_1, x_2, \cdots, x_{n-d})$ and so $T/(x_1, x_2, \cdots, x_{n-d})$ is a regular local ring. Let $S = T/(x_1, x_2, \cdots, x_{n-d})$. Then clearly $\dim S = d$. Let now $\mathfrak{M}_S$ be the maximal ideal of S and $E_S(S/\mathfrak{M}_S)$ be the injective hull of the residue field $S/\mathfrak{M}_S$ of S and so again by the Independence of Base,

$\mathrm{H}^d_{\mathfrak{M}}(M) \cong \mathrm{H}^d_{\mathfrak{M}_S}(M)$. But then by the local duality theorem [4, Theorem 4.4], we have

$$\mathrm{H}^d_{\mathfrak{M}}(M) \cong \mathrm{H}^d_{\mathfrak{M}_S}(M) \cong \mathrm{Hom}_S(\mathrm{Hom}_S(M,S), E_S(S/\mathfrak{M}_S)).$$

Since every regular local ring is an integral domain, $0 \in \mathrm{Ass}(\mathrm{Hom}_S(M,S))$ which implies that $\mathrm{Hom}_S(M,S)$ is nonzero and then again by local duality $\mathrm{H}^d_{\mathfrak{M}}(M)$ is nonzero. On the other hand, by Corollary 3.3

$$\mathrm{Hom}_S(\mathrm{Hom}_S(M,S), E_S(S/\mathfrak{M}_S))$$

is a divisible S-module. Then by Proposition 3.6, $\mathrm{H}^d_{\mathfrak{M}}(M)$ is finitely generated only if S is Artinian and so $d = \dim S = 0$.

Conversely, suppose $\dim M = 0$. Then the result follows from the fact that $\mathrm{H}^0_{\mathfrak{M}}(M) \subseteq M$. □

THEOREM 4.2. *Let R be a reduced Noetherian local ring and M be a finitely generated R-module of dimension d with the property that $Z(M) \subseteq Z(R)$. Then for any ideal I of R, $\mathrm{H}^d_I(M)$ is a nonzero finitely generated R-module if only if R is a field.*

Proof. Suppose $\mathrm{H}^d_I(M)$ is a nonzero finitely generated R-module and $x \in R - Z(R) \subseteq R - Z(M)$. Then the short exact sequence

$$0 \longrightarrow M \xrightarrow{.x} M \longrightarrow M/xM \longrightarrow 0$$

yields the following long exact sequence

$$\begin{aligned} 0 \longrightarrow \ & \mathrm{H}^0_I(M) \xrightarrow{.x} \mathrm{H}^0_I(M) \longrightarrow \mathrm{H}^0_I(M/xM) \longrightarrow \cdots \\ \longrightarrow \ & \mathrm{H}^{d-1}_I(M/xM) \longrightarrow \mathrm{H}^d_I(M) \xrightarrow{.x} \mathrm{H}^d_I(M) \\ \longrightarrow \ & \mathrm{H}^d_I(M/xM) \longrightarrow \cdots \end{aligned}$$

Since $\dim M/xM < d$, $\mathrm{H}^d_I(M/xM)$ is zero and so the map $\mathrm{H}^d_I(M) \xrightarrow{.x} \mathrm{H}^d_I(M)$ is surjective. Therefore for any nonzero divisor x of R, we have $\mathrm{H}^d_I(M) = x\,\mathrm{H}^d_I(M)$ which implies that $\mathrm{H}^d_I(M)$ is divisible. Then by Proposition 3.10, R is Artinian. The result now follows from the fact that a reduced Artinian ring is nothing but a field.

The converse is obvious. □

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