FUZZY SUBGROUPS OF AN INVARIANT GROUP IN FINITELY SUPPORTED MATHEMATICS

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Invariant groups are those groups defined in a newly developed framework called Finitely Supported Mathematics which represents a reformulation of Zermelo-Fraenkel mathematics in the world of finitely supported objects. More precisely, invariant groups are groups all of whose elements are fixed under some special automorphism of atoms. In this paper, we prove that in Finitely Supported Mathematics, for an invariant group G, the family of all finitely supported fuzzy subgroups of G forms an invariant complete lattice, and the family of all finitely supported fuzzy normal subgroups of G forms an invariant modular lattice. The proof includes specific techniques derived from Finitely Supported Mathematics.

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1. INTRODUCTION

In order to provide a computational description of infinite structures, we defined a mathematics called *Finitely Supported Mathematics* (FSM) which deals with a more relaxed notion of infiniteness [3]. Intuitively, in Finitely Supported Mathematics we are able to model infinite structures by using only a finite number of characteristics. More precisely, in FSM we accept the existence of infinite structures, but for an infinite structure we find out that only of a finite family of its elements is 'really important' in order to characterize the related structure, while the other elements are somehow 'similar'. In this sense, we associate to each object a finite family of elements characterizing it, which is called its 'finite support'. As proved in [3], FSM has strong connections with the *Fraenkel-Mostowski* (FM) permutative model of Zermelo-Fraenkel set theory with atoms [9], with Fraenkel-Mostowski axiomatic set theory [8] and with the theory of nominal sets introduced by Gabbay and Pitts [12]. In order to define FSM, the theory of nominal sets over a fixed countable set of atoms is extended to a theory of invariant sets over a fixed infinite (possible non-countable) set of atoms. The theory of invariant sets allows us to define invariant algebraic structures (as invariant sets endowed with invariant algebraic laws) which are used to construct FSM. More generally, we can see FSM sets as finitely supported subsets of invariant sets, and FSM algebraic structures as FSM sets equipped with finitely supported internal operations or with finitely supported relations. Concretely, FSM represents a reformulation of the Zermelo-Fraenkel (ZF) algebra obtained by replacing '(infinite) set' with either 'finitely supported set' or 'invariant set'; all the structures of FSM must be defined according to the finite support requirement. The principles of constructing FSM have historical roots both in the definition of logical notions by Alfred Tarski [15] and in the Erlangen Program of Felix Klein for the classification of various geometries according to invariants under suitable groups of transformations [10]. We have also presented in [3] several similarities between FSM, admissible sets introduced by Barwise [6] and Gandy machines used for describing computability [7].

The main idea of translating a classical ZF result into FSM is to analyze if there exists a valid result obtained by replacing 'set' with 'FSM set' in the ZF result. As proved in [3], not every ZF result can be directly rephrased in the world of invariant sets in terms of finitely supported objects according to arbitrary permutation actions. This is because, given an invariant set X, there could exist some subsets of X which fail to be finitely supported. A related example is represented by the subsets of the set A of atoms which are in the same time infinite and coinfinite. Examples of consistent ZF results that cannot be translated into FSM can be found in [3]. We particularly mention the axiom of choice, all its weaker forms and the Stone duality. Therefore, the translation of classical ZF results into FSM is not trivial and deserves a special attention. The algorithmic techniques for such a translation are described in [3], and are applied in this paper.

In [3] we presented a theory of invariant partially ordered sets (and particularly, invariant lattices) that were involved in order to describe a theory of abstract interpretation in FSM and in order to define in FSM a consistent theory of rough sets. Invariant partially ordered sets had also been used in [14] in order to solve the Scott recursive domain equation $D \cong (D \to D)$ within invariant sets. By applying this last result, Shinwell implemented a functional programming language incorporating facilities for manipulating syntax involving names and binding operations. Since there exist invariant complete lattices failing to be ZF complete, in [4] we proved that there may exist abstract interpretations of some programming languages that can be easier described by using invariant sets than in the classical ZF framework. In this paper, our goal is to connect the theory of invariant partially ordered sets with the theory of fuzzy groups. More precisely, we intend to prove that the ZF order properties over the family of fuzzy subgroups of a group remain valid when translating them into FSM.

2. INVARIANT SETS

Let A be a fixed infinite ZF-set.

A *transposition* is a function $(a\,b) : A \to A$ defined by $(a\,b)(a) = b$, $(a\,b)(b) = a$, and $(a\,b)(n) = n$ for $n \neq a, b$. A *permutation* of A is generated by composing finitely many transpositions. Let S_A be the set of all permutations of A (*i.e.* the set of all bijections on A which leave unchanged all but finitely many elements). As proved in [3], in FSM any bijection of A has to be a permutation of A.

Definition 2.1. Let X be a ZF set.

1. An S_A-*action* on X is a function $\cdot : S_A \times X \to X$ having the properties that $Id \cdot x = x$ and $\pi \cdot (\pi' \cdot x) = (\pi \circ \pi') \cdot x$ for all $\pi, \pi' \in S_A$ and $x \in X$. An S_A-*set* is a pair $(X, \cdot)$ where X is a ZF set, and $\cdot : S_A \times X \to X$ is an S_A-action on X.
2. Let $(X, \cdot)$ be an S_A-set. We say that $S \subset A$ *supports* x (or x is S-supported) whenever for each $\pi \in Fix(S)$ we have $\pi \cdot x = x$, where $Fix(S) = \{\pi \,|\, \pi(a) = a, \text{ for all } a \in S\}$.
3. Let $(X, \cdot)$ be an S_A-set. We say that X is an *invariant set* if for each $x \in X$ there exists a finite set $S_x \subset A$ which supports x. Invariant sets are also called *nominal sets* if we work in the ZF framework [12], or *equivariant sets* if they are defined as empty-supported elements in the Cumulative Hierarchy Fraenkel-Mostowski universe $FM(A)$ [8].
4. Let X be an S_A-set and let $x \in X$. If there exists a finite set supporting x (particularly, if X is an invariant set), then there exists a least finite set $supp(x)$ supporting x [8] which is called *the support of* x. An empty supported element is called *equivariant*.

