INTEGRABILITY RESULTS FOR WEAK SOLUTIONS TO ULTRAPARABOLIC EQUATIONS

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The aim of this paper is to consider a class of linear ultraparabolic equations with bounded and VMO coefficients $a_{ij}(z)$. We first establish higher L^p estimate and higher Morrey estimates for gradients of weak solutions. And then we give a Campanato estimate for weak solutions.

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1. INTRODUCTION

In the paper, we consider the ultraparabolic equation of the form

$$Lu = \sum_{i,j=1}^{m_0} \partial_{x_i} \left(a_{ij}(z)\partial_{x_j} u(z)\right) + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} u(z) - \partial_t u(z) = g(z) + \sum_{j=1}^{m_0} \partial_{x_j} f_j(z), \tag{1.1}$$

where $z = (x, t) \in \mathbb{R}^{N+1}$, $1 \le m_0 \le N$, $b_{ij} \in \mathbb{R}$ $(i, j = 1, \dots, N)$, $g, f_j \in L^p(\Omega)$ or $L^{p,\lambda}(\Omega)$, $L^{p,\lambda}(\Omega)$ is a Morrey space, Ω is a bounded domain in $\mathbb{R}^{N+1}$, $p \ge 2$, $0 \le \lambda < Q + 2$, Q is the homogeneous dimension, see Section 2.

The assumptions to (1.1) are:

(H1) (ellipticity condition on $\mathbb{R}^{m_0}$) Let coefficients $a_{ij}(z) \in VMO \cap L^\infty(\Omega)$(see next section for the definition of VMO), $a_{ij}(z) = a_{ji}(z)$. Assume that there exists a constant $\Lambda > 1$ such that for any $z \in \mathbb{R}^{N+1}$, $\xi \in \mathbb{R}^{m_0}$,

$$\Lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^{m_0} a_{ij}(z)\xi_i\xi_j \le \Lambda|\xi|^2.$$

(H2) The constant matrix $B = (b_{ij})_{i,j=1,\dots,N}$ in (1.1) has the form

$$B = \begin{pmatrix} 0 & B_1 & 0 & \cdots & 0 \\ 0 & 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_r \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $B_k (k = 1, 2, \ldots, r)$ is a $m_{k-1} \times m_k$ matrix with rank m_k and

$$m_0 \geq m_1 \geq \cdots \geq m_r \geq 1, \quad \sum_{k=0}^{r} m_k = N.$$

The equation (1.1) can be written as

$$Lu = div\,(A(z)D_0 u) + Yu = g + div f,$$

where $D_0 = \big(\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_{m_0}}, 0, \ldots, 0\big)$, $Yu = \langle x, BDu\rangle - \partial_t u$, $D = (\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_N})$, $f = (f_1, f_2, \ldots, f_{m_0}, 0, \ldots, 0)$, $A(z)$ is a $N \times N$ matrix with the form

$$A(z) = \begin{pmatrix} A_0(z) & 0 \\ 0 & 0 \end{pmatrix}, \quad A_0(z) = (a_{ij}(z))_{i,j=1,\ldots,m_0}.$$

Regularity for weak solutions to parabolic equations were provided by many mathematicians including DiBenedetto [6], Friedman [11], Krylov [19], Ladyzhenskaya-Solonnikov-Ural'tseva [20], Lieberman [22] and references therein.

In recent decades, many scholars have been concerned with regularity of weak solutions to ultraparabolic equations. These equations are closely related to finance, Brown motion, particle physics and human vision, etc. In contrast to the classic linear parabolic equation

$$\sum_{i=1}^{N} \partial_{x_i x_j} u(x,t) - \partial_t u(x,t) = f(x,t),$$

(1.1) is strongly degenerate if $1 \leq m_0 < N$ and owns a drift Yu. These differences give rise to several new difficulties to research of regularity to (1.1) and new tools have to be drawn.

For the homogeneous ultraparabolic equation

$$Lu = \sum_{i,j=1}^{m_0} \partial_{x_i}\big(a_{ij}(z)\partial_{x_j} u(z)\big) + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} u(z) - \partial_t u(z) = 0, \tag{1.2}$$

Polidoro in [28] gave global lower bound of the fundamental solution to (1.2). The boundedness of weak solutions to (1.2) with measurable coefficients was investigated by Pascucci and Polidoro in [27] with Moser's iteration method based on a combination of a Caccioppoli type estimate and the classical embedding Sobolev inequality. Wang and Zhang in [30] derived Hölder estimates for weak solutions to (1.2) with measurable coefficients by establishing local a priori estimate to (1.2) and a Poincaré inequality of nonnegative weak lower solution.

To the following ultraparabolic equation

(1.3)

$$Lu = \sum_{i,j=1}^{m_0} \partial_{x_i} \left(a_{ij}(z)\partial_{x_j} u(z)\right) + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} u(z) - \partial_t u(z) = \sum_{j=1}^{m_0} \partial_{x_j} F_j(x,t)$$

with $F_j \in L^p_{loc}\left(\mathbb{R}^{N+1}\right)$ $(1 < p < \infty)$, $a_{ij}(z)$ belonging to VMO spaces, Manfredini and Polidoro in [24] concluded L^p estimates and Hölder continuity for weak solutions $u \in L^p_{loc}\left(\mathbb{R}^{N+1}\right)$. If $F_j \in L^{p,\lambda}_{loc}\left(\mathbb{R}^{N+1}\right)$ $(1 < p < \infty, 0 \le \lambda < Q+2)$ and $a_{ij}(z)$ belong to some VMO spaces, Polidoro and Ragusa in [29] deduced Hölder regularity for weak solution $u \in L^p_{loc}\left(\mathbb{R}^{N+1}\right)$ to (1.3). Bramanti, Cerutti and Manfredini [2] proved local L^p estimates for second order derivatives $\partial_{x_i x_j} u\,(i,j = 1,\ldots,m_0)$ of strong solutions to the nondivergence ultraparabolic equation

$$\sum_{i,j=1}^{m_0} a_{ij}(z)\partial_{x_i x_j} u + \langle x, BD\rangle\, u - \partial_t u = f$$

with $a_{ij}(z)$ being in VMO and $f \in L^p$. The methods in [2, 24, 29] are based on the representation formulae for solutions and estimates of singular integral operators. More related results also see Cinti, Passcucci and Polidoro [5], Xin and Zhang [31], Zhang [32] and references therein.

The aim of this paper is to establish integrability for weak solution $u \in W_2^{1,1}(\Omega)$ to (1.1) with the method of a priori estimates. For results on higher integrability of parabolic equations, see Byun and Wang [3], Fugazzda [12], Palagachev and Softova [26] and references therein. The first statement in our paper is the higher $L^p (p > 2)$ estimates. For this purpose, an appropriate frame is homogeneous spaces. Bramanti, Cerutti and Manfredini [2] pointed out that the ball related to a quasidistance of (1.1) (see Section 2 below) is a homogeneous space and Gianazza [14] showed a reverse Hölder inequality on homogeneous spaces. These facts will play important roles in our proof and in spite of this, some new preliminary conclusions are needed to supply. Inspired by the method in [27], we deduce a Caccioppoli type inequality and a Sobolev type inequality for weak solution to (1.1). Following to [8], a Poincaré type inequality for weak solution to (1.1) is obtained. And then we prove higher L^p estimates for gradients of weak solutions to (1.1) by using these new inequalities and the reverse Hölder inequality in [14].

The second result is higher integrability in Morrey spaces for gradients of weak solution $u \in W_2^{1,1}(\Omega)$ to (1.1). With the aid of the approach appeared in parabolic equations (*e.g.*, see [15]), we consider a homogeneous ultraparabolic equation of variable coefficients with a nonhomogeneous boundary value condition, *i.e.*, (6.1) below, and a nonhomogeneous ultraparabolic equation of variable coefficients with homogeneous boundary value condition, *i.e.*, (6.2). The L^p estimates for gradients of weak solutions to (6.1) is gained by proving a local L^∞ estimate and a local L^2 estimate of weak solutions to homogeneous ultraparabolic equation of constant coefficient, (5.1). We also establish a local L^p estimate for gradients of weak solutions to (6.2). These results are of independent interest and allow us to deduce higher integrability in Morrey spaces for gradients of weak solutions to (1.1) by combining a known iteration lemma in [16, 25].

Finally, we derive a Campanato estimate for weak solutions to (1.1) with the Poincaré type inequality in Section 3 and the estimates for (5.1) and (6.2).

The following is the notion of weak solution to (1.1).

Definition 1.1. If $u \in W_2^{1,1}(\Omega)$ and for any $\psi \in C_0^\infty(\Omega)$,

$$-\int_\Omega AD_0uD_0\psi \mathrm{d}z + \int_\Omega \psi Yu\mathrm{d}z = \int_\Omega (g\psi - fD_0\psi)\mathrm{d}z,$$

then u is said a weak solution to (1.1).

The main results of this paper are stated as follows.

THEOREM 1.1. *Under assumptions (H1) and (H2), if $u \in W_2^{1,1}(\Omega)$ is a weak solution to (1.1), $g, f_j \in L^p(\Omega)$, then there exists a constant $\varepsilon_0 > 0$ such that for any $p \in \left[2, 2 + \frac{2Q}{Q+2}\varepsilon_0\right)$, we have $D_0u \in L^p_{loc}(\Omega)$ and for any $\Omega' \subset\subset \Omega'' \subset\subset \Omega$,*

$$\|D_0u\|_{L^p(\Omega')} \le c\left(\|D_0u\|_{L^2(\Omega'')} + \|g\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}\right). \tag{1.4}$$

THEOREM 1.2. *Under (H1) and (H2), let $u \in W_2^{1,1}(\Omega)$ be a weak solution to (1.1), $g, f_j \in L^{p,\lambda}(\Omega)$, then for any $p \in \left[2, 2 + \frac{2Q}{Q+2}\varepsilon_0\right)$, ε_0 as in Theorem 1.1, we have $D_0u \in L^{p,\lambda}_{loc}(\Omega)$ $(p < \lambda < Q + 2)$ and for any $\Omega' \subset\subset \Omega'' \subset\subset \Omega$,*

$$\|D_0u\|_{L^{p,\lambda}(\Omega')} \le c\left(\|D_0u\|_{L^2(\Omega'')} + \|g\|_{L^{p,\lambda}(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}\right). \tag{1.5}$$

THEOREM 1.3. *Under the assumptions of Theorem 1.2, we have $u \in \mathcal{L}_{loc}^{2,Q+4-\frac{2\lambda}{p}}$, and for any $\Omega' \subset\subset \Omega'' \subset\subset \Omega$,*

$$[u]_{2,Q+4-\frac{2\lambda}{p};\Omega'} \le c\left(\|u\|_{L^p(\Omega'')} + \|f\|_{L^{p,\lambda}(\Omega)} + \|g\|_{L^{p,\lambda}(\Omega)}\right). \tag{1.6}$$

This paper is organized as follows. In Section 2, we describe some basic knowledge on the fundamental solution of L_0 the frozen operator of L, and collect several useful lemmas which will be used later on. Section 3 is devoted to proofs of a Caccioppoli type inequality, a Sobolev type inequality and a

Poincaré type inequality for weak solutions. In Section 4, the proof of Theorem 1.1 is given by using the inequalities in Section 3. In Section 5, we derive a higher L^p estimate for gradient of weak solutions to (5.1). In Section 6, the proof of Theorem 1.2 is ended by proving local L^p estimate for gradient of weak solutions to (6.1) and (6.2), and the proof of Theorem 1.3 is given.

2. PRELIMINARIES

For any $z_0 \in \Omega \subset \mathbb{R}^{N+1}$, we denote the frozen operator of L by

$$L_0 = \sum_{i,j=1}^{m_0} \partial_{x_i}\left(a_{ij}(z_0)\partial_{x_j}\right) + \sum_{i,j=1}^{N} b_{ij}x_i\partial_{x_j} - \partial_t. \tag{2.1}$$

Now let us introduce the following.

