GENERALIZED Q-GAUSSIAN VON NEUMANN ALGEBRAS WITH COEFFICIENTS, II. ABSENCE OF CENTRAL SEQUENCES

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Communicated by Dan Timotin

We show that the generalized q-gaussian von Neumann algebras with coefficients $\Gamma_q(B, S \otimes H)$ with B a finite dimensional factor, $\dim(D_k(S))$ sub-exponential and the dimension of H finite and larger than a constant depending on q, have no non-trivial central sequences.

AMS 2010 Subject Classification: 46L10.

Key words: von Neumann algebras, type II_1 factors, property Gamma.

1. INTRODUCTION.

In this short note, which is a sequel to [5], we investigate the lack of non-trivial central sequences in the generalized q-gaussian von Neumann algebras with coefficients introduced in [5]. Specifically, we prove that the von Neumann algebras $M = \Gamma_q(B, S \otimes H)$ are factors without the property Γ of Murray and von Neumann when B is a finite dimensional factor, the dimensions (over $\mathbb{C}$) of the spaces $D_k(S)$ (see Def. 3.18 in [5]) are sub-exponential and the dimension of H is finite and larger than a constant depending on q. A type II_1 factor (M, τ) has property Γ, according to Murray and von Neumann, if there exists a sequence (u_n) of unitaries in M such that $\|xu_n - u_n x\|_2 \to 0$ for all $x \in M$ and $\tau(u_n) = 0$ for all n (see [9]). Murray and von Neumann used this property to distinguish between the hyperfinite factor R and $L(\mathbb{F}_2)$. Central sequences in type II_1 factors were further studied by Dixmier ([3]) and Lance ([4]). In the 70's, property Γ played an important role in the work of McDuff ([8]) and Connes ([2]) regarding the classification of injective factors. The absence of central sequences in the context of q-gaussian von Neumann algebras was investigated by Sniady ([14], see also [6, 7, 13] for the factoriality of these algebras).

In [5] we introduced a new class of von Neumann algebras, the so-called generalized q-gaussian von Neumann algebras with coefficients $\Gamma_q(B, S \otimes H)$ associated to a sequence of symmetric independent copies (π_j, B, A, D), and we proved that under certain assumptions they display a powerful structural property, namely strong solidity relative to B. We continue our investigation of the generalized q-gaussians by proving that, under the same assumptions, they do not possess the property Γ if B is finite dimensional and the dimension of H is finite and exceeds a constant depending on q.

2. THE MAIN THEOREM.

Throughout this section we use the notations and results from Section 3 of [5].

THEOREM 2.1. *Let (π_j, B, A, D) a sequence of symmetric independent copies with B amenable, $1 \in S = S^* \subset A$ and assume that there exist constants $C, d > 0$ such that $dim_B(D_k(S)) \leq Cd^k$, for all $k \geq 0$. Let H be a Hilbert space with $2 \leq dim(H) < \infty$ and $M = \Gamma_q(B, S \otimes H)$. Assume that M is a factor. For $k \geq 0$, denote by $P_{\leq k}$ the orthogonal projection of $L^2(M)$ onto $\bigoplus_{s \leq k} L^2_s(M)$. Let $(x_n) \in M' \cap M^\omega$. Then for every $\delta > 0$, there exists a $k \geq 0$ such that*

$$\lim_{n \to \omega} \|x_n - P_{\leq k}(x_n)\|_2 \leq \delta.$$

If moreover B is finite dimensional, then $M' \cap M^\omega = \mathbb{C}$, i.e. M does not have the property Γ.