Proposition 2.2. *Let $(X, \cdot)$ be an S_A-set, and $\pi \in S_A$. If $x \in X$ is finitely supported, then $\pi \cdot x$ is finitely supported and $supp(\pi \cdot x) = \pi(supp(x))$.*

Example 2.3. 1. The set A of atoms is an S_A-set with the S_A-action $\cdot : S_A \times A \to A$ defined by $\pi \cdot a := \pi(a)$ for all $\pi \in S_A$ and $a \in A$. $(A, \cdot)$ is an invariant set because for each $a \in A$ we have that $\{a\}$ supports a. Moreover, $supp(a) = \{a\}$ for each $a \in A$.

2. The set S_A is an S_A-set with the S_A-action $\cdot : S_A \times S_A \to S_A$ defined by $\pi \cdot \sigma := \pi \circ \sigma \circ \pi^{-1}$ for all $\pi, \sigma \in S_A$. $(S_A, \cdot)$ is an invariant set because for each $\sigma \in S_A$ we have that the finite set $\{a \in A \,|\, \sigma(a) \neq a\}$ supports σ. Moreover, $supp(\sigma) = \{a \in A \,|\, \sigma(a) \neq a\}$ for each $\sigma \in S_A$.

3. Any ordinary ZF-set X ($\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $[0,1]$, for example) is an S_A-set with the trivial S_A-action $\cdot : S_A \times X \to X$ defined by $\pi \cdot x := x$ for all $\pi \in S_A$ and $x \in X$. Furthermore, X is an invariant set because for each $x \in X$ we have that $\emptyset$ supports x. Moreover, $supp(x) = \emptyset$ for each $x \in X$. The trivial actions are the single S_A-actions that can be defined on ordinary ZF sets [12].
4. Let $(X, \cdot)$ and $(Y, \diamond)$ be S_A-sets. The Cartesian product $X \times Y$ is also an S_A-set with the S_A-action $\star : S_A \times (X \times Y) \to (X \times Y)$ defined by $\pi \star (x, y) = (\pi \cdot x, \pi \diamond y)$ for all $\pi \in S_A$ and all $x \in X$, $y \in Y$. If $(X, \cdot)$ and $(Y, \diamond)$ are invariant sets, then $(X \times Y, \star)$ is also an invariant set.
5. If $(X, \cdot)$ is an S_A-set, then the powerset $\wp(X) = \{Y \mid Y \subseteq X\}$ is also an S_A-set with the S_A-action $\star : S_A \times \wp(X) \to \wp(X)$ defined by $\pi \star Y := \{\pi \cdot y \mid y \in Y\}$ for all $\pi \in S_A$, and all $Y \subseteq X$. For each invariant set $(X, \cdot)$, we denote by $\wp_{fs}(X)$ the set formed from those subsets of X which are finitely supported according to the action $\star$. According to Proposition 2.2, $(\wp_{fs}(X), \star|_{\wp_{fs}(X)})$ is an invariant set, where $\star|_{\wp_{fs}(X)}$ represents the action $\star$ restricted to $\wp_{fs}(X)$.

Definition 2.4. Let $(X, \cdot)$ be an invariant set. A subset Z of X is called *finitely supported* if and only if $Z \in \wp_{fs}(X)$.

It is worth noting that not any subset of an invariant set is finitely supported. For example, if $B \subset A$ and B is finite, then $supp(B) = B$. If $C \subseteq A$ and C is cofinite (*i.e.* its complementary is finite), then $supp(C) = A \setminus C$. However, if $D \subseteq A$ is neither finite nor cofinite, then D is not finitely supported.

Since functions are particular relations (*i.e.* particular subsets of a Cartesian product of two sets), we have the following results.

Definition 2.5. Let X and Y be invariant sets. A function $f : X \to Y$ is *finitely supported* if $f \in \wp_{fs}(X \times Y)$.

Let $Y^X = \{f \subseteq X \times Y \mid f$ is a function from the underlying set of X to the underlying set of $Y\}$.

PROPOSITION 2.6 ([3]). *Let $(X, \cdot)$ and $(Y, \diamond)$ be invariant sets. Then Y^X is an S_A-set with the S_A-action $\widetilde{\star} : S_A \times Y^X \to Y^X$ defined by $(\pi \widetilde{\star} f)(x) = \pi \diamond (f(\pi^{-1} \cdot x))$ for all $\pi \in S_A$, $f \in Y^X$ and $x \in X$. A function $f : X \to Y$ is finitely supported in the sense of Definition 2.5 if and only if it is finitely supported with respect to the permutation action $\widetilde{\star}$.*

PROPOSITION 2.7 ([3]). *Let $(X, \cdot)$ and $(Y, \diamond)$ be invariant sets. Let $f \in Y^X$ and $\sigma \in S_A$ be arbitrary elements. Let $\widetilde{\star} : S_A \times Y^X \to Y^X$ be the S_A-*

action on Y^X *defined by* $(\pi \widetilde{\star} f)(x) = \pi \diamond (f(\pi^{-1} \cdot x))$ *for all* $\pi \in S_A$, $f \in Y^X$ *and* $x \in X$. *Then* $\sigma \widetilde{\star} f = f$ *if and only if for all* $x \in X$ *we have* $f(\sigma \cdot x) = \sigma \diamond f(x)$.