Definition 2.1. For any $(x,t),(\xi,\tau) \in \mathbb{R}^{N+1}$, define a multiplication law

$$(x,t)\circ(\xi,\tau) = (\xi + E(\tau)x, t+\tau)\,, \quad E(\tau) = \exp(-\tau B^T).$$

We say that $\left(\mathbb{R}^{N+1},\circ\right)$ is a noncommutative Lie group with the neutral element $(0,0)$, the inverse of an element $(x,t) \in \mathbb{R}^{N+1}$ is

$$(x,t)^{-1} = (-E(-t)x, -t)\,.$$

Authors in [21] claimed that the frozen operator L_0 is hypoelliptic and left invariant about the groups of translations and dilations. Note that the dilations associated to L_0 are given by

$$\delta_\lambda = diag\left(\lambda I_{m_0}, \lambda^3 I_{m_1}, \ldots, \lambda^{2r+1} I_{m_r}, \lambda^2\right), \quad \lambda > 0,$$

here I_{m_k} denotes the $m_k \times m_k$ identity matrix, and

$$\det\left(\delta_\lambda\right) = \lambda^{Q+2},$$

with $Q+2 = m_0 + 3m_1 + \cdots + (2r+1)m_r + 2$. In this case, the number $Q+2$ is called the homogeneous dimension of $\mathbb{R}^{N+1}$, and Q the homogeneous dimension of $\mathbb{R}^N$. This implies that L_0 is δ_λ homogeneous of degree 2, namely, for any $\lambda > 0$,

$$L_0 \circ \delta_\lambda = \lambda^2\left(\delta_\lambda \circ L_0\right).$$

Due to [17], the fundamental solution $\Gamma_0(\cdot,\zeta)$ of L_0 has an explicit expression with respect to the pole $\zeta \in \mathbb{R}^{N+1}$: that is, for any $z,\zeta \in \mathbb{R}^{N+1}$, $z \neq \zeta$,

$$\Gamma_0(z,\zeta) = \Gamma_0(\zeta^{-1}\circ z, 0), \tag{2.2}$$

where

$$\Gamma_0\left((x,t),(0,0)\right)=\begin{cases}\frac{1}{\left((4\pi)^N\det C(t)\right)^{\frac{1}{2}}}\exp\left(-\frac{1}{4}\left\langle C^{-1}(t)x,x\right\rangle\right), t>0,\\ 0, \qquad t\le 0,\end{cases}$$

$$C(t)=\int_0^t E(s)A_0E^T(s)ds.$$

It is known that $C(t)$ is strictly positive for every positive t. In view of the invariance properties of L_0, we have that for any $z\in\mathbb{R}^{N+1}\backslash\{0\}$ and $\lambda>0$,

$$\Gamma_0\left(\delta_\lambda(z),0\right)=\lambda^{-Q}\Gamma_0(z,0),$$

and it means that Γ_0 is δ_λ homogeneous of degree $-Q$. For $i,j=1,\ldots,m_0$, $D_{x_i}\Gamma_0$ and $D_{x_ix_j}\Gamma_0$ are δ_λ homogeneous of degree $-(Q+1)$ and $-(Q+2)$, respectively.

For any $(x,t)\in\mathbb{R}^{N+1}$, the homogeneous norm of (x,t) with respect to δ_λ is defined by

$$\|(x,t)\|=\sum_{j=1}^{N}|x_j|^{\frac{1}{\alpha_j}}+|t|^{\frac{1}{2}},$$

where $\alpha_j=1$, if $1\le j\le m_0$; $\alpha_j=2k+3$, if $m_k<j\le m_{k+1}\,(0\le k\le r-1)$. For any $z,\zeta\in\mathbb{R}^{N+1}$, we denote the quasidistance between t and ζ by

$$d\left(z,\zeta\right)=\left\|\zeta^{-1}\circ z\right\|.$$

LEMMA 2.2 ([7], Lemma 2.1). *Let $\Omega\subset\mathbb{R}^{N+1}$ be a bounded domain. Then $d\left(z,\zeta\right)$ is a quasisymmetric quasidistance in Ω, if for any $z,z',\zeta\in\Omega$,*

$$d\left(z,\zeta\right)\le cd\left(\zeta,z\right),\quad d\left(z,\zeta\right)\le c\left(d\left(\zeta,z'\right)+d\left(z',\zeta\right)\right).$$

The ball with respect to d centered at z_0 is denoted by

$$B_R(z_0)=B(z_0,R)=\left\{\zeta\in\mathbb{R}^{N+1}:\ d\left(z_0,\zeta\right)<R\right\}.$$

Note clearly that $B(0,R)=\delta_RB(0,1)$.

Remark 2.3. Recalling [2, Remark 1.5], it holds that for any $z_0\in\mathbb{R}^{N+1}$, $R>0$,

$$|B(z_0,R)|=|B(0,R)|=|B(0,1)|\,R^{Q+2},$$
$$|B(z_0,2R)|=2^{Q+2}\,|B(z_0,R)|\,,$$

and therefore the space $\left(\mathbb{R}^{N+1},\mathrm{d}z,d\right)$ is a homogeneous space. The fact allows us to employ known conclusions in homogeneous spaces, for example, see [1].

If one does not need to concern the center of the ball, $B(z_0,R)$ can simply be written as B_R. For convenience, we usually consider the estimates on cubes instead of balls. Let us describe the notion of cubes. For any $t\in\mathbb{R}$,

$x = (x', \bar{x}) \in \mathbb{R}^N$ with $x' = (x_1, \cdots, x_{m_0})$, $\bar{x} = (x_{m_0+1}, \cdots, x_N)$, the cube of centered at (x_0, t_0) is defined by

$$Q_R = \Big\{ (x,t) \,\Big|\, t_0 - R^2/2 \le t \le t_0 + R^2/2, |x'| \le R, |x_{m_0+1}| \le \left(\Lambda N^2 R\right)^3, \cdots, \\ |x_N| \le \left(\Lambda N^2 R\right)^{2r+1} \Big\}$$

Also, we write

$$I_R = \left[t_0 - \frac{R^2}{2}, t_0 + \frac{R^2}{2} \right],$$
$$K_R = \left\{ x' \,\big|\, |x'| \le R \right\},$$
$$S_R = \left\{ \bar{x} \,\Big|\, |x_{m_0+1}| \le \left(\Lambda N^2 R\right)^3, \cdots, |x_N| \le \left(\Lambda N^2 R\right)^{2r+1} \right\}.$$

Then $Q_R = K_R \times S_R \times I_R$.

A cube of centered at $(0,0)$ is simply denoted by

$$Q_R(0,0) = \left\{ (x,t) \,\big|\, |t| \le R^2, |x_1| \le R^{\alpha_1}, \cdots, |x_N| \le R^{\alpha_N} \right\}.$$

It is easy to see that there exists a constant $c_0 = c_0(B, N) > 0$, such that

$$Q_{R/c_0}(0,0) \subset B_R(0,0) \subset Q_{c_0 R}(0,0).$$

We state a result on δ_λ homogeneous functions in [9, 27].

Lemma 2.4. *Let $\alpha \in [0, Q+2]$ and $G \in C\left(\mathbb{R}^{N+1} \backslash \{0\}\right)$ be a δ_λ homogeneous function of degree $\alpha - Q - 2$. If $f \in L^p\left(\mathbb{R}^{N+1}\right)$, $p \in [1, +\infty)$, then the function*

$$G_f(z) \equiv \int_{\mathbb{R}^{N+1}} G(\zeta^{-1} \circ z) f(\zeta) \mathrm{d}\zeta$$

is well defined almost everywhere and there exists a constant $c = c(Q, P) > 0$ such that

$$\|G_f\|_{L^q(\mathbb{R}^{N+1})} \le c \max_{\|z\|=1} |G(z)| \, \|f\|_{L^p(\mathbb{R}^{N+1})}, \tag{2.3}$$

where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q+2}$.

This lemma can be used to yield the following.

Lemma 2.5. *Let $f \in L^{\frac{2(Q+2)}{Q+4}}\left(\mathbb{R}^{N+1}\right)$. There exists a positive constant $c = c(Q)$ such that*

$$\|\Gamma_0(f)\|_{L^{\frac{2(Q+2)}{Q}}(\mathbb{R}^{N+1})} \le c \|f\|_{L^{\frac{2(Q+2)}{Q+4}}(\mathbb{R}^{N+1})}, \tag{2.4}$$

$$\|\Gamma_0(D_0 f)\|_{L^2(\mathbb{R}^{N+1})} \le c \|f\|_{L^{\frac{2(Q+2)}{Q+4}}(\mathbb{R}^{N+1})}, \tag{2.5}$$

where

$$\Gamma_0(f)(z) = \int_{\mathbb{R}^{N+1}} \Gamma_0(z,\zeta) f(\zeta)\mathrm{d}\zeta,$$

$$\Gamma_0(D_0 f)(z) = \int_{\mathbb{R}^{N+1}} \Gamma_0(z,\zeta) D_0 f(\zeta)\mathrm{d}\zeta.$$

Proof. Since Γ_0 is homogeneous of degree $-Q$ with respect to δ_λ, we immediately have (2.4) from Lemma 2.4 by taking $\alpha = 2$, $q = \frac{2(Q+2)}{Q}$ and $p = \frac{2(Q+2)}{Q+4}$. Noting that $\partial_{x_i}\Gamma_0$ is homogeneous of degree $-(Q+1)$ with respect to δ_λ, (2.5) holds by Lemma 2.4 with $\alpha = 1$, $q = 2$ and $p = \frac{2(Q+2)}{Q+4}$. $\square$

Definition 2.6 (Morrey space $L^{p,\lambda}$). Let Ω be an open bounded subset in $\mathbb{R}^{N+1}$, $1 \le p < +\infty$, $\lambda \ge 0$. We say that $f \in L^p(\Omega)$ belongs to the Morrey space $L^{p,\lambda}(\Omega)$, if

$$\|f\|_{L^{p,\lambda}} := \sup_{z_0\in\Omega, 0<\rho<d_0} \left(\frac{\rho^\lambda}{|\Omega \cap B_\rho(z_0)|} \int_{\Omega\cap B_\rho(z_0)} |f|^p \mathrm{d}z \right)^{\frac{1}{p}} < +\infty,$$

where d_0 is the diameter of Ω.

Definition 2.7 (Campanato space $\mathcal{L}^{p,\lambda}$). Let $1 \le p < +\infty, \lambda \ge 0$, if for any $f \in L^p(\Omega)$,

$$[f]_{p,\lambda} := \sup_{z_0\in\Omega, 0<\rho<d_0} \left(\frac{1}{\rho^\lambda} \int_{\Omega\cap B_\rho(z_0)} \left|f - f_{\Omega\cap B_\rho(z_0)}\right|^p \mathrm{d}z \right)^{\frac{1}{p}} < +\infty,$$

where d_0 is the diameter of Ω, $f_{\Omega\cap B_\rho(z_0)} = \frac{1}{|\Omega\cap B_\rho(z_0)|}\int_{\Omega\cap B_\rho(z_0)} f(z)\mathrm{d}z$, then we say that f belongs to the Campanato space $\mathcal{L}^{p,\lambda}(\Omega)$ endowed with the norm

$$\|f\|_{\mathcal{L}^{p,\lambda}} = [f]_{p,\lambda} + \|f\|_{L^p}.$$

Definition 2.8 (Sobolev space $W^{1,1}_p$). Let Ω be an open subset in $\mathbb{R}^{N+1}$. The Sobolev space with respect to D_0 and Y is defined by

$$W^{1,1}_p(\Omega) = \{u \in L^p(\Omega) : \partial_{x_i} u, Yu \in L^p(\Omega),\ i,j = 1,\dots,m_0\}$$

endowed with the norm

$$\|u\|^p_{W^{1,1}_p} = \|u\|^p_{L^p} + \sum_{i=1}^{m_0} \|\partial_{x_i} u\|^p_{L^p} + \|Yu\|^p_{L^p}.$$

The space $W^{1,1}_{p,0}(\Omega)$ is the closure of $C_0^\infty(\bar{\Omega})$ in $W^{1,1}_p(\Omega)$.

