PRIME CRITICALITY AND PRIME COVERING

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Given a *graph* G, a subset M of $V(G)$ is a module of G if for $a, b \in M$ and $x \in V(G) \setminus M$, $xa \in E(G)$ if and only if $xb \in E(G)$. A graph G with at least three vertices is *prime* if $\emptyset$, the single-vertex sets, and $V(G)$ are the only modules of G. A vertex x of a prime graph G is *critical* if $G - x$ is not prime. In this paper, we provide a new characterization of critical graphs, and of graphs admitting either only one non-critical vertex or at least two, in terms of coverings by prime induced subgraphs of a certain size.

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1. INTRODUCTION

All graphs in this paper are finite and simple. In a graph G, a subset M of $V(G)$ is a *module* [22] (or a *clan* [14], or an *interval* [13, 21]) of G provided that for all $a, b \in M$ and $x \in V(G) \setminus M$, $xa \in E(G)$ if and only if $xb \in E(G)$. The empty set, the singleton sets, and the full set of vertices are *trivial modules*. A graph is *indecomposable* [16, 21] if all its modules are trivial. An indecomposable graph with at least three vertices is a *prime graph*. The simplest prime graph with n vertices where $n \geq 4$ is the *path* P_n. Up to isomorphism, the graph P_4 is the only 4-vertex prime graph. In recent years, the concept of primality has become fundamental in the study of finite structures. This concept and other notions have been the subject of several papers, for example [1, 2, 4, 5, 7, 8, 9, 12, 15, 17, 18, 19, 20]. Let G be a graph. The order of G is denoted by $v(G)$ or $|V(G)|$. Let p be a partition of $V(G)$; the graph G is *multipartite* by p if for every $M \in p$, $G[M]$ is empty. It is *bipartite* when $|p| = 2$. Let G be a prime graph. A vertex x of $V(G)$ is called a *critical vertex* of G if $G - x$ is not prime. We denote by $\mathscr{N}(G)$ the set of the non-critical vertices of G. The graph G is *critical* [21] if $\mathscr{N}(G) = \emptyset$. For example, for each integer $n \geq 2$, the graph G_{2n} (see Figure 1) defined below is critical.

The vertex set of G_{2n} is $\{0, \ldots, 2n-1\}$ and for $i, j \in \{0, \ldots, 2n-1\}$, ij is an edge of G_{2n} if there exist $k \leq l \in \{0, \ldots, n-1\}$ such that $\{i, j\} = \{2k, 2l+1\}$. The notion of critical graphs was introduced by Schmerl and Trotter [21], they gave the following characterization.

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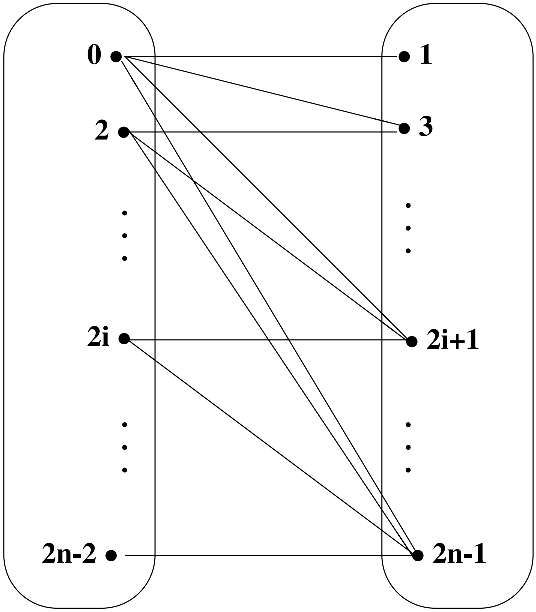

Figure 1: G_{2n}

THEOREM 1.1 (Schmerl and Trotter [21]). *A prime graph G is critical if and only if it is isomorphic to G_{2n} or to $\overline{G_{2n}}$, where $n \geq 2$.*

Recently, in [8] Boudabbous and Salhi give a complete morphologic description of "critically without duo graphs". Moreover, Belkhechine, Boudabbous and Baka Elayech [3] described the prime graphs G such that $|\mathcal{N}(G)| = 1$. Chudnovsky and Seymour [11] provide a characterization of a non-critical graphs in terms of growing prime graphs, starting from a prime induced subgraph and adding vertices one at a time in such a way that all the intermediate subgraphs are prime of which by the following.