Proof. We use the spectral gap principle of Popa (see [11, 12]). Let $\tilde{M} = \Gamma_q(B, S \otimes (H \oplus H))$ and for every $m \geq 1$ let $\mathcal{F}_m \subset L^2(\tilde{M})$ be the $M - M$ bimodule introduced in [5], Sections 6 and 7. Namely, $\mathcal{F}_m$ is the closed linear span of reduced Wick words $W_\sigma(x_1, \ldots, x_s, h_1, \ldots, h_t) \in \tilde{M}$ such that $h_i \in H \oplus \{0\} \cup \{0\} \oplus H$ and at least m of them are in $\{0\} \oplus H$. Also let (α_t) be the 1-parameter group of *-automorphisms of $\tilde{M}$ introduced in [5], Thm. 3.16. Let's note the following transversality property, due to Avsec (see [1], Prop. 5.1).

LEMMA 2.2. *There exists a constant $C_m > 0$ such that for $0 < t < 2^{-m-1}$ we have*

$$\|\alpha_{t^{m+1}}(\xi) - \xi\|_2 \leq C_m \|P_{\mathcal{F}_m} \alpha_t(\xi)\|_2 \quad \textit{for all} \quad \xi \in \bigoplus_{k \geq m+1} L^2_k(M) \subset L^2(\tilde{M}).$$

As noted in Section 6 of [5], since B is amenable, there exists an $m \geq 1$ such that $\mathcal{F}_m$ is weakly contained into the coarse bimodule $L^2(M) \otimes L^2(M)$. Fix such an m. Since M is a non-amenable factor, it follows that $L^2(M)$ is not weakly contained in $\mathcal{F}_m$. This means that for every $\delta > 0$ there exist a finite set $F \subset \mathcal{U}(M)$ and an $\varepsilon > 0$ such that if $\xi \in \mathcal{F}_m$ satisfies $\|u\xi - \xi u\|_2 \leq \varepsilon$ for all $u \in F$, then $\|\xi\|_2 \leq \delta$. Fix such a δ, set $\delta' = \frac{\delta}{2C_m+1}$ and take ε and F corresponding to δ'. Take $(x_n) \in M' \cap M^\omega$. There's no loss of generality in assuming that $\|x_n\|_\infty \leq 1$ for all n. Fix $0 < t < 2^{-m-1}$ such that $\|\alpha_t(u) - u\|_2 \leq \frac{\varepsilon}{4}$ for all $u \in F$ and $\|\alpha_t(\xi) - \xi\|_2 \leq \delta'$ for all $\xi \in \bigoplus_{k\leq m} L^2_k(M)$ with $\|\xi\|_2 \leq 1$. For all $n \in \mathbb{N}$ we have

$$\begin{aligned}
&\|u\alpha_t(x_n) - \alpha_t(x_n)u\|_2 = \|[\alpha_t(x_n), u]\|_2 = \|[x_n, \alpha_{-t}(u)]\|_2 \\
&\leq \|[x_n, \alpha_{-t}(u) - u]\|_2 + \|[x_n, u]\|_2 \\
&\leq 2\|\alpha_{-t}(u) - u\|_2 + \|[x_n, u]\|_2 \leq \frac{\varepsilon}{2} + \|[x_n, u]\|_2.
\end{aligned}$$

Since $(x_n) \in M' \cap M^\omega$ we see that for n large enough and for all $u \in F$ we have

$$\|\alpha_t(x_n)u - u\alpha_t(x_n)\|_2 \leq \varepsilon,$$

which further implies $\|P_{\mathcal{F}_m}\alpha_t(x_n)\|_2 \leq \delta'$. Write $x_n = x'_n + x''_n$, where

$$x'_n \in \bigoplus_{k\leq m} L^2_k(M) \text{ and } x''_n \in \bigoplus_{k\geq m+1} L^2_k(M).$$

Note that $\|x'_n\|_2 \leq 1$, $\|x''_n\|_2 \leq 1$. Due to our choice of t we see that, for n large enough,

$$\|P_{\mathcal{F}_m}\alpha_t(x'_n)\|_2 \leq \|P_{\mathcal{F}_m}(\alpha_t(x'_n) - x'_n)\|_2 + \|P_{\mathcal{F}_m}(x'_n)\|_2 \leq \delta'.$$