Definition 2.9 (BMO and VMO spaces). For any $a \in L^1_{loc}(\Omega)$, let

$$\eta_R(a) = \sup_{z_0 \in \Omega, 0 \le \rho \le R} \left(\frac{1}{|\Omega \cap B_\rho(z_0)|} \int_{\Omega \cap B_\rho(z_0)} \left| a(z) - a_{\Omega \cap B_\rho(z_0)}(z) \right| \mathrm{d}z \right),$$

where $a_{\Omega \cap B_\rho(z_0)} = \frac{1}{|\Omega \cap B_\rho(z_0)|} \int_{\Omega \cap B_\rho(z_0)} a(z)\mathrm{d}z$. If $\sup_{R>0} \eta_R(a) < \infty$, we say $a \in BMO(\Omega)$ (Bounded Mean Oscillation). Moreover, if $\eta_R(a) \to 0$ as $R \to 0$, we say $a \in VMO(\Omega)$ (Vanishing Mean Oscillation).

We mention two iterative lemmas.

LEMMA 2.10 ([4]). *Let $\varphi(t)$ be a bounded nonnegative function on $[T_0, T_1]$, where $T_1 > T_0 \ge 0$. Suppose that for any $s, t : T_0 \le t < s \le T_1$, φ satisfies*

$$\varphi(t) \le \theta_1 \varphi(s) + \frac{a_2}{(s-t)^\alpha} + b_2,$$

where θ_1, a_2, b_2 and α are nonnegative constants, and $\theta_1 < 1$. Then for any $T_0 \le \rho < R \le T_1$,

$$\varphi(\rho) \le c \left(\frac{a_2}{(R-\rho)^\alpha} + b_2 \right), \tag{2.6}$$

where c depends only on α and θ_1.

LEMMA 2.11 ([16,25]). *Let H be a nonnegative increasing function. Suppose that for any $\rho < R \le R_0 = dist(z_0, \partial\Omega)$,*

$$H(\rho) \le A_1 \left[\left(\frac{\rho}{R} \right)^{a_1} + \varepsilon_1 \right] H(R) + B_1 R^{b_1},$$

where A_1, ε_1, a_1 and b_1 are nonnegative constants with $A \geqslant 1, a_1 > b_1, \varepsilon_1 > 0$. Then there exist positive constants $\varepsilon_2 = \varepsilon_2(A_1, a_1, b_1)$ and $c = c(A_1, a_1, b_1)$, such that if $\varepsilon_1 < \varepsilon_2$, then

$$H(\rho) \le c \left[\left(\frac{\rho}{R} \right)^{b_1} H(R) + B_1 \rho^{b_1} \right]. \tag{2.7}$$

3. PRELIMINARY INEQUALITIES

LEMMA 3.1 (Caccioppoli type inequality). *Let $u \in W_2^{1,1}(\Omega)$ be a weak solution to (1.1). Then for any $B_R \subset \Omega$, $\rho < R$, there exists a positive constant c such that:*

$$\int_{B_\rho} |D_0 u|^2 \mathrm{d}z \le \frac{c}{(R-\rho)^2} \int_{B_R} |u|^2 \mathrm{d}z + c \int_{B_R} \left(|g|^2 + |f|^2 \right) \mathrm{d}z. \tag{3.1}$$

Proof. Let $\xi(z) \in C_0^\infty(B_R)$ be a cutoff function [27] satisfying

$$
\begin{aligned}
&\xi(z) = 1(|z| < \rho), \quad \xi(z) = 0(|z| \geq R), \quad 0 \leq \xi \leq 1, \\
&\left|\partial_{x_j}\xi\right|, |\partial_t \xi| \leq \frac{c}{R-\rho}(j = 1, \ldots, N).
\end{aligned} \tag{3.2}
$$

Hence

$$
|Y\xi| = |xBD\xi - \partial_t \xi| \leq c\,|D\xi| + c\,|\partial_t \xi| \leq \frac{c}{R-\rho},
$$

and by the divergence theorem,

$$
\int_{B_R} Y\left(u^2\xi^2\right)\mathrm{d}z = 0.
$$

Multiplying both sides of (1.1) by $u\xi^2$ and integrating on B_R, we have

$$
\int_{B_R}\left[-AD_0uD_0\left(u\xi^2\right) + u\xi^2 Yu\right]\mathrm{d}z = \int_{B_R}\left[gu\xi^2 - fD_0\left(u\xi^2\right)\right]\mathrm{d}z
$$

and

$$
\begin{aligned}
&\int_{B_R} A\xi^2 D_0uD_0u\mathrm{d}z \\
&= -2\int_{B_R} Au\xi D_0uD_0\xi\mathrm{d}z - \int_{B_R} u^2\xi Y\xi\mathrm{d}z - \int_{B_R} gu\xi^2\mathrm{d}z \\
&\quad + \int_{B_R} f\xi^2 D_0u\mathrm{d}z + 2\int_{B_R} fu\xi D_0\xi\mathrm{d}z.
\end{aligned} \tag{3.3}
$$

It follows from (H1) and Young's inequality with $\varepsilon > 0$,

$$
\begin{aligned}
&\Lambda^{-1}\int_{B_R}|D_0u|^2\xi^2\mathrm{d}z \\
&\leq c_\varepsilon\int_{B_R}|u|^2|D_0\xi|^2\mathrm{d}z + \varepsilon\int_{B_R}|D_0u|^2\xi^2\mathrm{d}z + \int_{B_R}|u|^2\,|Y\xi|\,\xi\mathrm{d}z \\
&\quad + c_\varepsilon\int_{B_R}|g|^2\xi^2\mathrm{d}z + \varepsilon\int_{B_R}|u|^2\xi^2\mathrm{d}z + c_\varepsilon\int_{B_R}|f|^2\xi^2\mathrm{d}z \\
&\quad + \varepsilon\int_{B_R}|D_0u|^2\xi^2\mathrm{d}z + c_\varepsilon\int_{B_R}|f|^2\xi^2\mathrm{d}z + \varepsilon\int_{B_R}|u|^2|D_0\xi|^2\mathrm{d}z \\
&\leq \int_{B_R}|u|^2\left(c_\varepsilon|D_0\xi|^2 + |Y\xi|\,\xi + \varepsilon\xi^2 + \varepsilon|D_0\xi|^2\right)\mathrm{d}z \\
&\quad + 2\varepsilon\int_{B_R}|D_0u|^2\xi^2\mathrm{d}z + c_\varepsilon\int_{B_R}\left(|g|^2 + |f|^2\right)\xi^2\mathrm{d}z.
\end{aligned} \tag{3.4}
$$

Now we choose ε small enough such that $\Lambda^{-1}-2\varepsilon>0$ to derive

$$\begin{aligned}&\int_{B_R}|D_0u|^2\xi^2\mathrm{d}z\\ \le&\int_{B_R}|u|^2\left(c_\varepsilon|D_0\xi|^2+|Y\xi|\,\xi+\varepsilon\xi^2+\varepsilon|D_0\xi|^2\right)\mathrm{d}z+c_\varepsilon\int_{B_R}\left(|g|^2+|f|^2\right)\xi^2\mathrm{d}z\\ \le&\int_{B_R}|u|^2\left(\frac{c_\varepsilon}{(R-\rho)^2}+\frac{c\xi}{R-\rho}+\varepsilon\xi^2+\frac{\varepsilon}{(R-\rho)^2}\right)\mathrm{d}z\\ &+c_\varepsilon\int_{B_R}\left(|g|^2+|f|^2\right)\xi^2\mathrm{d}z.\end{aligned}$$

So (3.1) is proved. □

There are many observations on Sobolev type inequalities and Poincaré type inequalities with respect to vector fields [10, 13, 18, 23]. Here we provide some similar inequalities for weak solutions to (1.1).

Lemma 3.2 (Sobolev type inequality). *Let $u\in W_2^{1,1}(\Omega)$ be a weak solution to (1.1). Then for any $B_R\subset\Omega$, $\rho<R$, it follows*

$$\begin{aligned}\|u\|_{L^2(B_\rho)}\le\frac{c}{R-\rho}\Bigg(&\|u\|_{L^{\frac{2(Q+2)}{Q+4}}(B_R)}+\|D_0u\|_{L^{\frac{2(Q+2)}{Q+4}}(B_R)}\\ &+\|g\|_{L^{\frac{2(Q+2)}{Q+4}}(B_R)}+\|f\|_{L^{\frac{2(Q+2)}{Q+4}}(B_R)}\Bigg).\end{aligned}\tag{3.5}$$

Proof. We use the fundamental solution Γ_0 of L_0 and the cutoff function ξ in (3.2) to write that for any $z\in B_R$,

$$(\xi u)(z)=\int_{B_R}[\langle A_0D_0(\xi u),D_0\Gamma_0\rangle-\Gamma_0Y(\xi u)]\mathrm{d}\zeta\triangleq I_1(z)+I_2(z)+I_3(z),\tag{3.6}$$

where

$$I_1(z)=\int_{B_R}[A_0uD_0\xi D_0\Gamma_0-\Gamma_0uY\xi]\mathrm{d}\zeta,$$

$$I_2(z)=\int_{B_R}[(A_0-A)\,\xi D_0uD_0\Gamma_0-\Gamma_0AD_0uD_0\xi]\mathrm{d}\zeta$$

and

$$I_3(z)=\int_{B_R}[AD_0uD_0(\xi\Gamma_0)-\xi\Gamma_0Yu]\mathrm{d}\zeta.$$

It yields by (2.4) and (2.5) that

$$\begin{aligned}&\|I_1\|_{L^2(B_R)}\\ \le&\,2\left\|\int_{B_R}A_0uD_0\xi D_0\Gamma_0\mathrm{d}\zeta\right\|_{L^2(B_R)}+2\left\|\int_{B_R}\Gamma_0uY\xi\mathrm{d}\zeta\right\|_{L^2(B_R)}\end{aligned}$$

$$
\begin{aligned}
&\leq c\|\Gamma_0\left(D_0\left(uD_0\xi\right)\right)\|_{L^2(B_R)}+c\|\Gamma_0\left(uY\xi\right)\|_{L^2(B_R)}\\
&\leq c\|uD_0\xi\|_{L^{\frac{2(Q+2)}{Q+4}}(B_R)}+c\|\Gamma_0\left(uY\xi\right)\|_{L^{\frac{2(Q+2)}{Q}}(B_R)}|B_R|^{\frac{1}{Q+2}}\\
&\leq c\|uD_0\xi\|_{L^{\frac{2(Q+2)}{Q+4}}(B_R)}+c\|uY\xi\|_{L^{\frac{2(Q+2)}{Q+4}}(B_R)}R\\
&\leq \frac{c}{R-\rho}\|u\|_{L^{\frac{2(Q+2)}{Q+4}}(B_R)}+\frac{cR}{R-\rho}\|u\|_{L^{\frac{2(Q+2)}{Q+4}}(B_R)}\\
&\leq \frac{c}{R-\rho}\|u\|_{L^{\frac{2(Q+2)}{Q+4}}(B_R)} \qquad (3.7)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|I_2\|_{L^2(B_R)}\\
&\leq 2\left\|\int_{B_R}\left(A_0-A\right)\xi D_0uD_0\Gamma_0\mathrm{d}\zeta\right\|_{L^2(B_R)}\\
&\quad+2\left\|\int_{B_R}\Gamma_0AD_0uD_0\xi\mathrm{d}\zeta\right\|_{L^2(B_R)}\\
&\leq c\|\Gamma_0\left(D_0\left(\xi D_0u\right)\right)\|_{L^2(B_R)}+c\|\Gamma_0\left(D_0uD_0\xi\right)\|_{L^2(B_R)}\\
&\leq c\|\xi D_0u\|_{L^{\frac{2(Q+2)}{Q+4}}(B_R)}+c\|\Gamma_0\left(D_0uD_0\xi\right)\|_{L^{\frac{2(Q+2)}{Q}}(B_R)}|B_R|^{\frac{1}{Q+2}}\\
&\leq c\|\xi D_0u\|_{L^{\frac{2(Q+2)}{Q+4}}(B_R)}+c\|D_0uD_0\xi\|_{L^{\frac{2(Q+2)}{Q+4}}(B_R)}R\\
&\leq \frac{c}{R-\rho}\|D_0u\|_{L^{\frac{2(Q+2)}{Q+4}}(B_R)}. \qquad (3.8)
\end{aligned}
$$