THEOREM 1.2 (Chudnovsky and Seymour [11]). *Let G be a graph, and let H be a proper induced subgraph of G. If both G and H are prime and G is not critical, then there are $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_k \subseteq V(G)$ satisfying*

- *$G[X_0]$ is isomorphic to H;*
- *$X_k = V(G)$;*
- *for $0 \leq i \leq k$, $G[X_i]$ is prime;*
- *for $0 \leq i \leq k-1$, $|X_{i+1} \setminus X_i| = 1$.*

In this work, we give a classification of the prime graphs based on their prime subgraphs. In order to introduce our main result, we need the following notations. For a prime graph G, and for a positive integer i, we denote by $W_i(G)$ the set of vertices x of $V(G)$ such that there is a subset X of $V(G)$ satisfying: $x \in X$, $|X| = i$ and $G[X]$ is prime. It is clear that $W_i(G) = V(G)$ when $i = v(G)$. Given a prime graph G, denote $cp(G) = 8$ if the order of G is odd and $cp(G) = 7$ otherwise. For a proper subset X of $V(G)$ and for each integer j, with $1 \leq j \leq |V(G) \setminus X|$, we denote by $I_G^X(j)$ the family of the subsets Y of $V(G) \setminus X$ such that $|Y| = j$ and $G[X \cup Y]$ is prime.

Our main result is:

THEOREM 1.3. *Let G be a prime graph with $v(G) \geq 8$, and $a \in V(G)$. The following three assertions hold.*

1. *$\mathscr{N}(G) = \emptyset$ if and only if $W_5(G) = \emptyset$.*
2. *$\mathscr{N}(G) = \{a\}$ if and only if $W_{cp(G)}(G) = V(G) \setminus \{a\}$.*
3. *$|\mathscr{N}(G)| \geq 2$ if and only if $W_{cp(G)}(G) = V(G)$.*

Notice that for the case of tournaments, a similar result was obtained by Boudabbous [6].

The rest of the paper is organized as follows. In Section 2, we review relevant properties of prime graphs. Section 3 proves Theorem 1.3. In section 4, we give some consequences from the Theorem 1.3.

2. PRIME GRAPHS

In this section, we recall some properties of prime graphs that will be useful in the remainder of the present paper.

THEOREM 2.1 (Sumner [23]). *For every prime graph G with $v(G) \geq 4$, there exists $X \subseteq V(G)$ such that $G[X]$ is isomorphic to P_4.*

Before we state the following theorem which improves Theorem 2.1, we need to define the following graph. Let B_5 denote the *bull*, that is, the prime graph defined on $\{1,\ldots,5\}$ such that ij is an edge of B_5 if $\{i,j\} \in \{\{1,3\}, \{1,4\}, \{2,3\}, \{3,4\}, \{4,5\}\}$.

THEOREM 2.2 (Cournier and Ille [13]). *Given a prime graph G with $v(G) \geq 5$, then $|V(G) \setminus W_4(G)| \leq 1$ and $V(G) \setminus W_4(G) \subseteq W_5(G)$. Furthermore, if $V(G) \setminus W_4(G) = \{x\}$, then there is a subset X of $V(G)$ containing x and an isomorphism f from $G[X]$ onto B_5 such that $f(x) = 1$.*

THEOREM 2.3 (Schmerl and Trotter [21]). *Given a prime graph G with $v(G) \geq 7$, there exist $x \neq y \in V(G)$ such that $G[V(G) \setminus \{x,y\}]$ is prime.*

Ille [16] improves Theorem 2.3 by the following.

THEOREM 2.4 (Ille [16]). *Let G be a prime graph and let X be a subset of $V(G)$ such that $G[X]$ is prime. If $|V(G) \setminus X| \geq 6$, then there exist $x \neq y \in V(G) \setminus X$ such that $G[V(G) \setminus \{x,y\}]$ is prime.*

We close this section by the following notations and results.

Notation 2.5. Given a graph G, let X be a proper subset of $V(G)$ such that $G[X]$ is prime. We consider the following subsets of $V(G) \setminus X$:

- Ext(X) is the set of $x \in V(G) \setminus X$ such that $G[X \cup \{x\}]$ is prime;
- $\langle X \rangle$ is the set of $x \in V(G) \setminus X$ such that X is a module of $G[X \cup \{x\}]$;
- given $u \in X$, $X(u)$ is the set of $x \in V(G) \setminus X$ such that $\{u,x\}$ is a module of $G[X \cup \{x\}]$.

The family $\{$Ext(X), $\langle X \rangle\} \cup \{X(u)\}_{u \in X}$ is denoted by $p_{(G,X)}$. Furthermore, $\langle X \rangle$ is divided onto X^- and X^+ as follows.

- X^- is the set of elements x of $V(G) \setminus X$ such that for every $y \in X$, $xy \notin E(G)$.
- X^+ is the set of elements x of $V(G) \setminus X$ such that for every $y \in X$, $xy \in E(G)$.

Similarly, for each $u \in X$, $X(u)$ is divided onto $X^-(u)$ and $X^+(u)$ as follows.