3. FUZZY SUBGROUPS OF A CLASSICAL GROUP

In this section, we remind some basic results in the classical Zermelo-Fraenkel theory of fuzzy groups.

Definition 3.1. A *fuzzy set over the ZF set U* is a function $\mu : U \to [0,1]$.

Definition 3.2. Let $(G, \cdot, 1)$ be a group. On the family of all fuzzy sets over G we define a partial order relation $\sqsubseteq$, called *fuzzy sets inclusion*, by $\mu \sqsubseteq \eta$ if and only if $\mu(x) \leq \eta(x)$ for all $x \in G$.

Definition 3.3. Let $(G, \cdot, 1)$ be a group. A fuzzy set μ over G is called *fuzzy subgroup* of G if the following conditions are satisfied.

- $\mu(x \cdot y) \geq min\{\mu(x), \mu(y)\}$ for all $x, y \in G$;
- $\mu(x^{-1}) \geq \mu(x)$ for all $x \in G$.

Definition 3.4. Let $(G, \cdot, 1)$ be a group. A fuzzy subgroup η of G that satisfies the additional condition $\eta(x \cdot y) = \eta(y \cdot x)$ for all $x, y \in G$ is called a *fuzzy normal subgroup* of G.

THEOREM 3.5 ([1,2]). *Let* $(G, \cdot, 1)$ *be a group.*

- *The set* $FL(G)$ *consisting of all fuzzy subgroups of* G *forms a complete lattice with respect to fuzzy sets inclusion.*
- *The set* $FN(G)$ *consisting of all fuzzy normal subgroups of* G *forms a modular lattice with respect to fuzzy sets inclusion.*

4. FUZZY SUBGROUPS OF AN INVARIANT GROUP IN FSM

Invariant groups were studied in [3]. An invariant group is an invariant set equipped with an equivariant internal group law.

Definition 4.1. An *invariant group* is a triple $(G, \cdot, \diamond)$ such that the following conditions are satisfied:

- $(G, \cdot)$ is a group;
- $(G, \diamond)$ is a non-trivial invariant set;
- for each $\pi \in S_A$ and each $x, y \in G$ we have $\pi \diamond (x \cdot y) = (\pi \diamond x) \cdot (\pi \diamond y)$, which means that the internal law on G is equivariant.

PROPOSITION 4.2. $(G, \cdot, \diamond)$ *be an invariant group. We have*

1. $\pi \diamond 1 = 1$ *for all* $\pi \in S_A$, *where 1 is the neutral element of* G.

2. $\pi \diamond x^{-1} = (\pi \diamond x)^{-1}$ *for all* $\pi \in S_A$ *and* $x \in G$.

Proof. 1. Since $\pi \diamond 1 = \pi \diamond (1 \cdot 1) = (\pi \diamond 1) \cdot (\pi \diamond 1)$, it follows $\pi \diamond 1 = 1$.
2. We have $(\pi \diamond x) \cdot (\pi \diamond x^{-1}) = \pi \diamond (x \cdot x^{-1}) = \pi \diamond 1 = 1$, and analogously $(\pi \diamond x^{-1}) \cdot (\pi \diamond x) = 1$. Therefore, $(\pi \diamond x)^{-1} = \pi \diamond x^{-1}$. □

The following examples of invariant groups were presented in [3].

Example 4.3. 1. The group $(S_A, \circ, \cdot)$ is an invariant group, where $\circ$ is the usual composition of permutations and $\cdot$ is the S_A-action on S_A defined as in Example 2.3(2). Since the composition law on S_A is associative, one can easily verify that $\pi \cdot (\sigma \circ \tau) = (\pi \cdot \sigma) \circ (\pi \cdot \tau)$ for all $\pi, \sigma, \tau \in S_A$.

2. With the notations above, the group $(P_A, \circ, \cdot)$ of all FSM bijections of A onto A is an invariant group. This is because P_A coincides to S_A in FSM according to Proposition 2.6 from [3].
3. The free group $(F(X), \top, \widetilde{\star})$ over an invariant set $(X, \diamond)$ (formed by those classes $[w]$ of words w, where two words are in the same class if one can be obtained from another by repeatedly cancelling or inserting terms of the form $x^{-1}x$ or xx^{-1} for $x \in X$) is an invariant group, where $\widetilde{\star} : S_A \times F(X) \to F(X)$ is defined by $\pi \widetilde{\star} [x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_l^{\varepsilon_l}] = [(\pi \diamond x_1)^{\varepsilon_1} \dots (\pi \diamond x_l)^{\varepsilon_l}]$, and $[x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n}] \top [y_1^{\delta_1} y_2^{\delta_2} \dots y_m^{\delta_m}] = [x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n} y_1^{\delta_1} y_2^{\delta_2} \dots y_m^{\delta_m}]$.

Definition 4.4. Let $(G, \cdot, \diamond)$ be an invariant group. An *FSM subgroup* of G is a subgroup of G which is finitely supported as an element of $\wp(G)$.

Example 4.5. 1. Let $(G, \cdot, \diamond)$ be an invariant group. The centre of G, namely $Z(G) := \{g \in G \mid g \cdot x = x \cdot g, \text{ for all } x \in G\}$, is an FSM subgroup of G, and it is itself an invariant group because it is empty supported as an element of $\wp(G)$.

2. Let X be a finitely supported subset of G. Then the subgroup generated by X is an FSM subgroup of G, but it is not itself an invariant group.