Since u is a weak solution to (1.1), we infer that

$$
I_3(z)=\int_{B_R}\left[fD_0\left(\xi\Gamma_0\right)-g\xi\Gamma_0\right]\mathrm{d}\zeta=\int_{B_R}\left[f\xi D_0\Gamma_0+f\Gamma_0D_0\xi-g\xi\Gamma_0\right]\mathrm{d}\zeta
$$

and

$$
\begin{aligned}
&\|I_3\|_{L^2(B_R)}\\
&\leq c\left\|\int_{B_R}f\xi D_0\Gamma_0\mathrm{d}\zeta\right\|_{L^2(B_R)}+c\left\|\int_{B_R}f\Gamma_0D_0\xi\mathrm{d}\zeta\right\|_{L^2(B_R)}\\
&\quad+c\left\|\int_{B_R}g\xi\Gamma_0\mathrm{d}\zeta\right\|_{L^2(B_R)}\\
&\leq c\|\Gamma_0\left(D_0\left(f\xi\right)\right)\|_{L^2(B_R)}+c\|\Gamma_0\left(fD_0\xi\right)\|_{L^2(B_R)}\quad+c\|\Gamma_0\left(g\xi\right)\|_{L^2(B_R)}\\
&\leq c\|f\xi\|_{L^{\frac{2(Q+2)}{Q+4}}(B_R)}
\end{aligned}
$$

$$+ cR\left(\|\Gamma_0 (fD_0\xi)\|_{L^{\frac{2(Q+2)}{Q}}(B_R)} + c\|\Gamma_0 (g\xi)\|_{L^{\frac{2(Q+2)}{Q}}(B_R)}\right)$$

$$\le c\|f\xi\|_{L^{\frac{2(Q+2)}{Q+4}}(B_R)} + cR\left(\|fD_0\xi\|_{L^{\frac{2(Q+2)}{Q+4}}(B_R)} + \|g\xi\|_{L^{\frac{2(Q+2)}{Q+4}}(B_R)}\right)$$

$$\le c\|f\|_{L^{\frac{2(Q+2)}{Q+4}}(B_R)} + cR\left(\frac{c}{R-\rho}\|f\|_{L^{\frac{2(Q+2)}{Q+4}}(B_R)} + \|g\|_{L^{\frac{2(Q+2)}{Q+4}}(B_R)}\right)$$

$$\le \frac{c}{R-\rho}\left(\|f\|_{L^{\frac{2(Q+2)}{Q+4}}(B_R)} + \|g\|_{L^{\frac{2(Q+2)}{Q+4}}(B_R)}\right). \tag{3.9}$$

Inserting (3.7), (3.8) and (3.9) into (3.6), it obtains (3.5). □

LEMMA 3.3 (Poincaré type inequality). *Let $u \in W_2^{1,1}(\Omega)$ be a weak solution to (1.1). Then for any $B_R \subset \Omega$, $\rho < R$, one has*

$$\int_{B_\rho} |u|^2 \mathrm{d}z \le \frac{cR^4}{(R-\rho)^2}\int_{B_R} |D_0 u|^2 \mathrm{d}z + cR^2 \int_{B_R}\left(|g|^2 + |f|^2\right)\mathrm{d}z. \tag{3.10}$$

Proof. Introduce two cutoff functions $\varsigma(x)$ and $\eta(t) \in C_0^\infty(Q_R)$ [8] satisfying

$$\varsigma(x) = 1(|x| < \rho), \quad \varsigma(x) = 0(|x| \ge R),$$
$$0 \le \varsigma \le 1, \quad \left|\partial_{x_j}\varsigma\right| \le \frac{c}{R-\rho}(j = 1, \dots, N);$$
$$\eta(t) = \begin{cases} \frac{2t - 2\left(t_0 - R^2/2\right)}{R^2 - \rho^2}, & t \in \left[t_0 - \frac{R^2}{2}, t_0 - \frac{\rho^2}{2}\right), \\ 1, & t \in \left[t_0 - \frac{\rho^2}{2}, t_0 + \frac{R^2}{2}\right]. \end{cases}$$

Multiplying both sides of (1.1) by $u\varsigma^2(x)\eta(t)$ and integrating on $Q_R{}' = K_R \times S_R \times I_R{}'$ ($I_R{}' = \left[t_0 - \frac{R^2}{2}, s\right]$, $s \le t_0 + \frac{R^2}{2}$), we have

$$\int_{Q_R{}'} \left[-AD_0 u D_0\left(u\varsigma^2\eta\right) + xBu\varsigma^2\eta Du - u\varsigma^2\eta\partial_t u\right]\mathrm{d}z$$
$$= \int_{Q_R{}'} \left[gu\varsigma^2\eta - fD_0\left(u\varsigma^2\eta\right)\right]\mathrm{d}z. \tag{3.11}$$

Noting

$$\int_{Q_R{}'} u\varsigma^2\eta\partial_t u \mathrm{d}z = \frac{1}{2}\int_{Q_R{}'} \varsigma^2\left(u^2\eta\right)_t \mathrm{d}z - \frac{1}{2}\int_{Q_R{}'} u^2\varsigma^2\eta_t \mathrm{d}z, \tag{3.12}$$

$$\int_{Q_R{}'} xBu\varsigma^2\eta Du \mathrm{d}z = \frac{1}{2}\int_{Q_R{}'} xBD\left(u^2\varsigma^2\eta\right)\mathrm{d}z - \int_{Q_R{}'} xBu^2\varsigma\eta D\varsigma \mathrm{d}z, \tag{3.13}$$

it implies by inserting (3.12) and (3.13) into (3.11) that

$$\begin{aligned}
&\frac{1}{2}\int_{Q_{R'}} u^2\varsigma^2\eta_t \mathrm{d}z\\
=&\int_{Q_{R'}} A\varsigma^2\eta D_0uD_0u\mathrm{d}z+2\int_{Q_{R'}} Au\varsigma\eta D_0uD_0\varsigma\mathrm{d}z\\
&+\int_{Q_{R'}} xBu^2\varsigma\eta D\varsigma\mathrm{d}z+\frac{1}{2}\int_{Q_{R'}}\varsigma^2\left(u^2\eta\right)_t\mathrm{d}z+\int_{Q_{R'}} gu\varsigma^2\eta\mathrm{d}z\\
&-\int_{Q_{R'}} f\varsigma^2\eta D_0u\mathrm{d}z-2\int_{Q_{R'}} fu\varsigma\eta D_0\varsigma-\frac{1}{2}\int_{Q_{R'}} xBD\left(u^2\varsigma^2\eta\right)\mathrm{d}z\mathrm{d}z\\
=&\int_{Q_{R'}} A\varsigma^2\eta D_0uD_0u\mathrm{d}z+2\int_{Q_{R'}} Au\varsigma\eta D_0uD_0\varsigma\mathrm{d}z\\
&-\int_{Q_{R'}} Y\left(\frac{1}{2}u^2\varsigma^2\eta\right)\mathrm{d}z+\int_{Q_{R'}} xBu^2\varsigma\eta D\varsigma\mathrm{d}z+\int_{Q_{R'}} gu\varsigma^2\eta\mathrm{d}z\\
&-\int_{Q_{R'}} f\varsigma^2\eta D_0u\mathrm{d}z-2\int_{Q_{R'}} fu\varsigma\eta D_0\varsigma\mathrm{d}z.
\end{aligned}\tag{3.14}$$

By the divergence theorem and the property of ς, it follows

$$\int_{Q_{R'}} Y\left(\frac{1}{2}u^2\varsigma^2\eta\right)\mathrm{d}z=0.$$

Hence by Young's inequality,

$$\begin{aligned}
&\frac{1}{2}\int_{Q_{R'}} u^2\varsigma^2\eta_t\mathrm{d}z\\
\leq&\Lambda\int_{Q_{R'}}|D_0u|^2\varsigma^2\eta\mathrm{d}z+\varepsilon\int_{Q_{R'}}|u|^2|D_0\varsigma|^2\eta\mathrm{d}z+c_\varepsilon\int_{Q_{R'}}|D_0u|^2\varsigma^2\eta\mathrm{d}z\\
&+c\int_{Q_{R'}}|u|^2\left|D\varsigma\right|\varsigma\eta\mathrm{d}z+c_\varepsilon\int_{Q_{R'}}|g|^2\varsigma^2\eta\mathrm{d}z+\varepsilon\int_{Q_{R'}}|u|^2\varsigma^2\eta\mathrm{d}z\\
&+c_\varepsilon\int_{Q_{R'}}|f|^2\varsigma^2\eta\mathrm{d}z+\varepsilon\int_{Q_{R'}}|D_0u|^2\varsigma^2\eta\mathrm{d}z+c_\varepsilon\int_{Q_{R'}}|f|^2\varsigma^2\eta\mathrm{d}z\\
&+\varepsilon\int_{Q_{R'}}|u|^2|D_0\varsigma|^2\eta\mathrm{d}z\\
\leq&\int_{Q_{R'}}|u|^2\left(2\varepsilon|D_0\varsigma|^2\eta+c|D\varsigma|^2\varsigma\eta+\varepsilon\varsigma^2\eta\right)\mathrm{d}z+c\int_{Q_{R'}}|D_0u|^2\varsigma^2\eta\mathrm{d}z\\
&+c_\varepsilon\int_{Q_{R'}}\left(|g|^2+|f|^2\right)\varsigma^2\eta\mathrm{d}z.
\end{aligned}\tag{3.15}$$

In the light of properties of ς, η and (3.15), it yields

$$\begin{aligned}
&\int_{Q_\rho} |u|^2 \mathrm{d}z \leq \int_{Q_R{}'} |u|^2 \varsigma^2 \mathrm{d}z \leq c\left(R^2-\rho^2\right) \int_{Q_R{}'} |u|^2 \varsigma^2 \eta_t \mathrm{d}z \\
&\leq \left(R^2-\rho^2\right) \int_{Q_R{}'} |u|^2 \left(2\varepsilon |D_0\varsigma|^2 \eta + c|D\varsigma|^2 \varsigma\eta + \varepsilon\varsigma^2\eta\right) \mathrm{d}z \\
&\quad + c\left(R^2-\rho^2\right) \int_{Q_R{}'} |D_0 u|^2 \varsigma^2 \eta \mathrm{d}z \\
&\quad + c_\varepsilon \left(R^2-\rho^2\right) \int_{Q_R{}'} \left(|g|^2+|f|^2\right) \varsigma^2 \eta \mathrm{d}z \\
&\leq \int_{Q_R} |u|^2 \left(\frac{2\varepsilon\left(R^2-\rho^2\right)\eta}{(R-\rho)^2} + \frac{c\left(R^2-\rho^2\right)\varsigma\eta}{(R-\rho)^2} + \varepsilon\left(R^2-\rho^2\right)\varsigma^2\eta\right) \mathrm{d}z \\
&\quad + \frac{cR^2(R-\rho)^2}{(R-\rho)^2} \int_{Q_R} |D_0 u|^2 \mathrm{d}z + c_\varepsilon R^2 \int_{Q_R} \left(|g|^2+|f|^2\right) \mathrm{d}z
\end{aligned}$$

$$\leq \theta_1 \int_{Q_R} |u|^2 \mathrm{d}z + \frac{cR^4}{(R-\rho)^2} \int_{Q_R} |D_0 u|^2 \mathrm{d}z + c_\varepsilon R^2 \int_{Q_R} \left(|g|^2+|f|^2\right) \mathrm{d}z, \tag{3.16}$$

where $\theta_1 = \frac{2\varepsilon\left(R^2-\rho^2\right)\eta}{(R-\rho)^2} + \frac{c\left(R^2-\rho^2\right)\varsigma\eta}{(R-\rho)^2} + \varepsilon\left(R^2-\rho^2\right)\varsigma\eta$. Choosing ε small enough, it ensures $0 < \theta_1 < 1$ and attains from Lemma 2.10 that

$$\int_{Q_\rho} |u|^2 \mathrm{d}z \leq \frac{cR^4}{(R-\rho)^2} \int_{Q_R} |D_0 u|^2 \mathrm{d}z + cR^2 \int_{Q_R} \left(|g|^2+|f|^2\right) \mathrm{d}z. \tag{3.17}$$

Now (3.17) and $B_{\rho/c_0} \subset Q_\rho \subset Q_R \subset B_{c_0 R}$ imply (3.10). □

4. PROOF OF THEOREM 1.1

Let us first describe a known result.