- $X^-(u)$ is the set of elements x of $X(u)$ such that $ux \notin E(G)$.
- $X^+(u)$ is the set of elements x of $X(u)$ such that $ux \in E(G)$.

We then introduce the three families below :

- $q_{(G,X)} = \{Ext(X), X^-, X^+\} \cup \{X^-(u), X^+(u)\}_{u \in X}$
- $q^-_{(G,X)} = \{X^-\} \cup \{X^-(u)\}_{u \in X}$
- $q^+_{(G,X)} = \{X^+\} \cup \{X^+(u)\}_{u \in X}$

LEMMA 2.6 (Ehrenfeucht and Rozenberg [14]). *Given a graph G, let X be a proper subset of $V(G)$ such that $G[X]$ is prime. The family $p_{(G,X)}$ realizes a partition of $V(G) \setminus X$. Moreover, the following assertions are satisfied.*

1. *Given $x \neq y \in \mathrm{Ext(X)}$, if $G[X \cup \{x,y\}]$ is not prime, then $\{x,y\}$ is a module of $G[X \cup \{x,y\}]$.*

2. *Given $x \in X(u)$ and $y \in V(G) \setminus (X \cup X(u))$ where $u \in X$, if $G[X \cup \{x,y\}]$ is not prime, then $\{x,u\}$ is a module of $G[X \cup \{x,y\}]$.*

3. *Given $x \in \langle X \rangle$ and $y \in V(G) \setminus (X \cup \langle X \rangle)$, if $G[X \cup \{x,y\}]$ is not prime, then $X \cup \{y\}$ is a module of $G[X \cup \{x,y\}]$.*

The following result is a consequence of the preceding lemma.

THEOREM 2.7 (Ehrenfeucht and Rozenberg [14]). *Let X be a subset of a prime graph G such that $G[X]$ is prime. If $|V(G) \setminus X| \geq 2$, then there exist $x \neq y \in V(G) \setminus X$ such that $G[X \cup \{x,y\}]$ is prime.*

This theorem leads us to introduce the following graph. Given a graph G, consider a subset X of $V(G)$ such that $|V(G) \setminus X| \geq 2$ and $G[X]$ is prime. In [10] the *outside* graph G_X is defined on $V(G) \setminus X$ in the following manner. For any $x \neq y \in V(G) \setminus X$, $xy \in E(G_X)$ if $G[X \cup \{x,y\}]$ is prime.

LEMMA 2.8 (Breiner, Deogun and Ille [10]). *Given a graph G, consider a proper subset X of $V(G)$ such that $G[X]$ is prime, let $M \in p_{(G,X)}$ and $N \in q_{(G,X)}$ such that $N \subseteq M$. Assume that $I_G^X(1) = \emptyset$. If I is a module of G_X such that $I \subseteq N$ and if I is a module of $G[M]$, then I is a module of G.*

Consider a prime graph G and a proper subset X of $V(G)$ such that $G[X]$ is prime and $|V(G) \setminus X| \geq 4$. One may verify that some results of [10] remain valid if we replace the hypothesis "G is critical according to $G[X]$" by "$I_G^X(3) = \emptyset$ ". So we deduce the following.

LEMMA 2.9. *Let G be a prime graph. Given a subset X of $V(G)$ such that $G[X]$ is prime and $|V(G) \setminus X| \geq 4$. Assume that $I_G^X(3) = \emptyset$. Given distinct elements a,a' and b of $V(G) \setminus X$, if there is $M \in p_{(G,X)}$ such that $a,a' \in M$ and if $ab \in E(G_X)$ and $a'b \notin E(G_X)$, then either $M = \langle X \rangle$ and $X \cup \{a,b\}$ is a module of $G[X \cup \{a,a',b\}]$ or $M = X(u)$, where $u \in X$, and $\{u,a'\}$ is a module of $G[X \cup \{a,a',b\}]$.*

PROPOSITION 2.10. *Given a prime graph G, consider a subset X of $V(G)$ such that $G[X]$ is prime and $|V(G) \setminus X| \geq 4$. If $I_G^X(3) = \emptyset$, then the following assertions are satisfied.*

1. *For every connected component C of G_X, there exist distinct elements M and N of $p_{(G,X)}$ such that $G_X[C]$ is bipartite by $\{M \cap C, N \cap C\}$.*

2. *For every connected component C of G_X, $G[X \cup C]$ is prime.*

3. *The partitions $p_{(G,X)}$ and $q_{(G,X)}$ coincide.*

4. *For every $M \in q^-_{(G,X)}$, $G[M]$ is empty, and for every $M \in q^+_{(G,X)}$, $G[M]$ is complete.*

Remark 2.11. Consider a prime graph G and a proper subset X of $V(G)$ such that $G[X]$ is prime and $|V(G) \setminus X| \geq 4$. By Theorem 2.7, observe that if $I^X_G(3) = \emptyset$, then $I^X_G(1) = \emptyset$.