Definition 4.6. An *invariant partially ordered set (invariant poset)* is an invariant set $(E, \cdot)$ together with an equivariant (*i.e.* empty supported) partial order relation $\sqsubseteq$ on E. An invariant poset is denoted by $(E, \sqsubseteq, \cdot)$, or simply E.

Definition 4.7. An *invariant lattice* is an invariant set $(L, \cdot)$ together with an equivariant lattice order $\sqsubseteq$ on L.

Definition 4.8. An *invariant complete lattice* is an invariant poset $(L, \sqsubseteq, \cdot)$ such that every finitely supported subset $X \subseteq L$ has a least upper bound $\sqcup X$ with respect to the order relation $\sqsubseteq$.

Definition 4.9. An *invariant modular lattice* is an invariant lattice $(L, \sqsubseteq, \cdot)$ that satisfies the classical ZF modularity law: $x \sqsubseteq z$ implies $x \vee (y \wedge z) = (x \vee y) \wedge z$ for all $x, y, z \in L$.

If $(G, \cdot, \diamond)$ is an invariant group, we denote by $\mathcal{L}(G)_{inv}$ the family of all FSM subgroups of G ordered by inclusion. According to Theorem 4.6 from [5], we have that $(\mathcal{L}(G)_{inv}, \subseteq, \star)$ is an invariant complete lattice, where $\subseteq$ represents the usual inclusion relation on $\wp(G)$, and $\star$ is the S_A-action on $\wp(G)$ defined as in Example 2.3(5). The goal now is to prove that the family of all FSM fuzzy subgroups of an invariant group forms also an invariant complete lattice.

Definition 4.10. An *FSM fuzzy set over the invariant set* $(U, \cdot)$ is a finitely supported function $\mu : U \to [0, 1]$.

Definition 4.11. Let $(G, \cdot, \diamond)$ be an invariant group. An FSM fuzzy set μ over the invariant set G is called an *FSM fuzzy subgroup* of G if the following conditions are satisfied:

- $\mu(x \cdot y) \geq min\{\mu(x), \mu(y)\}$ for all $x, y \in G$;
- $\mu(x^{-1}) \geq \mu(x)$ for all $x \in G$.

Example 4.12. Let us consider the set A of atoms and $(F(A), \intercal, \widetilde{\star})$ the invariant free group over A described in Example 4.3(3). For an element in $F(A)$ of form $[w] = [x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_l^{\varepsilon_l}]$, we define $s([w]) = \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_l$. Whenever a word w is equivalent with w' modulo the reduction/inserting of terms of form xx^{-1} or $x^{-1}x$, *i.e.* whenever $[w] = [w']$, we obviously have $s([w]) = s([w'])$, and so s is well defined. It follows directly (using Proposition 4.2(2) for the second item below) that

1. $s([w] \intercal [w']) = s([w]) + s([w'])$ for all $[w], [w'] \in F(A)$;
2. $s(\pi \widetilde{\star} [w]) = s([w])$ for all $\pi \in S_A$ and $[w] \in F(A)$, meaning that s is an equivariant function from $F(A)$ to $\mathbb{Z}$;
3. $s([w]^{-1}) = -s([w])$.

We obtained that s is an equivariant group homomorphism between the invariant groups $F(A)$ and $\mathbb{Z}$. Let us consider $\mu : F(A) \to [0, 1]$ defined by

$$\mu([w]) = \begin{cases} 0, & \text{if } s([w]) \text{ is odd in } \mathbb{Z}\ ; \\ 1 - \frac{1}{n}, & \text{if } s([w]) = m \cdot 2^n \text{ with } m \text{ odd in } \mathbb{Z} \text{ and } n \in \mathbb{N}; \\ 1, & \text{if } s([w]) = 0\ . \end{cases}$$

Since every even integer k can be uniquely expressed as $k = m \cdot 2^n$ with m an odd integer and $n \in \mathbb{N}$, we have that μ is well defined. Clearly, from Proposition 2.7 we have that μ is equivariant because, according to item 2 above, we obtain $\mu(\pi \widetilde{\star} [w]) = \mu([w])$ for all $\pi \in S_A$ and $[w] \in F(A)$. Obviously, according to item 3 above, $\mu([w]^{-1}) = \mu([w])$ for all $[w] \in F(A)$. It remains to prove that $\mu([w] \top [w']) \geq min\{\mu([w]), \mu([w'])\}$ for all $[w], [w'] \in F(A)$. Let us fix $[w], [w'] \in F(A)$. The non-trivial case to be analyzed is the case when both $s([w])$ and $s([w'])$ are non-zero even integers. Then there exist the unique expressions $s([w]) = m_1 \cdot 2^{n_1}$ and $s([w']) = m_2 \cdot 2^{n_2}$ with m_1, m_2 integer and odd. Assume $n_1 \leq n_2$ (the other case is analogue). If $n_1 = n_2$, then $s([w] \top [w']) = s([w]) + s([w']) = (m_1 + m_2) \cdot 2^{n_1} = m \cdot 2^n$ with $m \in \mathbb{Z}$ odd and $n > n_1$ because $m_1 + m_2$ is non-zero and even (we considered only the case $m_1 \neq -m_2$ since the case $m_1 = -m_2$ leads to $s([w] \top [w']) = 0$, and so $\mu([w] \top [w']) = 1$, and the result follows trivially). Thus, $\mu([w] \top [w']) = 1 - \frac{1}{n} > 1 - \frac{1}{n_1} = min\{\mu([w]), \mu([w'])\}$. Suppose now $n_1 < n_2$. Then $s([w] \top [w']) = s([w]) + s([w']) = (m_1 + m_2 \cdot 2^{n_2 - n_1}) \cdot 2^{n_1}$ with $m_1 + m_2 \cdot 2^{n_2 - n_1}$ integer and odd. Thus, $\mu([w] \top [w']) = 1 - \frac{1}{n_1} = min\{1 - \frac{1}{n_1}, 1 - \frac{1}{n_2}\}$. We have that μ is an FSM fuzzy subgroup of $F(A)$.