Using Avsec's transversality property, this further implies, for n large enough,

$$\delta' \geq \|P_{\mathcal{F}_m}\alpha_t(x_n)\|_2 \geq \|P_{\mathcal{F}_m}\alpha_t(x''_n)\|_2 - \|P_{\mathcal{F}_m}\alpha_t(x'_n)\|_2 \geq \|P_{\mathcal{F}_m}\alpha_t(x''_n)\|_2 - \delta',$$

hence

$$2\delta' C_m \geq C_m\|P_{\mathcal{F}_m}\alpha_t(x''_n)\|_2 \geq \|\alpha_{t^{m+1}}(x''_n) - x''_n\|_2.$$

Thus, for $0 < s < t, t^{m+1}$ and n large enough we have

$$\|\alpha_s(x_n) - x_n\|_2 \leq \|\alpha_s(x'_n) - x'_n\|_2 + \|\alpha_s(x''_n) - x''_n\|_2 \leq (2C_m + 1)\delta'.$$

Using [5], Thm. 3.16, we see that there exists a $k = k(s, \delta)$ such that, for n large enough,

$$\|x_n - P_{\leq k}(x_n)\|_2 \leq (2C_m + 1)\delta' = \delta.$$

Taking the limit with respect to $n \to \omega$ establishes the first statement. For the moreover part, assume first that $B = \mathbb{C}$. Let's make the following general remark. Suppose (M, τ) is a type II_1 factor, ω a free ultrafilter on $\mathbb{N}$ and consider $M \subset M^\omega$ embedded in the canonical way, i.e. as constant sequences. Let $(x_n) \in M' \cap M^\omega$ such that $\tau(x_n) = 0$ for all n. Then for every $a \in M$ we have $\lim_{n\to\omega} \tau(ax_n) = 0$, i.e. $x_n \to 0$ ultraweakly as $n \to \omega$. To prove this, let $E_M : M^\omega \to M$ be the trace-preserving conditional expectation. Then

$E_M((x_n)) \in \mathbb{C}$. Indeed, since (x_n) is a central sequence, for every $a \in M$ we have

$$aE_M((x_n)) = E_M(a(x_n)) = E_M((x_n)a) = E_M((x_n))a.$$

Thus $E_M((x_n))$ is in the center of M, so there exists a scalar λ such that $E_M((x_n)) = \lambda 1$. Then $\lambda = \tau(E_M((x_n))) = \tau_\omega((x_n)) = \lim \tau(x_n) = 0$. Hence $E_M((x_n)) = 0$ and for every $a, b \in M$ we have

$$\lim \tau(ax_n b) = \tau_\omega((ax_n b)) = \tau(E_M((ax_n b))) = \tau(aE_M((x_n))b) = 0,$$

which proves the claim. Assume now that $(x_n) \in M' \cap M^\omega$ such that $x_n \in \mathcal{U}(M)$ and $\tau(x_n) = 0$ for all n. Fix $0 < \varepsilon < 1$ and k such that $\lim \|x_n - P_{\leq k}(x_n)\|_2 \leq \varepsilon$, according to the first part of the proof. Since $D_s(S)$ is finitely generated over $B = \mathbb{C}$ for every s, according to Prop. 3.20 in [5], the space $\bigoplus_{s \leq k} L^2_s(M)$ is finite dimensional (over $\mathbb{C}$). Choose an orthonormal basis $\{\xi_j\}_{1 \leq j \leq N(k)}$ of $\bigoplus_{s \leq k} L^2_s(M)$, then write $P_{\leq k}(x_n) = \sum_{j=1}^{N(k)} \lambda_j(n)\xi_j$, with $\lambda_j(n) \in \mathbb{C}$. Note that $\sum_{j=1}^{N(k)} |\lambda_j(n)|^2 = 1$ for all n. For all n large enough we have