3. PROOF OF THEOREM 1.3

To begin, we establish some properties that will be needed in the sequel.

The proof of the following result leads to the definition of the following graph. Let C_n where $n \geq 5$ denote the *cycle*, that is, the prime graph with distinct vertices $\{1, \ldots, n\}$ such that ij is an edge of C_n if $|i - j| = 1$ or $n - 1$.

LEMMA 3.1. *Let G be a prime graph, and X be a subset of $V(G)$ such that $G[X]$ is prime, $I^X_G(3) = \emptyset$. If $|V(G) \setminus X| = 5$, then G_X is isomorphic to P_5, $|q_{(G,X)}| = 2$ and G_X is bipartite by $q_{(G,X)}$.*

Proof. Since $|V(G) \setminus X|$ is odd, there exists a connected component C of G_X such that $|C|$ is odd. By Assertion 2 of Proposition 2.10, $G[X \cup C]$ is prime. Then, $C = V(G) \setminus X$ and G_X is connected because $|V(G) \setminus X| = 5$ and $I^X_G(3) = \emptyset$. Prove that G_X is prime. Suppose to the contrary that G_X is not prime. Let I be a non trivial module of G_X. By Proposition 2.10, there exist distinct elements M and N of $p_{(G,X)}$ such that G_X is bipartite by $\{M, N\}$. Since G_X is connected, we have either $I \subseteq M$ or $I \subseteq N$. Without loss of generality, we may assume that $I \subseteq M$. By Assertions 3 and 4 of Proposition 2.10, M, $N \in q_{(G,X)}$ and I is a module of $G[M]$. Lemma 2.8 implies that I is a non trivial module of G; which contradicts the fact that G is prime. Consequently, G_X is prime. Up to isomorphic, notice that C_5, P_5, B_5 and $\overline{P}_5$ are the only 5-vertex prime graphs. Therefore, G_X is isomorphic to P_5 because G_X is bipartite graph. □

PROPOSITION 3.2. *Let G be a prime graph with $v(G) \geq 7$. Then, both assertions below hold.*

1. *If $v(G)$ is odd, then $W_7(G) = V(G)$.*

2. *If $v(G)$ is even, then $W_8(G) = V(G)$.*

Proof. Let G be a prime graph with $v(G) \geq 7$.

For the first assertion, the result is obvious when $v(G) = 7$. Hence, assume that $v(G) \geq 9$. Suppose to the contrary that there exists $x \in V(G)$ such that $x \notin W_7(G)$. By Theorem 2.7, $x \notin W_5(G)$. It follows from Theorem 2.2 that there exist a subset

X of $V(G)$ containing x such that $G[X]$ is isomorphic to P_4. By interchanging G and its complement $\overline{G}$, we only need to consider that $|N_{G[X]}(x)| = 1$. Without loss of generality, we may assume that $G[X]$ defined on $\{1,\ldots,4\}$ such that for $i, j \in \{1,\ldots.,4\}$, ij is an edge of $G[X]$ if $|i-j| = 1$, and assume that $x = 1$. We apply Theorem 2.4 several times, we obtain a subset Z of $V(G)$ such that $|Z| = 9$, $X \subset Z$ and $G[Z]$ is prime. Let $G' = G[Z]$. Since $1 \notin W_7(G')$, we have $I^X_{G'}(3) = \emptyset$. By Lemma 3.1, $G'_X \simeq P_5$ and there exist distinct elements M and N of $q_{(G,X)} \setminus \{\mathrm{Ext(X)}\}$ such that G'_X is bipartite by $\{M \cap (Z\setminus X), N \cap (Z\setminus X)\}$. Since $\mathrm{Ext(X)} \cap (\mathrm{Z} \setminus \mathrm{X}) = \emptyset$ and for every u, $v \in \langle X\rangle \cap (Z\setminus X)$, $G'[X \cup \{u,v\}]$ is not prime, there exists $u \in X$ such that $X(u) \cap (Z\setminus X) \neq \emptyset$.