Lemma 4.1 (reverse Hölder inequality, [14]). *Let $\hat{g}$ and $\hat{f}$ be nonnegative functions on Ω and satisfy*

$$\hat{g} \in L^{\hat{q}}(\Omega), \quad \hat{f} \in L^r(\Omega), \quad 1 < \hat{q} < r.$$

If there exist constants $b_2 > 1$ and $\theta_2 \in [0,1)$ such that for any $B_{2R} \subset \Omega$, the inequality holds

$$\frac{1}{|B_R|} \int_{B_R} \hat{g}^{\hat{q}} \mathrm{d}z \leq b_2 \left[\left(\frac{1}{|B_{4R/3}|} \int_{B_{4R/3}} \hat{g} \mathrm{d}z\right)^{\hat{q}} + \frac{1}{|B_{4R/3}|} \int_{B_{4R/3}} \hat{f}^{\hat{q}} \mathrm{d}z\right]$$

$$+\theta_2\frac{1}{|B_{4R/3}|}\int_{B_{4R/3}}\hat{g}^{\hat{q}}\mathrm{d}z,$$

then there exist positive constants $\theta_0=\theta_0(\hat{q},\Omega)$ *and* ε_0 *such that if* $\theta_2<\theta_0$, *then for any* $\hat{p}\in[\hat{q},\hat{q}+\varepsilon_0)$, *it follows* $\hat{g}\in L^{\hat{p}}_{loc}(\Omega)$ *and*

$$\left(\frac{1}{|B_R|}\int_{B_R}\hat{g}^{\hat{p}}\mathrm{d}z\right)^{\frac{1}{\hat{p}}}\le c\left[\left(\frac{1}{|B_{2R}|}\int_{B_{2R}}\hat{g}^{\hat{q}}\mathrm{d}z\right)^{\frac{1}{\hat{q}}}+\left(\frac{1}{|B_{2R}|}\int_{B_{2R}}\hat{f}^{\hat{p}}\mathrm{d}z\right)^{\frac{1}{\hat{p}}}\right], \tag{4.1}$$

where c *and* ε_0 *depend on* $b_2,\hat{q},\theta_2$ *and* Q.

The following result is essential to prove Theorem 1.1.

LEMMA 4.2. *Let* $u\in W_2^{1,1}(\Omega)$ *be a weak solution to (1.1) in* Ω. *Then for any* $p\in\left[2,2+\frac{2Q}{Q+2}\varepsilon_0\right)$, *we have* $D_0u\in L^p_{loc}(\Omega)$ *and for any* $B_R\subset B_{2R}\subset\Omega$,

$$\begin{aligned}&\left(\frac{1}{|B_R|}\int_{B_R}|D_0u|^p\mathrm{d}z\right)^{\frac{1}{p}}\\ &\le c\left[\left(\frac{1}{|B_{2R}|}\int_{B_{2R}}|D_0u|^2\mathrm{d}z\right)^{\frac{1}{2}}+\left(\frac{1}{|B_{2R}|}\int_{B_{2R}}\left(|g|^2+|f|^2\right)^{\frac{p}{2}}\mathrm{d}z\right)^{\frac{1}{p}}\right].\end{aligned} \tag{4.2}$$

Proof. By using Hölder's inequality, it implies

$$\begin{aligned}&\int_{B_{11R/9}}|D_0u|^{\frac{2(Q+2)}{Q+4}}\mathrm{d}z\\ &\le\left(\int_{B_{11R/9}}|D_0u|^2\mathrm{d}z\right)^{\frac{1}{2}}\left(\int_{B_{11R/9}}|D_0u|^{\frac{2Q}{Q+4}}\mathrm{d}z\right)^{\frac{1}{2}}\\ &\le\left(\int_{B_{11R/9}}|D_0u|^2\mathrm{d}z\right)^{\frac{1}{2}}\left|B_{11R/9}\right|^{\frac{1}{Q+4}}\left(\int_{B_{11R/9}}|D_0u|^{\frac{2Q}{Q+2}}\mathrm{d}z\right)^{\frac{Q+2}{2(Q+4)}}.\end{aligned} \tag{4.3}$$

Combining (3.5) and (4.3),

$$\begin{aligned}&\int_{B_{10R/9}}|u|^2\mathrm{d}z\\ &\le\frac{c}{R^2}\left[\left(\int_{B_{11R/9}}|u|^{\frac{2(Q+2)}{Q+4}}\mathrm{d}z\right)^{\frac{Q+4}{2(Q+2)}}+\left(\int_{B_{11R/9}}|D_0u|^{\frac{2(Q+2)}{Q+4}}\mathrm{d}z\right)^{\frac{Q+4}{2(Q+2)}}\right]^2\\ &\quad+\frac{c}{R^2}\left[\left(\int_{B_{11R/9}}|f|^{\frac{2(Q+2)}{Q+4}}\mathrm{d}z\right)^{\frac{Q+4}{2(Q+2)}}+\left(\int_{B_{11R/9}}|g|^{\frac{2(Q+2)}{Q+4}}\mathrm{d}z\right)^{\frac{Q+4}{2(Q+2)}}\right]^2\end{aligned}$$

$$\leq \frac{c}{R^2}\left[|B_{11R/9}|^{\frac{1}{Q+2}}\left(\int_{B_{11R/9}}|u|^2\mathrm{d}z\right)^{\frac{1}{2}}\right]^2$$

$$+\left[\left(\int_{B_{11R/9}}|D_0u|^2\mathrm{d}z\right)^{\frac{Q+4}{4(Q+2)}}|B_{11R/9}|^{\frac{1}{2(Q+2)}}\left(\int_{B_{11R/9}}|D_0u|^{\frac{2Q}{Q+2}}\mathrm{d}z\right)^{\frac{1}{4}}\right]^2$$

$$+\frac{c}{R^2}\left[|B_{11R/9}|^{\frac{1}{Q+2}}\left(\int_{B_{11R/9}}|f|^2\mathrm{d}z\right)^{\frac{1}{2}}+|B_{11R/9}|^{\frac{1}{Q+2}}\left(\int_{B_{11R/9}}|g|^2\mathrm{d}z\right)^{\frac{1}{2}}\right]^2$$

$$\leq c\int_{B_{11R/9}}|u|^2\mathrm{d}z+\frac{c}{R}\left(\int_{B_{11R/9}}|D_0u|^2\mathrm{d}z\right)^{\frac{Q+4}{2(Q+2)}}\left(\int_{B_{11R/9}}|D_0u|^{\frac{2Q}{Q+2}}\mathrm{d}z\mathrm{d}z\right)^{\frac{1}{2}}$$

$$+c\int_{B_{11R/9}}\left(|f|^2+|g|^2\right)\mathrm{d}z. \tag{4.4}$$

Noting (3.1), (3.10) and (4.4), it follows

$$\int_{B_R}|D_0u|^2\mathrm{d}z$$

$$\leq \frac{c}{R^2}\int_{B_{11R/9}}|u|^2\mathrm{d}z+\frac{c}{R^3}\left(\int_{B_{11R/9}}|D_0u|^2\mathrm{d}z\right)^{\frac{Q+4}{2(Q+2)}}\left(\int_{B_{11R/9}}|D_0u|^{\frac{2Q}{Q+2}}\mathrm{d}z\right)^{\frac{1}{2}}$$

$$+\frac{c}{R^2}\int_{B_{11R/9}}\left(|f|^2+|g|^2\right)\mathrm{d}z+c\int_{B_{10R/9}}\left(|g|^2+|f|^2\right)\mathrm{d}z\leq c\int_{B_{4R/3}}|D_0u|^2\mathrm{d}z$$

$$+\frac{c}{R^3}|B_{4R/3}|^{\frac{Q+3}{Q+2}}\left(\frac{1}{|B_{4R/3}|}\int_{B_{4R/3}}|D_0u|^2\mathrm{d}z\right)^{\frac{Q+4}{2(Q+2)}}$$

$$\left(\frac{1}{|B_{4R/3}|}\int_{B_{4R/3}}|D_0u|^{\frac{2Q}{Q+2}}\mathrm{d}z\right)^{\frac{1}{2}}+\frac{c}{R^2}\int_{B_{4R/3}}\left(|g|^2+|f|^2\right)\mathrm{d}z$$

and hence

$$\frac{1}{|B_R|}\int_{B_R}|D_0u|^2\mathrm{d}z$$

$$\leq \frac{c}{|B_{4R/3}|}\int_{B_{4R/3}}|D_0u|^2\mathrm{d}z+\varepsilon\left(\frac{1}{|B_{4R/3}|}\int_{B_{4R/3}}|D_0u|^2\mathrm{d}z\right)$$

$$+c_\varepsilon R^{-\frac{4(Q+2)}{Q}}\left(\frac{1}{|B_{4R/3}|}\int_{B_{4R/3}}|D_0u|^{\frac{2Q}{Q+2}}\mathrm{d}z\right)^{\frac{Q+2}{Q}}$$

$$
\begin{aligned}
&+\frac{c}{R^2}\frac{1}{|B_{4R/3}|}\int_{B_{4R/3}}\left(|g|^2+|f|^2\right)\mathrm{d}z\\
\le\ & c\left(\frac{1}{|B_{4R/3}|}\int_{B_{4R/3}}|D_0u|^2\mathrm{d}z\right)\\
&+c_\varepsilon R^{-\frac{4(Q+2)}{Q}}\left(\frac{1}{|B_{4R/3}|}\int_{B_{4R/3}}|D_0u|^{\frac{2Q}{Q+2}}\mathrm{d}z\right)^{\frac{Q+2}{Q}}\\
&+\frac{c}{R^2}\frac{1}{|B_{4R/3}|}\int_{B_{4R/3}}\left(|g|^2+|f|^2\right)\mathrm{d}z. \qquad (4.5)
\end{aligned}
$$

Let $\hat{g}=|D_0u|^{\tilde{q}}$, $\tilde{q}=\frac{2Q}{Q+2}$, $\hat{q}=\frac{2}{\tilde{q}}=\frac{Q+2}{Q}>1$, $\hat{f}=\left(|g|^2+|f|^2\right)^{\frac{Q}{Q+2}}$, then we rewrite (4.5) in the form

$$
\begin{aligned}
&\frac{1}{|B_R|}\int_{B_R}\hat{g}^{\hat{q}}\mathrm{d}z\\
\le\ & c\left[\left(\frac{1}{|B_{4R/3}|}\int_{B_{4R/3}}\hat{g}\mathrm{d}z\right)^{\hat{q}}+\frac{1}{|B_{4R/3}|}\int_{B_{4R/3}}\hat{f}^{\hat{q}}\mathrm{d}z\right]\\
&+\frac{c}{|B_{4R/3}|}\int_{B_{4R/3}}\hat{g}^{\hat{q}}\mathrm{d}z. \qquad (4.6)
\end{aligned}
$$