THEOREM 4.13. *Let $(G, \cdot, \diamond)$ be an invariant group. The set $FL_{inv}(G)$ consisting of all FSM fuzzy subgroups of G forms an invariant complete lattice with respect to fuzzy sets inclusion.*

Proof. We prove first that $FL_{inv}(G)$ is an invariant poset. Clearly, $FL_{inv}(G)$ is a subset of the invariant set formed by those finitely supported functions from G to $[0, 1]$. We have to prove that $FL_{inv}(G)$ is itself invariant, that is $\pi \widetilde{\star} \mu$ is an FSM fuzzy subgroup of G for all $\pi \in S_A$ and $\mu \in FL_{inv}(G)$, where $\widetilde{\star}$ is the S_A-action on $[0, 1]^G$ defined in Proposition 2.6. Let us fix $\pi \in S_A$ and $\mu \in FL_{inv}(G)$. We have $\mu(x \cdot y) \geq min\{\mu(x), \mu(y)\}$ for all $x, y \in G$, and $\mu(x^{-1}) \geq \mu(x)$ for all $x \in G$. According to Proposition 2.2, $\pi \widetilde{\star} \mu$ is a finitely supported function from G to $[0, 1]$. According to Proposition 2.6 and since $[0, 1]$ is a trivial invariant set (*i.e.* $[0, 1]$ is an S_A-set equipped with the trivial S_A-action $(\sigma, x) \mapsto x$ for all $(\sigma, x) \in S_A \times [0, 1]$), we have $(\pi \widetilde{\star} \mu)(x \cdot y) = \mu(\pi^{-1} \diamond (x \cdot y)) = \mu((\pi^{-1} \diamond x) \cdot (\pi^{-1} \diamond y)) \geq min\{\mu(\pi^{-1} \diamond x), \mu(\pi^{-1} \diamond y)\} = min\{(\pi \widetilde{\star} \mu)(x), (\pi \widetilde{\star} \mu)(y)\}$ for all $x, y \in G$; the second identity holds because G is an invariant group, and so the internal law on G is equivariant. Moreover, $(\pi \widetilde{\star} \mu)(x^{-1}) = \mu(\pi^{-1} \diamond (x^{-1})) = \mu((\pi^{-1} \diamond x)^{-1}) \geq \mu(\pi^{-1} \diamond x) = (\pi \widetilde{\star} \mu)(x)$; the second identity holds from Proposition 4.2(2). This means that $(FL_{inv}(G), \widetilde{\star})$ is an invariant set.

Now we prove that $(FL_{inv}(G), \sqsubseteq)$ is an invariant poset where $\sqsubseteq$ is the classical inclusion order on the family of all FSM fuzzy subgroups of G, defined by $\mu \sqsubseteq \eta$ if and only if $\mu(x) \leq \eta(x)$ for all $x \in G$. We have to prove that $\sqsubseteq$ is equivariant. Indeed, let $\pi \in S_A$, and μ, η be two FSM fuzzy subgroups of G such that $\mu \sqsubseteq \eta$. Since $\mu(x) \leq \eta(x)$ for all $x \in G$, we have $\mu(\pi^{-1} \cdot x) \leq \eta(\pi^{-1} \cdot x)$ for all $x \in G$, namely $(\pi \widetilde{\star} \mu)(x) \leq (\pi \widetilde{\star} \eta)(x)$ for all $x \in G$. It follows that $\pi \widetilde{\star} \mu \sqsubseteq \pi \widetilde{\star} \eta$.

For each $\alpha \in [0,1]$ and each $\mu \in [0,1]^G_{fs}$, we define $G^\mu_\alpha = \{x \in G \,|\mu(x) \geq \alpha\}$, and prove that each G^μ_α is a finitely supported subset of G. Moreover, we have $supp(G^\mu_\alpha) \subseteq supp(\mu)$ for all $\alpha \in [0,1]$. Indeed, let us fix $\alpha \in [0,1]$, $\mu \in FL_{inv}(G)$, and consider $\pi \in Fix(supp(\mu))$. Recall that $[0,1]$ is a trivial invariant set. According to Proposition 2.7, we have $\mu(\pi \diamond x) = \mu(x)$ for all $x \in G$. Thus, for each $x \in G^\mu_\alpha$ and each $\pi \in Fix(supp(\mu))$ we have $\pi \diamond x \in G^\mu_\alpha$. Therefore, $\pi \star G^\mu_\alpha \subseteq G^\mu_\alpha$ for each $\pi \in Fix(supp(\mu))$, where $\star$ is the S_A-action on $\wp(G)$ defined as in Example 2.3(5). By contradiction, let us assume that there is a $\pi \in Fix(supp(\mu))$ such that $\pi \star G^\mu_\alpha \subsetneq G^\mu_\alpha$. By induction, we get $\pi^n \star G^\mu_\alpha \subsetneq G^\mu_\alpha$ for all $n \geq 1$. However, π is a finitary permutation, and so there exists $k \in \mathbb{N}$ such that $\pi^k = Id$. We obtain $G^\mu_\alpha \subsetneq G^\mu_\alpha$, a contradiction. Therefore, $\pi \star G^\mu_\alpha = G^\mu_\alpha$ for all $\pi \in Fix(supp(\mu))$, and so G^μ_α is supported by $supp(\mu)$. Since the support of G^μ_α is the least finite set supporting G^μ_α, we obtain that $supp(G^\mu_\alpha) \subseteq supp(\mu)$.