First, assume that $u \neq 1$. Since G'_X is bipartite by $\{M\cap(Z\setminus X), N\cap(Z\setminus X)\}$ and $G'_X \simeq P_5$, there exist $a \in M \cap (Z\setminus X)$ and b, $c \in N \cap (Z\setminus X)$ such that $ab \in E(G'_X)$ and $ac \notin E(G'_X)$. Without loss of generality, we may assume that $M = X(u)$. Therefore, either $N = \langle X\rangle$ or $N = X(v)$, where $v \in X\setminus\{u\}$. Set $H = (X\setminus\{u\}) \cup \{a\}$ so that $G'[H] \simeq P_4$. Clearly, $I^H_{G'}(3) = \emptyset$. Thus, it rfollows from Lemma 3.1 that $|q_{(G',H)}| = 2$. We distinguish the following two cases depending on N. At the beginning, assume that $N = \langle X\rangle$. Clearly, $u \in H(a)$. Moreover, we have by Lemma 2.9 that $c \sim X \cup \{a,b\}$, and in particular, $c \in \langle H\rangle$. We also have by Lemma 2.6 that $b \nsim X \cup \{a\}$ and $b \notin \langle H\rangle$. Show that $b \notin H(a)$. Suppose to the contrary that $b \in H(a)$. Thus, $\{b,u\}$ is a module of $G'[X \cup \{b\}]$ and $b \in X(u)$, which contradicts the fact that $b \in \langle X\rangle$. Consequently, $u \in H(a)$, $c \in \langle H\rangle$, $b \notin \langle H\rangle \cup H(a)$ and $|q_{(G',H)}| \geq 3$, which contradicts the fact that $|q_{(G',H)}| = 2$. Now, assume that $N = X(v)$, where $v \in X\setminus\{u\}$. We prove as previous that $u \in H(a)$, $c \in H(v)$, $b \notin H(a) \cup H(v)$ and $|q_{(G',H)}| \geq 3$, which contradicts the fact that $|q_{(G',H)}| = 2$.

Second, assume that $u = 1$. Thus, G'_X is bipartite by either $\{X(1)\cap(Z\setminus X), X(v)\cap(Z\setminus X)\}$, where $v \in X\setminus\{1\}$ or $\{X(1)\cap(Z\setminus X), \langle X\rangle\cap(Z\setminus X)\}$. In the first case, we return to the first step. Thus, we may assume that G'_X is bipartite by $\{X(1)\cap(Z\setminus X), \langle X\rangle\cap(Z\setminus X)\}$. We distinguish the following three cases. To start, assume that G'_X is bipartite by $\{X^-(1)\cap(Z\setminus X), X^+\cap(Z\setminus X)\}$. Since $G'_X \simeq P_5$, there exist $a \in X^+ \cap (Z\setminus X)$ and $b \in X^-(1) \cap (Z\setminus X)$ such that $ab \in E(G'_X)$. Thus, there is an isomorphism f from $G'[\{1,2,4,a,b\}]$ onto B_5 defined by $f(1) = 1$, $f(2) = 4$, $f(4) = 2$, $f(a) = 3$ and $f(b) = 5$; which is impossible because $1 \notin W_5(G')$. Now, assume that G'_X is bipartite by $\{X^+(1)\cap(Z\setminus X), X^-\cap(Z\setminus X)\}$. As previous, there is $a \in X^- \cap (Z\setminus X)$ and $b \in X^+(1) \cap (Z\setminus X)$ such that $ab \in E(G'_X)$, and then there is an isomorphism f from $G'[\{1,2,3,a,b\}]$ onto B_5 defined by $f(1) = 1$, $f(2) = 3$, $f(3) = 2$, $f(a) = 5$ and $f(b) = 4$, which is impossible because $1 \notin W_5(G')$. Finally, assume that G'_X is bipartite by either $\{X^+\cap(Z\setminus X), X^+(1)\cap(Z\setminus X)\}$ or $\{X^-\cap(Z\setminus X), X^-(1)\cap(Z\setminus X)\}$. Let $V(G'_X) = \{5,6,7,8,9\}$ such that for i, $j \in \{5,\ldots.,9\}$, $ij \in E(G'_X)$ if $|i-j| = 1$, and assume that $5,7,9 \in M$ and $6,8 \in N$. If $M = X^+$ and $N = X^+(1)$ (resp. $M = X^+(1)$ and $N = X^+$), then by using Assertion 4 of Proposition 2.10 and Lemma 2.6, it is easy to prove that $G'[\{1,3,5,6,7,8,9\}]$ is prime, which contradicts the fact that $1 \notin W_7(G')$. If $M = X^-$ and $N = X^-(1)$ (resp. $M = X^-(1)$ and $N = X^-$), then by using Assertion 4 of Proposition 2.10 and Lemma 2.6, it is easy to prove that $G'[\{1,2,5,6,7,8,9\}]$ is prime, which contradicts the fact that $1 \notin W_7(G')$.

For the second assertion, the proof is analogous to that of the first assertion. $\square$

Proof of THEOREM 1.3. Let G be a prime graph with $v(G) \geq 8$ and consider $a \in V(G)$.