It shows from Lemma 4.1 that for any $\hat{p}\in[\hat{q},\hat{q}+\varepsilon_0)$,

$$
\left(\frac{1}{|B_R|}\int_{B_R}\hat{g}^{\hat{p}}\mathrm{d}z\right)^{1/\hat{p}}\le c\left[\left(\frac{1}{|B_{2R}|}\int_{B_{2R}}\hat{g}^{\hat{q}}\mathrm{d}z\right)^{1/\hat{q}}+\left(\frac{1}{|B_{2R}|}\int_{B_{2R}}\hat{f}^{\hat{p}}\mathrm{d}z\right)^{1/\hat{p}}\right],
$$

which means

$$
\begin{aligned}
&\left(\frac{1}{|B_R|}\int_{B_R}|D_0u|^{\tilde{q}\hat{p}}\mathrm{d}z\right)^{\frac{1}{\hat{p}}}\\
\le\ & c\left[\left(\frac{1}{|B_{2R}|}\int_{B_{2R}}|D_0u|^2\mathrm{d}z\right)^{\frac{Q}{Q+2}}+\left(\frac{1}{|B_{2R}|}\int_{B_{2R}}\left(|g|^2+|f|^2\right)^{\frac{\hat{p}\tilde{q}}{2}}\mathrm{d}z\right)^{\frac{1}{\hat{p}}}\right]. \qquad (4.7)
\end{aligned}
$$

Setting $p=\hat{p}\tilde{q}\in\left[2,2+\frac{2Q}{Q+2}\varepsilon_0\right)$, we finish the proof. □

Proof of Theorem 1.1. The conclusion follows from Lemma 4.2 and the cutoff function technique. □

5. HOMOGENEOUS ULTRAPARABOLIC EQUATION

In this section, we consider the following homogeneous ultraparabolic equation

$$div\,(AD_0u) + Yu = 0. \tag{5.1}$$

To obtain L^p estimates for gradients of weak solutions to (5.1), we divide (5.1) into two equations. In fact, let v be a weak solution to the following Dirichlet boundary value problem of the homogeneous ultraparabolic equation with constant principal part:

$$\begin{cases} div\,(A_RD_0v) + Yv = 0, & in\ B_R, \\ v = u, & on\ \partial_pB_R, \end{cases} \tag{5.2}$$

where $A_R = \frac{1}{|B_R|}\int_{B_R} A\mathrm{d}z$. Then $w = u - v$ satisfies the Dirichlet boundary value problem of the nonhomogeneous ultraparabolic equation with constant principal part:

$$\begin{cases} div\,(A_RD_0w) + Yw = div\,((A_R - A)\,D_0u)\,, & in\ B_R, \\ w = 0, & on\ \partial B_R. \end{cases} \tag{5.3}$$

LEMMA 5.1. *Let $v \in W_2^{1,1}(\Omega)$ be a weak solution to (5.2). Then for any $B_R \subset \Omega$, one has*

$$\sup_{B_{R/2}} |v|^2 \le \frac{c}{R^{Q+2}} \int_{B_R} |v|^2\mathrm{d}z. \tag{5.4}$$

Proof. It is true from Corollary 1.4 of [27]. □

Furthermore, we have:

LEMMA 5.2. *Let $v \in W_2^{1,1}(\Omega)$ be a weak solution to (5.2). Then for any $B_R \subset \Omega$, $\rho < R$, it follows*

$$\int_{B_\rho} |v|^2\mathrm{d}z \le c\Big(\frac{\rho}{R}\Big)^{Q+2} \int_{B_R} |v|^2\mathrm{d}z. \tag{5.5}$$

Proof. When $\frac{R}{2} \le \rho < R$, the result is evident. Now it is enough to treat the case $\rho < \frac{R}{2}$. But by Lemma 5.1, it yields

$$\begin{aligned} \int_{B_\rho} |v|^2\mathrm{d}z &\le |B_\rho| \sup_{B_\rho} |v|^2 \le |B_\rho| \sup_{B_{R/2}} |v|^2 \\ &\le |B_\rho| \frac{c}{R^{Q+2}} \int_{B_R} |v|^2\mathrm{d}z \le c\Big(\frac{\rho}{R}\Big)^{Q+2} \int_{B_R} |v|^2\mathrm{d}z. \quad \square \end{aligned}$$

On the gradient of v, we have:

Lemma 5.3. *Let $v \in W_2^{1,1}(\Omega)$ be a weak solution to (5.2). Then for any $B_R \subset \Omega$, $\rho < R$, it follows*

$$\int_{B_\rho} |D_0 v|^2 \mathrm{d}z \le c\Big(\frac{\rho}{R}\Big)^Q \int_{B_R} |D_0 v|^2 \mathrm{d}z. \tag{5.6}$$

Proof. Combining Lemma 3.1, Lemma 3.3 (g=f=0) and (5.5), we arrive at

$$\int_{B_{\rho/2}} |D_0 v|^2 \mathrm{d}z \le \frac{c}{\rho^2} \int_{B_\rho} |v|^2 \mathrm{d}z \le \frac{c}{\rho^2}\Big(\frac{\rho}{R}\Big)^{Q+2} \int_{B_R} |v|^2 \mathrm{d}z$$
$$\le \frac{c}{\rho^2}\Big(\frac{\rho}{R}\Big)^{Q+2} R^2 \int_{B_{2R}} |D_0 v|^2 \mathrm{d}z \le c\Big(\frac{\rho}{R}\Big)^Q \int_{B_{2R}} |D_0 v|^2 \mathrm{d}z. \quad \square$$

Lemma 5.4. *Let $v \in W_2^{1,1}(\Omega)$ be a weak solution to (5.2). Then for any $p \in \Big[2, 2 + \frac{2Q}{Q+2}\varepsilon_0\Big)$, $B_R \subset \Omega$, $\rho < R$, we have*

$$\int_{B_\rho} |D_0 v|^p \mathrm{d}z \le c\Big(\frac{\rho}{R}\Big)^{Q+2-p} \int_{B_R} |D_0 v|^p \mathrm{d}z. \tag{5.7}$$

Proof. By Lemma 4.2 (g=f=0) and (5.6),

$$\left(\frac{1}{|B_{\rho/2}|} \int_{B_{\rho/2}} |D_0 v|^p \mathrm{d}z\right)^{\frac{1}{p}}$$
$$\le c\left(\frac{1}{|B_\rho|} \int_{B_\rho} |D_0 v|^2 \mathrm{d}z\right)^{\frac{1}{2}} \le c\left(\frac{1}{|B_\rho|}\Big(\frac{\rho}{R}\Big)^Q \int_{B_R} |D_0 v|^2 \mathrm{d}z\right)^{\frac{1}{2}}.$$

From Hölder's inequality, it implies

$$\int_{B_{\rho/2}} |D_0 v|^p \mathrm{d}z \le c\left|B_{\rho/2}\right| \left(\frac{1}{|B_\rho|}\Big(\frac{\rho}{R}\Big)^Q \int_{B_R} |D_0 v|^2 \mathrm{d}z\right)^{\frac{p}{2}}$$
$$\le c\left|B_{\frac{\rho}{2}}\right| \frac{1}{|B_\rho|^{\frac{p}{2}}}\Big(\frac{\rho}{R}\Big)^{\frac{pQ}{2}} |B_R|^{\frac{p-2}{2}} \int_{B_R} |D_0 v|^p \mathrm{d}z$$
$$\le c\Big(\frac{|B_\rho|}{|B_R|}\Big)^{\frac{2-p}{2}} \Big(\frac{\rho}{R}\Big)^{\frac{pQ}{2}} \int_{B_R} |D_0 v|^p \mathrm{d}z$$
$$\le c\Big(\frac{\rho}{R}\Big)^{Q+2-p} \int_{B_R} |D_0 v|^p \mathrm{d}z$$

and the proof is ended. $\square$

Lemma 5.5. *Let $u \in W_2^{1,1}(\Omega)$ be a weak solution to (5.1). Then for any $p \in \left[2, 2 + \frac{2Q}{Q+2}\varepsilon_0\right)$, $\frac{p-2}{p}(Q+2) < \mu < Q$, $B_R \subset \Omega$, $\rho < R$, one has*

$$\int_{B_\rho} |D_0 u|^p \mathrm{d}z \le c\left(\frac{\rho}{R}\right)^{\frac{2(Q+2)-p(Q+2-\mu)}{2}} \int_{B_R} |D_0 u|^p \mathrm{d}z. \tag{5.8}$$

Proof. When $\frac{R}{2} \le \rho < R$, (5.8) is clearly true. The remainder is to treat $\rho < \frac{R}{2}$.

Multiplying both sides of (5.3) by w and integrating on B_R, it observes

$$-\int_{B_R} A_R D_0 w D_0 w \mathrm{d}z + \int_{B_R} wYw \mathrm{d}z = -\int_{B_R} (A_R - A) D_0 u D_0 w \mathrm{d}z, \tag{5.9}$$

and from the divergence theorem,

$$\int_{B_R} wYw \mathrm{d}z = \frac{1}{2}\int_{B_R} Y\left(w^2\right)\mathrm{d}z = 0.$$

By (H1) and Young's inequality, we have by (5.9) that

$$\Lambda^{-1}\int_{B_R} |D_0 w|^2 \mathrm{d}z \le c_\varepsilon \int_{B_R} |A_R - A|^2 |D_0 u|^2 \mathrm{d}z + \varepsilon \int_{B_R} |D_0 w|^2 \mathrm{d}z. \tag{5.10}$$

Choosing ε small enough such that $\Lambda^{-1} - \varepsilon > 0$, then (5.10) implies

$$\begin{aligned}
\int_{B_R} |D_0 w|^2 \mathrm{d}z &\le c\int_{B_R} |A_R - A|^2 |D_0 u|^2 \mathrm{d}z \\
&\le c\left(\int_{B_R} |A_R - A|^{\frac{2p}{p-2}} \mathrm{d}z\right)^{\frac{p-2}{p}} \left(\int_{B_R} |D_0 u|^p \mathrm{d}z\right)^{\frac{2}{p}} \\
&\le c(|B_R|\,\eta_R(a_{ij}))^{\frac{p-2}{p}} \left(\int_{B_R} |D_0 u|^p \mathrm{d}z\right)^{\frac{2}{p}}
\end{aligned} \tag{5.11}$$

and applying (5.6) and (5.11) leads to

$$\begin{aligned}
\int_{B_{2\rho}} |D_0 u|^2 \mathrm{d}z &\le 2\int_{B_{2\rho}} |D_0 v|^2 \mathrm{d}z + 2\int_{B_{2\rho}} |D_0 w|^2 \mathrm{d}z \\
&\le c\left(\frac{\rho}{R}\right)^Q \int_{B_R} |D_0 v|^2 \mathrm{d}z + c\int_{B_R} |D_0 w|^2 \mathrm{d}z \\
&\le c\left(\frac{\rho}{R}\right)^Q \int_{B_R} |D_0 u|^2 \mathrm{d}z + c\int_{B_R} |D_0 w|^2 \mathrm{d}z
\end{aligned}$$

$$\leq c\Big(\frac{\rho}{R}\Big)^{Q}|B_R|^{\frac{p-2}{p}}\Big(\int_{B_R}|D_0u|^p\mathrm{d}z\Big)^{\frac{2}{p}}$$

$$+c(|B_R|\,\eta_R\,(a_{ij}))^{\frac{p-2}{p}}\Big(\int_{B_R}|D_0u|^p\mathrm{d}z\Big)^{\frac{2}{p}}$$

$$\leq c\left[\Big(\frac{\rho}{R}\Big)^{Q}+(\eta_R\,(a_{ij}))^{\frac{p-2}{p}}\right]\Big(|B_R|^{\frac{p-2}{2}}\int_{B_R}|D_0u|^p\mathrm{d}z\Big)^{\frac{2}{p}}. \tag{5.12}$$

It shows owing to Lemma 4.2 ($g=f=0$) that

$$\Big(|B_\rho|^{\frac{p-2}{2}}\int_{B_\rho}|D_0u|^p\mathrm{d}z\Big)^{\frac{2}{p}}\leq c\int_{B_{2\rho}}|D_0u|^2\mathrm{d}z$$

$$\leq c\left[\Big(\frac{\rho}{R}\Big)^{Q}+(\eta_R\,(a_{ij}))^{\frac{p-2}{p}}\right]\Big(|B_R|^{\frac{p-2}{2}}\int_{B_R}|D_0u|^p\mathrm{d}z\Big)^{\frac{2}{p}}. \tag{5.13}$$