Let $[G^\mu_\alpha]$ be the subgroup of G generated by G^μ_α, *i.e.* the smallest subgroup of G containing G^μ_α. Every element from $[G^\mu_\alpha]$ can be expressed as a finite product of elements of G^μ_α and inverses of elements of G^μ_α. We have to prove that $[G^\mu_\alpha]$ is a finitely supported subgroup of G. We claim that $[G^\mu_\alpha]$ is supported by $supp(G^\mu_\alpha)$. Indeed, let us consider $\pi \in Fix(supp(G^\mu_\alpha))$. Let $x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot \ldots \cdot x_n^{\varepsilon_n}$, $x_i \in G^\mu_\alpha$, $\varepsilon_i = \pm 1$, $i = 1, \ldots, n$ be an arbitrary element from $[G^\mu_\alpha]$. Since $\pi \in Fix(supp(G^\mu_\alpha))$, we have $\pi \diamond x_i \in G^\mu_\alpha$ for all $i \in \{1, \ldots, n\}$. Therefore, because the internal law on G is equivariant, we have $\pi \diamond (x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot \ldots \cdot x_n^{\varepsilon_n}) = (\pi \diamond x_1^{\varepsilon_1}) \cdot (\pi \diamond x_2^{\varepsilon_2}) \cdot \ldots \cdot (\pi \diamond x_n^{\varepsilon_n}) = (\pi \diamond x_1)^{\varepsilon_1} \cdot (\pi \diamond x_2)^{\varepsilon_2} \cdot \ldots \cdot (\pi \diamond x_n)^{\varepsilon_n} \in [G^\mu_\alpha]$. Therefore, $\pi \star [G^\mu_\alpha] = [G^\mu_\alpha]$ where $\star$ is the S_A-action on $\wp(G)$ defined as in Example 2.3(5), and so $supp(G^\mu_\alpha)$ supports $[G^\mu_\alpha]$. Since $supp(G^\mu_\alpha) \subseteq supp(\mu)$, we also have that $supp([G^\mu_\alpha]) \subseteq supp(\mu)$ for all $\alpha \in [0,1]$.

For any finitely supported function $\nu : G \to [0,1]$, we consider the function $\nu^* : G \to [0,1]$ defined by $\nu^*(x) = supremum\{\alpha \in [0,1] \mid x \in [G^\nu_\alpha]\}$ for any $x \in G$. We claim that $\nu^\star$ is supported by $supp(\nu)$. Let $\pi \in Fix(supp(\nu))$. We have $\nu^*(\pi \diamond x) = supremum\{\alpha \in [0,1] \mid \pi \diamond x \in [G^\nu_\alpha]\} = supremum\{\alpha \in [0,1] \mid x \in \pi^{-1} \star [G^\nu_\alpha]\}$. However, $\pi^{-1} \in Fix(supp(\nu))$, $supp([G^\nu_\alpha]) \subseteq supp(\nu)$ for all $\alpha \in [0,1]$ (meaning that there exists a set of atoms not depending on α which supports all $[G^\nu_\alpha]$), and so $\pi^{-1} \star [G^\nu_\alpha] = [G^\nu_\alpha]$. Therefore, $\nu^*(\pi \diamond x) = supremum\{\alpha \in [0,1] \mid x \in [G^\nu_\alpha]\} = \nu^*(x)$ for all $x \in G$. Thus, ν^* is finitely supported. Furthermore, as in the standard fuzzy groups theory we have that ν^* is a fuzzy subgroup of G [1], and so it is an FSM fuzzy subgroup of G.

In order to prove that $FL_{inv}(G)$ is an invariant complete lattice it remains to establish that any finitely supported family of elements from $FL_{inv}(G)$ has a least upper bound. Let us consider now $\mathcal{F} = (\mu_i)_{i \in I}$ a finitely supported family of elements from $FL_{inv}(G)$. We define $\underset{i \in I}{\cup} \mu_i : G \to [0,1]$ by $\underset{i \in I}{\cup} \mu_i(x) =$