For the first assertion, assume that $\mathscr{N}(G) \neq \emptyset$. There exists $v \in V(G)$ such that $G-v$ is prime. Set $Y = V(G)$ if $v(G)$ is odd and $Y = V(G) \setminus \{v\}$ otherwise. We have $Y \subseteq V(G)$ such that $|Y| \geq 7$, $|Y|$ is odd and $G[Y]$ is prime. By applying Theorem 2.3 several times, a subset Z of $V(G)$ is obtained such that $Z \subseteq Y$, $|Z| = 5$ and $G[Z]$ is prime. Thus, $W_5(G) \neq \emptyset$.
Conversely, assume that $W_5(G) \neq \emptyset$, and consider a subset X of $V(G)$ such that $|X| = 5$ and $G[X]$ is prime. If $v(G)$ is even, then by applying Theorem 2.7 several times, we obtain a vertex x such that $G-x$ is prime. If $v(G)$ is odd, then by Theorem 2.1, there is a subset Y of $V(G)$ such that $G[Y] \simeq P_4$. By applying Theorem 2.7 several times, there is a vertex x such that $G-x$ is prime. Consequently, G is not critical.

For the second assertion, assume that $\mathscr{N}(G) = \{a\}$. To begin, we prove that $a \notin W_{cp(G)}(G)$. To the contrary, suppose that $a \in W_{cp(G)}(G)$ and consider a subset X of $V(G)$ such that $a \in X$, $|X| = cp(G)$ and $G[X]$ is prime. By using Theorem 2.7 serval times, we obtain a vertex x such that $x \neq a$ and $x \in \mathscr{N}(G)$, which contradicts our assumption. Now, we prove that for each $x \in V(G) \setminus \{a\}$, $x \in W_{cp(G)}(G)$. Let $x \in V(G) \setminus \{a\}$. Since $G-a$ is prime and $|V(G) \setminus \{a\}| \geq 7$, Proposition 3.2 implies that there exists a subset X of $V(G) \setminus \{a\}$ such that $x \in X$, $|X| = cp(G)$ and $G[X]$ is prime. Thus, $x \in W_{cp(G)}(G)$.
Conversely, to the contrary, suppose that $W_{cp(G)}(G) = V(G) \setminus \{a\}$ and $\mathscr{N}(G) \neq \{a\}$. First, prove that $\mathscr{N}(G) \neq \emptyset$. By Theorem 1.1, we can assume that $v(G)$ is even. Since $W_7(G) \neq \emptyset$, there is a subset Y of $V(G) \setminus \{a\}$ such that $|Y| = 7$ and $G[Y]$ is prime. Using Theorem 2.3, we obtain a subset Z such that $Z \subset Y$, $|Z| = 5$ and $G[Z]$ is prime. By Assertion 1 of Theorem 1.3, $\mathscr{N}(G) \neq \emptyset$. Moreover, since $\mathscr{N}(G) \neq \{a\}$, there is $x \in V(G) \setminus \{a\}$ such that $G-x$ is prime. Since $|V(G) \setminus \{x\}| \geq 7$, Proposition 3.2 implies that there exists a subset X of $V(G) \setminus \{x\}$ such that $a \in X$, $|X| = cp(G)$ and $G[X]$ is prime, which contradicts our assumption.

For the last assertion, assume that $|\mathscr{N}(G)| \geq 2$. Then, there exist $x \neq y \in V(G)$ such that $G-x$ and $G-y$ are prime. By applying Proposition 3.2 to $G-x$ and to $G-y$, we obtain that for each $z \in V(G)$, there exists a subset X of $V(G)$ such that $z \in X$, $G[X]$ is prime and $|X| = cp(G)$. Thus, $W_{cp(G)}(G) = V(G)$.
Conversely, assume that $W_{cp(G)}(G) = V(G)$. The Assertion 2 of Theorem 1.3 implies that $|\mathscr{N}(G)| \neq 1$. To verify that $\mathscr{N}(G) \neq \emptyset$, we proceed as in the proof of the