Denoting $H(\rho)=\Big(|B_\rho|^{\frac{p-2}{2}}\int_{B_\rho}|D_0u|^p\mathrm{d}z\Big)^{\frac{2}{p}}$, $H(R)=\Big(|B_R|^{\frac{p-2}{2}}\int_{B_R}|D_0u|^p\mathrm{d}z\Big)^{\frac{2}{p}}$, $a_1=Q$, $B_1=0$ in Lemma 2.11, we know that there exists $b_1=\mu(\frac{p-2}{p}(Q+2)<\mu<Q)$ such that

$$\Big(|B_\rho|^{\frac{p-2}{2}}\int_{B_\rho}|D_0u|^p\mathrm{d}z\Big)^{\frac{2}{p}}\leq c\Big(\frac{\rho}{R}\Big)^{\mu}\Big(|B_R|^{\frac{p-2}{2}}\int_{B_R}|D_0u|^p\mathrm{d}z\Big)^{\frac{2}{p}}. \tag{5.14}$$

Inserting $\frac{|B_R|}{|B_\rho|}\leq c\big(\frac{\rho}{R}\big)^{-Q-2}$ into (5.14), it attains (5.8). □

Lemma 5.6. *Let $v\in W_2^{1,1}(\Omega)$ be a weak solution to (5.2). Then for any $B_R\subset\Omega$, $\rho<R$,*

$$\int_{B_\rho}\left|v-v_{B_\rho}\right|^2\mathrm{d}z\leq c\left(\frac{\rho}{R}\right)^{Q+2}\int_{B_R}|v-v_{B_R}|^2\mathrm{d}z.$$

Proof. Since $v-v_{B_{2R}}$ is a weak solution to (5.2), we see that (3.1)(f=g=0) is true to $v-v_{B_{2R}}$, that is

$$\int_{B_R}|D_0v|^2\mathrm{d}z\leq\frac{c}{R^2}\int_{B_{2R}}|v-v_{B_{2R}}|^2\mathrm{d}z. \tag{5.15}$$

By (3.10)(f=g=0), (5.6) and (5.15),

$$\int_{B_{\rho/2}} \left|v - v_{B_{\rho/2}}\right|^2 \mathrm{d}z \le c \int_{B_{\rho/2}} |v|^2 \mathrm{d}z \le c\rho^2 \int_{B_\rho} |D_0 v|^2 \mathrm{d}z$$
$$\le c\rho^2 \left(\frac{\rho}{R}\right)^Q \int_{B_R} |D_0 v|^2 \mathrm{d}z \le c \left(\frac{\rho}{R}\right)^{Q+2} \int_{B_{2R}} |v - v_{B_{2R}}|^2 \mathrm{d}z. \quad \square$$

LEMMA 5.7. *Let $w \in W^{1,1}_{2,0}(\Omega)$ be a weak solution to (5.3). Then for any $B_{3R} \subset \Omega$, $\rho < R$,*

$$\int_{B_{R/2}} |w|^2 \mathrm{d}z \le c\,(\eta_R)^{\frac{p-2}{p}} \int_{B_{3R}} |u - u_{B_{3R}}|^2 \mathrm{d}z.$$

Proof. Using the proof of Lemma 3.3 we have

$$\int_{B_{R/2}} |w|^2 \mathrm{d}z \le cR^2 \int_{B_R} |D_0 w|^2 \mathrm{d}z + cR^2 \int_{B_R} |A - A_R|^2 \, |D_0 u|^2 \mathrm{d}z.$$

By (5.11) and (4.2) (f=g=0),

$$\int_{B_{R/2}} |w|^2 \mathrm{d}z \le cR^2 \int_{B_R} |A - A_R|^2 \, |D_0 u|^2 \mathrm{d}z$$
$$\le cR^2 \left(|B_R|\,\eta_R\right)^{\frac{p-2}{p}} \left(\int_{B_R} |D_0 u|^p \mathrm{d}z\right)^{\frac{2}{p}} \le cR^2 \left(\eta_R\right)^{\frac{p-2}{p}} \int_{B_{2R}} |D_0 u|^2 \mathrm{d}z. \tag{5.16}$$

Noting $u - u_{B_{3R}}$ is also a weak solution to (5.1), and using (3.1)(f=g=0) to $u - u_{B_{3R}}$,

$$\int_{B_{2R}} |D_0 u|^2 \mathrm{d}z \le \frac{c}{R^2} \int_{B_{3R}} |u - u_{B_{3R}}|^2 \mathrm{d}z.$$

Putting the above into (5.16), the desired estimate is obtained. $\square$

LEMMA 5.8. *Let $u \in W^{1,1}_{2}(\Omega)$ be a weak solution to (5.1). Then for any $B_R \subset \Omega$, $\rho < R$,*

$$\int_{B_\rho} \left|u - u_{B_\rho}\right|^2 \mathrm{d}z \le c \left[\left(\frac{\rho}{R}\right)^{Q+2} + (\eta_R)^{\frac{p-2}{p}}\right] \int_{B_R} |u - u_{B_R}|^2 \mathrm{d}z.$$

Proof. By Lemma 5.6 and Lemma 5.7,

$$\int_{B_\rho} \left|u - u_{B_\rho}\right|^2 \mathrm{d}z \le c \int_{B_\rho} \left|v - v_{B_\rho}\right|^2 \mathrm{d}z + c \int_{B_\rho} \left|w - w_{B_\rho}\right|^2 \mathrm{d}z$$

$$\leq c\Big(\frac{\rho}{R}\Big)^{Q+2}\int_{B_R}|v-v_{B_R}|^2\mathrm{d}z+c\int_{B_\rho}|w|^2\mathrm{d}z$$
$$\leq c\Big(\frac{\rho}{R}\Big)^{Q+2}\int_{B_R}|u-u_{B_R}|^2\mathrm{d}z+c\int_{B_R}|w|^2\mathrm{d}z$$
$$\leq c\Big(\frac{\rho}{R}\Big)^{Q+2}\int_{B_R}|u-u_{B_R}|^2\mathrm{d}z+c(\eta_R)^{\frac{p-2}{p}}\int_{B_{6R}}|u-u_{B_{6R}}|^2\mathrm{d}z$$
$$\leq c\left[\Big(\frac{\rho}{R}\Big)^{Q+2}+(\eta_R)^{\frac{p-2}{p}}\right]\int_{B_{6R}}|u-u_{B_{6R}}|^2\mathrm{d}z. \quad \square$$

6. PROOF OF THEOREM 1.2 AND THEOREM 1.3

Let v be a weak solution to the following problem

$$\begin{cases} div\,(AD_0v)+Yv=0, & in \quad B_R, \\ v{=}u, & on \quad \partial B_R, \end{cases} \tag{6.1}$$

then $w=u-v$ satisfies

$$\begin{cases} div\,(AD_0w)+Yw=g+div f, & in\ B_R, \\ w=0, & on\ \partial B_R. \end{cases} \tag{6.2}$$

LEMMA 6.1. *Let $w\in W^{1,1}_{2,0}(\Omega)$ be a weak solution to (6.2). Then for any $B_{2R}\subset\Omega$, one has*

$$\int_{B_R}|D_0w|^2\mathrm{d}z\leq c\int_{B_{2R}}\Big(|g|^2+|f|^2\Big)\mathrm{d}z. \tag{6.3}$$

Proof. Multiplying both sides of (6.2) by w and integrating on B_R,

$$-\int_{B_R}AD_0wD_0w\mathrm{d}z+\int_{B_R}wYw\mathrm{d}z=\int_{B_R}gw\mathrm{d}z-\int_{B_R}fD_0w\mathrm{d}z. \tag{6.4}$$

By (H1), the divergence theorem and Young's inequality with ε, we have

$$\Lambda^{-1}\int_{B_R}|D_0w|^2\mathrm{d}z$$
$$\leq c_\varepsilon\int_{B_R}|g|^2\mathrm{d}z+\varepsilon\int_{B_R}|w|^2\mathrm{d}z+c_\varepsilon\int_{B_R}|f|^2\mathrm{d}z+\varepsilon\int_{B_R}|D_0w|^2\mathrm{d}z. \tag{6.5}$$

Since by using (3.10),

$$\int_{B_R}|w|^2\mathrm{d}z\leq cR^2\int_{B_{2R}}|D_0w|^2\mathrm{d}z+cR^2\int_{B_{2R}}\Big(|g|^2+|f|^2\Big)\mathrm{d}z, \tag{6.6}$$

it implies

$$\begin{aligned}
&\int_{B_R} |D_0 w|^2 \mathrm{d}z \\
&\leq c\varepsilon R^2 \int_{B_{2R}} |D_0 w|^2 \mathrm{d}z + c\varepsilon R^2 \int_{B_{2R}} \left(|g|^2 + |f|^2\right) \mathrm{d}z \\
&\quad + c_\varepsilon \int_{B_R} \left(|g|^2 + |f|^2\right) \mathrm{d}z + \varepsilon \int_{B_R} |D_0 w|^2 \mathrm{d}z \\
&\leq \varepsilon \int_{B_{2R}} |D_0 w|^2 \mathrm{d}z + c_\varepsilon \int_{B_{2R}} \left(|g|^2 + |f|^2\right) \mathrm{d}z.
\end{aligned}$$

Then for any $\rho \leq R$,

$$\begin{aligned}
&\int_{B_\rho} |D_0 w|^2 \mathrm{d}z \leq \int_{B_R} |D_0 w|^2 \mathrm{d}z \\
&\leq \varepsilon \int_{B_{2R}} |D_0 w|^2 \mathrm{d}z + \frac{c_\varepsilon (2R-\rho)^2}{(2R-\rho)^2} \int_{B_{2R}} |g|^2 \mathrm{d}z + c_\varepsilon \int_{B_{2R}} |f|^2 \mathrm{d}z \\
&\leq \varepsilon \int_{B_{2R}} |D_0 w|^2 \mathrm{d}z + \frac{c_\varepsilon R^2}{(2R-\rho)^2} \int_{B_{2R}} |g|^2 \mathrm{d}z + c_\varepsilon \int_{B_{2R}} |f|^2 \mathrm{d}z.
\end{aligned}$$

Now due to Lemma 2.10, it infers

$$\int_{B_\rho} |D_0 w|^2 \mathrm{d}z \leq \frac{cR^2}{(2R-\rho)^2} \int_{B_{2R}} |g|^2 \mathrm{d}z + c \int_{B_{2R}} |f|^2 \mathrm{d}z,$$

and the conclusion holds with $\rho = R$. □

LEMMA 6.2. *Let $w \in W^{1,1}_{2,0}(\Omega)$ be a weak solution to (6.2). Then for any $p \in \left[2, 2 + \frac{2Q}{Q+2}\varepsilon_0\right)$, we have $D_0 w \in L^p_{loc}(\Omega)$, and for any $B_R \subset B_{4R} \subset \Omega$,*

$$\int_{B_R} |D_0 w|^p \mathrm{d}z \leq c \int_{B_{4R}} (|g|^p + |f|^p) \mathrm{d}z. \tag{6.7}$$

Proof. By (4.2) and (6.3), it follows

$$\begin{aligned}
&\int_{B_R} |D_0 w|^p \mathrm{d}z \\
&\leq c\,|B_R| \left[\left(\frac{1}{|B_{2R}|} \int_{B_{2R}} |D_0 w|^2 \mathrm{d}z \right)^{\frac{1}{2}} + \left(\frac{1}{|B_{2R}|} \int_{B_{2R}} \left(|g|^2 + |f|^2\right)^{\frac{p}{2}} \mathrm{d}z \right)^{\frac{1}{p}} \right]^p \\
&\leq c\,|B_R| \left[\left(\frac{c}{|B_{2R}|} \int_{B_{4R}} \left(|g|^2 + |f|^2\right) \mathrm{d}z \right)^{\frac{1}{2}} + \left(\frac{1}{|B_{2R}|} \int_{B_{2R}} \left(|g|^2 + |f|^2\right)^{\frac{p}{2}} \mathrm{d}z \right)^{\frac{1}{p}} \right]^p
\end{aligned}$$

$$\leq c\,|B_R|\left[\left(\frac{1}{|B_{2R}|}\int_{B_{4R}}(|g|^p+|f|^p)\mathrm{d}z\right)^{\frac{1}{p}}+\left(\frac{1}{|B_{2R}|}\int_{B_{2R}}(|g|^p+|f|^p)\mathrm{d}z\right)^{\frac{1}{p}}\right]^p$$

$$\leq c\int_{B_{4R}}(|g|^p+|f|^p)\mathrm{d}z. \quad \square$$

LEMMA 6.3. *Let $u \in W_2^{1,1}(\Omega)$ be a weak solution to (1.1). Then for any $p \in \left[2, 2+\frac{2Q}{Q+2}\varepsilon_0\right)$, we have $D_0u \in L^p_{loc}(\Omega)$ and for any $p < \lambda < Q+2$, $B_R \subset B_{4R} \subset \Omega$,*