$\underset{i\in I}{supremum}\{\mu_i(x) \mid i \in I\}$ for all $x \in G$. Since $[0,1]$ is a closed set in the classical topology of $\mathbb{R}$, we have that $\underset{i\in I}{\cup}\mu_i$ is well defined in the ZF framework (its codomain is contained in $[0,1]$). We claim that $supp(\mathcal{F})$ supports $\underset{i\in I}{\cup}\mu_i$. Let $\pi \in Fix(supp(\mathcal{F}))$. According to Proposition 2.7 and because $[0,1]$ is a trivial invariant set, in order to prove that $\pi\widetilde{\star} \underset{i\in I}{\cup}\mu_i = \underset{i\in I}{\cup}\mu_i$ we should prove the relation $\underset{i\in I}{supremum}\{\mu_i(\pi \diamond x) \mid i \in I\}=\underset{i\in I}{supremum}\{\mu_i(x) \mid i \in I\}$ for all $x \in G$. Since $\pi \in Fix(supp(\mathcal{F}))$, we also have $\pi^{-1} \in Fix(supp(\mathcal{F}))$, and so the action of π^{-1} on $\mathcal{F}$ leaves $\mathcal{F}$ unchanged. Therefore, for each $j \in I$ there exists a unique $i \in I$ such that $\mu_j = \pi^{-1}\widetilde{\star}\mu_i$. Thus, for each $j \in I$ there exists a unique $i \in I$ such that $\mu_j(x) = \mu_i(\pi \diamond x)$ for all $x \in G$. Fix some $y \in G$; it follows that $\{\mu_i(\pi \diamond y) \mid i \in I\} = \{\mu_j(y) \mid j \in I\}$. Thus, $\underset{i\in I}{supremum}\{\mu_i(\pi\diamond x)|i \in I\}=\underset{i\in I}{supremum}\{\mu_i(x) \mid i \in I\}$ for all $x \in G$. Therefore, we obtain that $supp(\mathcal{F})$ supports $\underset{i\in I}{\cup}\mu_i$. Since $\underset{i\in I}{\cup}\mu_i$ is finitely supported by $supp(\mathcal{F})$, we have that $(\underset{i\in I}{\cup}\mu_i)^*$ is finitely supported by $supp(\mathcal{F})$. Furthermore, as in the standard fuzzy groups theory we have that $(\underset{i\in I}{\cup}\mu_i)^*$ is the least upper bound of $\mathcal{F}$ in $FL(G)$ with respect to the order relation $\sqsubseteq$ [1]. Since $(\underset{i\in I}{\cup}\mu_i)^*$ is also finitely supported, it follows that $(\underset{i\in I}{\cup}\mu_i)^*$ is the least upper bound of $\mathcal{F}$ in $FL_{inv}(G)$ with respect to the equivariant order relation $\sqsubseteq$. □

According to Theorems 3.33 and 3.34 from [3], we get the following result.

COROLLARY 4.14. *Let $(G,\cdot,\diamond)$ be an invariant group and $f : FL_{inv}(G) \to FL_{inv}(G)$ a finitely supported monotone function.*

1. *There exist a greatest fixed point of f and a least fixed point of f.*
2. *If f is equivariant, then the set of all fixed points of f is itself an invariant complete sublattice of $FL_{inv}(G)$.*

Invariant Galois connections were introduced in [3]. Let $(P, \sqsubseteq_P, \cdot_P)$ and $(Q, \sqsubseteq_Q, \cdot_Q)$ be two invariant posets, and $f : P \to Q$, $g : Q \to P$ two functions. The pair (f, g) is an invariant Galois connection between P and Q if and only if both f and g are equivariant and for all $p \in P$ and $q \in Q$ we have that $f(p) \sqsubseteq_Q q \Leftrightarrow p \sqsubseteq_P g(q)$. If (f, g) is an invariant Galois connection, then we say that f has an invariant adjoint g, and g has an invariant co-adjoint f.

According to Proposition 3.19 from [3] we obtain the following result.

COROLLARY 4.15. *Let $(G,\cdot,\diamond)$ be an invariant group and $f : FL_{inv}(G) \to FL_{inv}(G)$ an equivariant function.*

1. *f has an invariant adjoint if and only if $f((\underset{i\in I}{\cup}\mu_i)^*) = (\underset{i\in I}{\cup} f(\mu_i))^*$ for every finitely supported family $(\mu_i)_{i\in I}$ of elements from $FL_{inv}(G)$.*
2. *f has an invariant co-adjoint if and only if $f(\underset{i\in I}{\cap}\mu_i) = \underset{i\in I}{\cap} f(\mu_i)$ for every finitely supported family $(\mu_i)_{i\in I}$ of elements from $FL_{inv}(G)$.*

According to [5], the family $(\mathcal{L}(G)_{inv}, \subseteq, \star)$ of all finitely supported subgroups of an invariant group G is an invariant complete lattice, with the operations join and meet defined by $H, K \mapsto [H \cup K]$ and $H, K \mapsto H \cap K$, where $[H \cup K]$ represents the subgroup of G generated by the set $H \cup K$ (*i.e.* the smallest subgroup of G containing both H and K). According to Proposition 3.19 in [3], we also have the following results.

COROLLARY 4.16. *Let $(G, \cdot, \diamond)$ be an invariant group and $f : FL_{inv}(G) \to \mathcal{L}(G)_{inv}$, $g : \mathcal{L}(G)_{inv} \to FL_{inv}(G)$ equivariant functions.*

1. *f has an invariant adjoint if and only if $f((\underset{i\in I}{\cup}\mu_i)^*) = [\underset{i\in I}{\cup} f(\mu_i)]$ for every finitely supported family $(\mu_i)_{i\in I}$ of elements from $FL_{inv}(G)$.*
2. *g has an invariant co-adjoint if and only if $g(\underset{i\in I}{\cap} H_i) = \underset{i\in I}{\cap} g(H_i)$ for every finitely supported family $(H_i)_{i\in I}$ of finitely supported subgroups of G.*

COROLLARY 4.17. *Let $(G, \cdot, \diamond)$ be an invariant group and $f : \mathcal{L}(G)_{inv} \to FL_{inv}(G)$, $g : FL_{inv}(G) \to \mathcal{L}(G)_{inv}$ equivariant functions.*

1. *f has an invariant adjoint if and only if $f([\underset{i\in I}{\cup} H_i]) = (\underset{i\in I}{\cup} f(H_i))^*$ for every finitely supported family $(H_i)_{i\in I}$ of finitely supported subgroups of G.*
2. *g has an invariant co-adjoint if and only if $g(\underset{i\in I}{\cap}\mu_i) = \underset{i\in I}{\cap} g(\mu_i)$ for every finitely supported family $(\mu_i)_{i\in I}$ of elements from $FL_{inv}(G)$.*