Assertion 2. Consequently, $|\mathscr{N}(G)| \geq 2$. $\square$

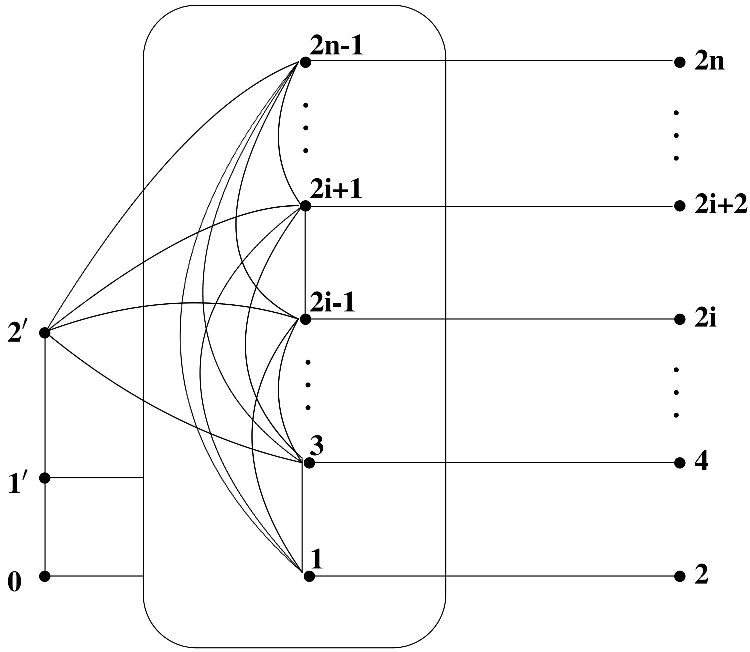

Figure 2: Φ_{2n+3}

4. SOME CONSEQUENCES

From what proceeds, we deduce the followings results.

As consequence of Theorem 2.3 and Assertion 1 of Theorem 1.3, we obtain:

COROLLARY 4.1. *Let G be a prime graph with $v(G) \geq 8$.*
$\mathscr{N}(G) = \emptyset$ if and only if $W_i(G) = \emptyset$, for each odd integer i with $5 \leq i \leq v(G)$.

From Theorems 2.1, 2.7 and Assertion 1 of Theorem 1.3, we may directly obtain:

COROLLARY 4.2. *Let G be a prime graph with $v(G) \geq 8$. If G is not critical, then for each integer m with $4 \leq m \leq v(G)$, there exists a subset X of $V(G)$ such that $|X| = m$ and $G[X]$ is prime.*

The following corollary is a consequence of Assertion 2 of Theorem 1.3 and Proposition 3.2.

COROLLARY 4.3. *Let G be a prime graph with $v(G) \geq 8$, and $a \in V(G)$. $\mathcal{N}(G) = \{a\}$ if and only if $W_{cp(G)+2i}(G) = V(G) \setminus \{a\}$, for each integer $i \geq 0$, with $cp(G)+2i \leq v(G)$.*

Proof. First, prove that $W_{cp(G)}(G) = W_{cp(G)+2i}(G)$, for each integer $i \geq 0$, with $cp(G)+2i \leq v(G)$. By Theorem 2.7, it suffices to prove that $W_{cp(G)+2i}(G) \subseteq W_{cp(G)}(G)$. Let $x \in W_{cp(G)+2i}(G)$, and consider a subset X of $V(G)$ such that $x \in X$, $|X| = cp(G)+2i$ and $G[X]$ is prime. Set $H = G[X]$. Applying Proposition 3.2 to H, we have $W_{cp(G)}(H) = V(H)$. Thus, $x \in W_{cp(G)}(G)$ and $W_{cp(G)}(G) = W_{cp(G)+2i}(G)$. To conclude, it is enough to use Assertion 2 of Theorem 1.3. $\square$

Using Theorems 1.3 and 2.7 and Proposition 3.2, we obtain:

COROLLARY 4.4. *Let G be a prime graph with $v(G) \geq 8$. $|\mathcal{N}(G)| \geq 2$ if and only if $W_m(G) = V(G)$ for each integer m with $7 \leq m \leq v(G)$.*

Proof. Let G be a prime graph with $v(G) \geq 8$ and m be an integer such that $7 \leq m \leq v(G)$, and assume that $|\mathcal{N}(G)| \geq 2$. First, assume that m is even. If $v(G)$ is even (resp. odd), then by Proposition 3.2 and Theorem 2.7 (resp. Assertion 3 of Theorem 1.3 and Theorem 2.7), we have $W_m(G) = V(G)$. Second, assume that m is odd. If $v(G)$ is even (resp. odd), then by Assertion 3 of Theorem 1.3 and Theorem 2.7 (resp. Proposition 3.2 and Theorem 2.7), we have $W_m(G) = V(G)$.
Conversely, assume that $W_m(G) = V(G)$ for each integer m with $7 \leq m \leq v(G)$. In particular, we have $W_{cp(G)}(G) = V(G)$. By Assertion 3 of Theorem 1.3, $|\mathcal{N}(G)| \geq 2$. $\square$

The following remark proves the optimality of some values in Theorem 1.3 and Corollary 4.4.

Remark 4.5. Let G be a prime graph with $v(G) \geq 8$.

1. The value $cp(G)$ in the second assertion of Theorem 1.3 is the smallest possible.

2. The value $m = 7$ in the Corollary 4.4 is the smallest possible.

We establish this in what follows. To begin, we introduce the following graphs.