(6.8)

$$\int_{B_\rho}|D_0u|^p\mathrm{d}z \leq c\left[\left(\frac{\rho}{R}\right)^{Q+2-\lambda}\int_{B_{4R}}|D_0u|^p\mathrm{d}z+\rho^{Q+2-\lambda}\left(\|g\|^p_{L^{p,\lambda}}+\|f\|^p_{L^{p,\lambda}}\right)\right].$$

Proof. Combining Lemma 5.5 and Lemma 6.2 indicates

$$\begin{aligned}
&\int_{B_\rho}|D_0u|^p\mathrm{d}z\\
&\leq 2\int_{B_\rho}|D_0v|^p\mathrm{d}z+2\int_{B_\rho}|D_0w|^p\mathrm{d}z\\
&\leq c\Big(\frac{\rho}{R}\Big)^{\frac{2(Q+2)-p(Q+2-\mu)}{2}}\int_{B_R}|D_0v|^p\mathrm{d}z+2\int_{B_\rho}|D_0w|^p\mathrm{d}z\\
&\leq c\Big(\frac{\rho}{R}\Big)^{\frac{2(Q+2)-p(Q+2-\mu)}{2}}\int_{B_R}|D_0u|^p\mathrm{d}z+c\int_{B_R}|D_0w|^p\mathrm{d}z\\
&\leq c\Big(\frac{\rho}{R}\Big)^{\frac{2(Q+2)-p(Q+2-\mu)}{2}}\int_{B_R}|D_0u|^p\mathrm{d}z+c\int_{B_{4R}}(|g|^p+|f|^p)\mathrm{d}z\\
&\leq c\Big(\frac{\rho}{R}\Big)^{\frac{2(Q+2)-p(Q+2-\mu)}{2}}\int_{B_R}|D_0u|^p\mathrm{d}z+c\frac{|B_{4R}|}{R^\lambda}\left(\|g\|^p_{L^{p,\lambda}}+\|f\|^p_{L^{p,\lambda}}\right)\\
&\leq c\Big(\frac{\rho}{R}\Big)^{\frac{2(Q+2)-p(Q+2-\mu)}{2}}\int_{B_R}|D_0u|^p\mathrm{d}z+cR^{Q+2-\lambda}\left(\|g\|^p_{L^{p,\lambda}}+\|f\|^p_{L^{p,\lambda}}\right). \qquad (6.9)
\end{aligned}$$

Let $H(\rho)=\int_{B_\rho}|D_0u|^s\mathrm{d}z$, $H(R)=\int_{B_R}|D_0u|^s\mathrm{d}z$, $a_1=\frac{2(Q+2)-p(Q+2-\mu)}{2}$, $b_1=Q+2-\lambda$, $B_1=c\left(\|g\|^p_{L^{p,\lambda}}+\|f\|^p_{L^{p,\lambda}}\right)$, $p<\lambda<Q+2$. Taking μ, $Q+2-\frac{2\lambda}{p}<\mu<Q$ it ensures $a_1>b_1$. Hence we can conclude from Lemma 2.11 that

$$\int_{B_\rho}|D_0u|^p\mathrm{d}z \leq c\left[\left(\frac{\rho}{R}\right)^{Q+2-\lambda}\int_{B_R}|D_0u|^p\mathrm{d}z+\rho^{Q+2-\lambda}\left(\|g\|^p_{L^{p,\lambda}}+\|f\|^p_{L^{p,\lambda}}\right)\right]. \quad \square$$

Proof of Theorem 1.2. The result of Theorem 1.2 follows in virtue of Lemma 6.3 and the cutoff function technique. $\square$

Proof of Theorem 1.3. By Lemma 3.3, Lemma 5.8 and Lemma 6.1,

$$\int_{B_\rho}\left|u-u_{B_\rho}\right|^2\mathrm{d}z \le c\int_{B_\rho}\left|v-v_{B_\rho}\right|^2\mathrm{d}z + c\int_{B_\rho}\left|w-w_{B_\rho}\right|^2\mathrm{d}z$$

$$\le c\left[\left(\frac{\rho}{R}\right)^{Q+2}+(\eta_R)^{\frac{p-2}{p}}\right]\int_{B_R}|u-u_{B_R}|^2\mathrm{d}z + c\int_{B_R}|w|^2\mathrm{d}z$$

$$\le c\left[\left(\frac{\rho}{R}\right)^{Q+2}+(\eta_R)^{\frac{p-2}{p}}\right]\int_{B_R}|u-u_{B_R}|^2\mathrm{d}z$$

$$+cR^2\int_{B_{2R}}|D_0 w|^2\mathrm{d}z + cR^2\int_{B_{2R}}\left(|f|^2+|g|^2\right)\mathrm{d}z$$

$$\le c\left[\left(\frac{\rho}{R}\right)^{Q+2}+(\eta_R)^{\frac{p-2}{p}}\right]\int_{B_R}|u-u_{B_R}|^2\mathrm{d}z + cR^2\int_{B_{4R}}\left(|f|^2+|g|^2\right)\mathrm{d}z$$

$$\le c\left[\left(\frac{\rho}{R}\right)^{Q+2}+(\eta_R)^{\frac{p-2}{p}}\right]\int_{B_R}|u-u_{B_R}|^2\mathrm{d}z + cR^{Q+4-\frac{2\lambda}{p}}\left(\|f\|^2_{L^{p,\lambda}}+\|g\|^2_{L^{p,\lambda}}\right).$$

Since $p<\lambda<Q+2, Q+4-\frac{2\lambda}{p}<Q+2$, we have by Lemma 2.11,

$$\int_{B_\rho}\left|u-u_{B_\rho}\right|^2\mathrm{d}z$$

$$\le c\left(\frac{\rho}{R}\right)^{Q+4-\frac{2\lambda}{p}}\int_{B_R}|u-u_{B_R}|^2\mathrm{d}z + c\rho^{Q+4-\frac{2\lambda}{p}}\left(\|f\|^2_{L^{p,\lambda}}+\|g\|^2_{L^{p,\lambda}}\right). \quad \square$$

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REFERENCES

[1] M. Bramanti and L. Brandolini, *Schauder estimates for parabolic nondivergence operators of Hörmander type.* J. Differential Equations **234** (2007), 177–245.

[2] M. Bramanti, M. Cerutti and M. Manfredini, L^p *estimates for some ultraparabolic operators with discontinuous coefficients.* J. Math. Anal. Appl. **200** (1996), 332–354.

[3] S. Byun and L. Wang, L^p *estimates for parabolic equations in Reifenberg domains.* J. Funct. Anal. **223** (2005), *1*, 44–85.

[4] Y. Chen and L. Wu, *Second order elliptic equations and elliptic systems.* Transl. Math. Monogr. **174**, Amer. Math. Soc., Providence, Rhode Island, 1998.

[5] C. Cinti, A. Passcucci and S. Polidoro, *Pointwise estimates for a class of non-homogeneous Kolmogorov equations.* Math. Ann. **340** (2008), 237–264.

[6] E. DiBenedetto, *Degenerate Parabolic Equations.* Universitext, Springer-Verlag, New York, NY, 1993.

[7] M. Di Francesco and S. Polidoro, *Schauder estimates, Harnack inequality and Gaussian lower bound for Kolmogorov-type operators in non-divergence form.* Adv. Difference Equ. **11** (2006), 1261–1320.

[8] H. Dong and D. Kim, *Global regularity of weak solutions to quasilinear elliptic and parabolic equations with controlled growth.* Comm. Partial Differential Equations **36** (2011), 1750–1777.

[9] G. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups.* Ark. Mat. **13** (1975), *2*, 161–207.

[10] B. Franchi, G. Lu and R. Wheeden, *Weighted Poincaré inequalities for Hörmander's vector fields and local regularity for a class of degenerate elliptic equations.* Potential Anal. **4** (1995), 361–375.

[11] A. Friedman, *Partial Differential Equations of Parabolic Type.* Krieger, Malabar, FL, 1983.

[12] A. Fugazzda, *Reverse Hölder inequalities and higher integrability for subcritical parabolic equations.* Adv. Differential Equations **17** (2012), *1-2*, 151–172.

[13] N. Garofalo and D. Nhieu, *Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces.* Comm. Pure Appl. Math. **49** (1996), 1081–1144.

[14] U. Gianazza, *Regularity for nonlinear equations involving square Hörmander operators.* Nonlinear Anal. **23** (1994), 49–73.

[15] M. Giaquinta and M. Struwe, *On the partial regularity of weak solutions of nonlinear parabolic systems.* Math. Z. **179** (1982), 437–451.

[16] Q. Han and F. Lin, *Elliptic Partial Differential Equations.* Courant Lect. Notes Math. **1**, Amer. Math. Soc., Providence, Rhode Island, 1997.

[17] L. Hörmander, *Hypoelliptic second order differential equations.* Acts Math. **119** (1967), 147–171.

[18] D. Jerison, *The Poincaré inequality for vector fields satisfying Hörmander's condition.* Duke Math. **53** (1986), 503–523.

[19] N. Krylov, *Nonlinear Elliptic and Parabolic Equations.* Mathematics and its Applications (Soviet Series) **7**, D. Reidel Publ. Comp., Dordrecht, Holland, 1987.

[20] O. Ladyzhenskaya, V. Solonnikov and N. Ural'tseva, *Linear and quasilinear equations of parabolic type.* Math. Sci. Eng. **46**, American Math. Soc., Providence, RI, 1968.

[21] E. Lanconelli and S. Polidoro, *On a class of hypoelliptic evolution operators.* Rend. Semin. Mat. Univ. Politec. Torino **52** (1994), 29–63.

[22] G. Lieberman, *Second Order Parabolic Differential Equations.* World Scientific Publ., River Edge, NJ, 1996.

[23] G. Lu, *The sharp inequality for free vector fields: an endpoint result.* Rev. Mat. Iberoam. **10** (1994), 453–466.

[24] M. Manfredini and S. Polidoro, *Interior regularity for weak solutions of ultraparabolic equations in divergence form with discontinuous coefficients.* Boll. Unione Mat. Ital. **1** (1998), *3*, 651–675.

[25] C. Morrey, *Multiple Integrals in the Calculus of Variations.* Classics in Mathematics, Springer-Verlag, Heidelberg, 1966.

[26] D. Palagachev and L. Softova, *Quasilinear divergence form parabolic equations in Reifenberg flat domains.* Discrete Contin. Dyn. Syst. **31** (2011), *4*, 1397–1410.

[27] A. Pascucci and S. Polidoro, *The Moser's iterative method for a class of ultraparabolic equations.* Commun. Contemp. Math. **6** (2004), *3*, 395–417.

[28] S. Polidoro, *A global lower bound for the fundamental solution of Kolmogorov-Fokker-Planck equations.* Arch. Ration. Mech. Anal. **137** (1997), 321–340.

[29] S. Polidoro and M. Ragusa, *Hölder regularity for solutions of ultraparabolic equations in divergence form.* Potential Anal. **14** (2001), 341–350.

[30] W. Wang and L. Zhang, *The C^α regularity of a class of non-homogeneous ultraparabolic equations.* Sci. China Series A: Math. **52** (2009), 1589–1606.

[31] Z. Xin and L. Zhang, *On the global existence of solutions to the Prandtl's system.* Adv. Math. **181** (2004), 88–133.

[32] L. Zhang, *The C^α regularity of a class of ultraparabolic equations.* arXiv:0510405v2, 2006.

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