THEOREM 4.18. *Let $(G, \cdot, \diamond)$ be an invariant group. The set $FN_{inv}(G)$ consisting of all FSM fuzzy normal subgroups of G forms an invariant modular lattice with respect to fuzzy sets inclusion.*

Proof. We prove that $FN_{inv}(G)$ is an invariant subset of the invariant set $FL_{inv}(G)$, that is $\pi\widetilde{\star}\mu$ is an FSM fuzzy normal subgroup of G for all $\pi \in S_A$ and $\mu \in FN_{inv}(G)$, where $\widetilde{\star}$ is the S_A-action on $FL_{inv}(G)$ defined as in Proposition 2.6. Let us fix $\pi \in S_A$ and $\mu \in FN_{inv}(G)$. We have $\mu(x \cdot y) = \mu(y \cdot x)$ for all $x, y \in G$. According to Proposition 2.2, $\pi\widetilde{\star}\mu$ is a finitely supported function from G to $[0, 1]$. According to Proposition 2.6, because $[0, 1]$ is a trivial invariant set, and because the internal law in G is equivariant, we have $(\pi\widetilde{\star}\mu)(x\cdot y) = \mu(\pi^{-1}\diamond(x\cdot y)) = \mu((\pi^{-1}\diamond x)\cdot(\pi^{-1}\diamond y)) = \mu((\pi^{-1}\diamond y)\cdot(\pi^{-1}\diamond x)) = \mu(\pi^{-1}\diamond(y\cdot x)) = (\pi\widetilde{\star}\mu)(y\cdot x)$ for all $x, y \in G$. This means $\pi\widetilde{\star}\mu$ is an FSM fuzzy normal subgroup of G, and so $FN_{inv}(G)$ is an invariant subset of $FL_{inv}(G)$.

As in the proof of Theorem 4.13, it follows that the inclusion order on $FN_{inv}(G)$ is equivariant. Furthermore, for any finitely supported functions $\mu, \eta : G \to [0,1]$ we have that the functions $\mu \cup \eta : G \to [0,1]$ defined by $(\mu \cup \eta)(x) = max[\mu(x), \eta(x)]$ for all $x \in G$ and $\mu \cap \eta : G \to [0,1]$ defined by $(\mu \cap \eta)(x) = min[\mu(x), \eta(x)]$ for all $x \in G$ are both finitely supported by $supp(\mu) \cup supp(\eta)$. As in the proof of Theorem 4.13, we also have that $(\mu \cup \eta)^*$ is supported by $supp(\mu) \cup supp(\eta)$. Similar to the standard ZF theory of fuzzy sets, we have that $(\mu \cup \eta)^*$ and $\mu \cap \eta$ are the least upper bound and the greatest lower bound of μ and η in $FN(G)$, respectively. This means that $(\mu \cup \eta)^*$ and $\mu \cap \eta$ are the least upper bound and the greatest lower bound of μ and η in $FN_{inv}(G)$, respectively. Therefore, we have that $FN_{inv}(G)$ is an invariant lattice. The modularity property of $FN_{inv}(G)$ holds as in the ZF approach due to the special property of fuzzy normal subgroups described in Definition 3.4. □

5. CONCLUSION

Rosenfeld introduced the notion of a fuzzy group and proved that many concepts of group theory can naturally be extended in order to develop the theory of fuzzy groups [13]. Starting from 1971, the theory of fuzzy groups has a continuous evolution, and there exist several applications. A survey of the development of fuzzy group theory can be found in [11]. More recent combinatorial developments on this theory can be found in [16, 17].

In this paper, we developed the theory of fuzzy groups in an alternative framework named Finitely Supported Mathematics (FSM), where all the structures should be finitely supported; in FSM, 'sets' are replaced either by 'invariant sets' (sets endowed with some group actions satisfying a finite support requirement) or by 'finitely supported sets' (finitely supported elements in the powerset of an invariant set). The main motivation for developing FSM comes from both mathematics (by modelling infinite algebraic structures in a finitary manner) and computer science (where a class of FSM sets, namely nominal sets, are used in various areas such as semantics, automata theory, software verification, and functional programming) [3, 12]. We introduced the notion of FSM fuzzy subgroups of a given FSM group, and proved several order properties on the family of all these FSM fuzzy subgroups. The FSM properties obtained in this paper represent a natural generalization of the ZF order properties obtained in the classical fuzzy group theory. More exactly, for a ZF group G we know that the family of all fuzzy subgroups of G forms a complete lattice, and the family of all fuzzy normal subgroups of G forms a modular lattice. In FSM we proved that, for an invariant group G, the family of all finitely supported fuzzy subgroups of G forms an invariant complete lattice, and the family of all finitely supported fuzzy normal subgroups of G forms an invariant modular lattice.

The translation of a classical ZF property into FSM is not a trivial approach. In order to translate a general ZF result into FSM, the proof of the related FSM result should not brake the requirement that any construction has to be finitely supported. Therefore, in order to get a valid FSM result, it is necessary to prove that all the structures involved in the proof (including the proofs of the used results) are finitely supported. Three general methods of proving that a certain structure is finitely supported are presented in [3]. In this paper, we applied effectively the 'constructive' method, meaning that we construct the support of a certain structure by using an hierarchical (step-by-step) construction. The other two methods presented in [3] are described by using higher order logic, but the formal involvement of them is much difficult. The method known as the the Finite Support Principle states that any function or relation that is defined from finitely supported functions and relations using classical higher-order logic is itself finitely supported. The third method is represented by a refinement of the previous one, and allows us to provide hierarchical boundedness properties for supports; this method states that for any finite set S of atoms, anything that is definable (in higher order logic) from S-supported structures using S-supported constructions is S-supported.

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