(a) For each $n \geq 3$, the graph Φ_{2n+3} defined on $\{0, \ldots, 2n\} \cup \{1', 2'\}$, as follows (see Figure 2).
 • For any $i \neq j \in \{0, \ldots, 2n\}$, ij is an edge of Φ_{2n+3} if either i and j are odd or $\{i, j\} = \{2k+1, 2k+2\}$ for some $k \in \{0, \ldots, n-1\}$ or $\{i, j\} = \{0, 2k+1\}$ for some $k \in \{0, \ldots, n-1\}$.
 • $N_{\Phi_{2n+3}}(1') = \{2i+1; i \in \{0, \ldots, n-1\}\} \cup \{0, 2'\}$.
 • $N_{\Phi_{2n+3}}(2') = \{2i+1; i \in \{1, \ldots, n-1\}\} \cup \{1'\}$.

(b) For each $n \geq 3$, the graph Λ_{2n+4} defined on $\{0,\ldots,2n\} \cup \{1',2',3'\}$, as follows (see Figure 3).
• $X = \{0,1',2',3'\}$.
• For each $i,j \in \{1,\ldots,2n\}$, ij is an edge of Λ_{2n+4} if $|i-j| = 1$.
• For each $i,j \in X$, ij is an edge of Λ_{2n+4} if $\{i,j\} \in \{\{0,1'\},\{1',2'\},\{2',3'\}\}$.
• For $(i,j) \in X \times \{1,\ldots,2n\}$, ij is an edge of Λ_{2n+4} if $\{i,j\} \in \{\{1',2i\}; i \in \{1,\ldots,n\}\}$.

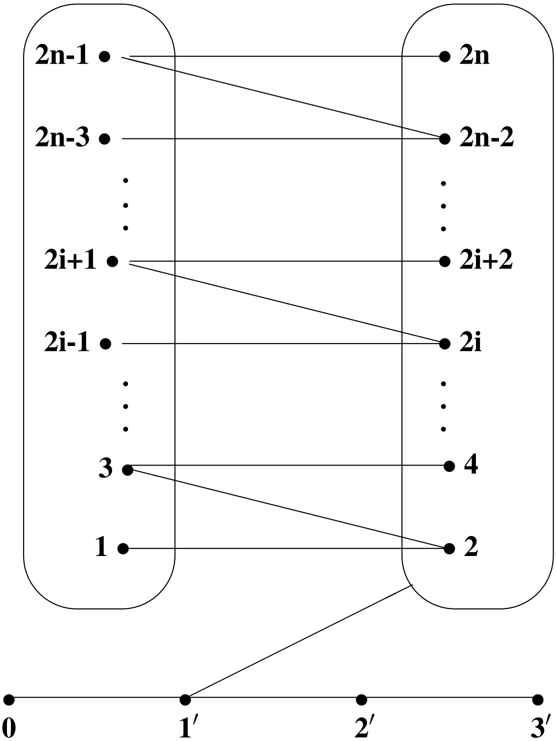

Figure 3: Λ_{2n+4}

Proof of Remark 4.5. For the first assertion, it is easy to verify that for each $n \geq 3$, Φ_{2n+3} and Λ_{2n+4} are prime, $|\mathscr{N}(\Phi_{2n+3})| \geq 2$, $|\mathscr{N}(\Lambda_{2n+4})| \geq 2$, $W_6(\Phi_{2n+3}) = V(\Phi_{2n+3}) \setminus \{0\}$, $W_5(\Lambda_{2n+4}) = V(\Lambda_{2n+4}) \setminus \{0\}$. Then, we cannot replace the value $cp(G)$ by $cp(G)-2$ in the second assertion of Theorem 1.3. Moreover, since $W_4 = V(\Phi_{2n+3}) \setminus \{0\}$, we cannot replace the value $cp(G)$ by 4 if $v(G)$ is odd in the second assertion of Theorem 1.3. Let G be a prime graph with $v(G) \geq 8$. Theorem 1.3 says that if $\mathscr{N}(G) = \{a\}$, then $a \notin W_{cp(G)}(G)$. It follows from Theorems 1.3, 2.2 and 2.7 that if $\mathscr{N}(G) = \{a\}$, then $a \in W_{cp(G)-1}(G) \cap W_{cp(G)-3}(G)$. So, we cannot replace the value $cp(G)$ by $cp(G)-1$ or by $cp(G)-3$ in the second assertion of Theorem 1.3. Consequently, the value $cp(G)$ in the second assertion of Theorem 1.3 is the smallest possible.

For the second assertion, since for each $n \geq 3$, Φ_{2n+3} and Λ_{2n+4} are prime, $W_6(\Phi_{2n+3}) = V(\Phi_{2n+3}) \setminus \{0\}$, $W_5(\Lambda_{2n+4}) = V(\Lambda_{2n+4}) \setminus \{0\}$, $W_4 = V(\Phi_{2n+3}) \setminus \{0\}$, $|\mathscr{N}(\Lambda_{2n+4})| \geq 2$ and $|\mathscr{N}(\Phi_{2n+3})| \geq 2$, then the value $m = 7$ in the Corollary 4.4 is the smallest possible. $\square$

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