$$1 - \varepsilon \leq |\langle x_n, P_{\leq k}(x_n)\rangle| = |\langle x_n, \sum_{j=1}^{N(k)} \lambda_j(n)\xi_j\rangle| \leq \sum_{j=1}^{N(k)} |\lambda_j(n)||\langle x_n, \xi_j\rangle|$$

$$= \sum_{j=1}^{N(k)} |\lambda_j(n)||\tau(\xi_j^* x_n)| \to 0,$$

which produces a contradiction. When B is finite dimensional, the same argument applies since $\bigoplus_{s \leq k} L^2_s(M)$ is again finitely generated over $\mathbb{C}$, and this finishes the proof. □

Remark 2.3. The moreover statement in Thm. 2.1 can also be obtained as a consequence of [10] and Cor. 7.5 in [5]. Indeed, due to Cor. 7.5 in [5], the von Neumann algebras $M = \Gamma_q(B, S \otimes H)$ are strongly solid under the assumptions of Thm. 2.1, hence they are also solid. Ozawa remarked in [10], based on a result of Popa, that a non-amenable solid factor is automatically non-Γ, which reproves the second statement of Thm. 2.1.

Corollary 2.4. *Let $-1 < q < 1$ be fixed. There exists $d = d(q)$ such that the following von Neumann algebras are non-Γ factors as soon as $\infty > dim(H) \geq d$:*

1. *$\Gamma_q(H)$;*

2. *$B \bar{\otimes} \Gamma_q(H)$, for B a type II_1 non-Γ factor;*

3. $\Gamma_q(\mathbb{C}, S \otimes K)$ *associated to the symmetric copies*

$$(\pi_j, B = \mathbb{C}, A = \Gamma_{q_0}(H), D = \Gamma_q(\ell^2 \otimes H)),$$

where $-1 < q_0 < 1$, *the symmetric copies are given by* $\pi_j(s_{q_0}(h)) = s_q(e_j \otimes h)$ *(*(e_j) *an orthonormal basis of* ℓ^2*) and* K *is a finite dimensional Hilbert space (see Example 4.4.1 in [5]);*

4. $\Gamma_q(\mathbb{C}, S \otimes H)$ *associated to the symmetric copies* $(\pi_j, B_0 = \mathbb{C}, A_0 = L(\Sigma_{[0,1]}), D_0 = L(\Sigma_{[0,\infty)}))$ *and* $S = \{1, u_{(01)}\}$*; the symmetric copies are defined by* $\pi_j(a) = u_{(1j)} a u_{(1j)}, a \in A_0$*, where* $u_\sigma, \sigma \in \Sigma_{[0,\infty)}$ *are the canonical generating unitaries for* D_0 *(see Example 4.4.2 in [5]);*

5. $\Gamma_q(\mathbb{C}, S \otimes H)$ *associated to the symmetric copies* $(\pi_j, \mathbb{C}, A, D)$*, where* $D = *_{\mathbb{N}} L(\mathbb{Z})$*, the* j*-th copy of* $L(\mathbb{Z})$ *is generated by the Haar unitary* u_j*,* $A = \{u_1\}''$*, the copies* π_j *are defined by* $\pi_j(u_1) = u_j$ *and* $S = \{1, u_1, u_1^*\}$ *(see Example 4.4.3 in [5]).*

Proof. The von Neumann algebras in (3), (4) and (5) are factors due to Prop. 3.23 in [5]. The second statement is a consequence of Cor. 2.3 in [2], while the rest follow from Thm. 2.1. Let's remark that (1) has been first proved by Sniady ([14]), and that for the examples in (1) and (2) the restriction on the dimension is not necessary, due to the fact that $\Gamma_q(H)$ is a factor for $\dim(H) \geq 2$ (see [13]) and to Remark 2.3 above. $\square$

REFERENCES

[1] S. Avsec, *Gaussian-like Von Neumann algebras and noncommutative brownian motion.* Thesis (Ph.D.)-University of Illinois at Urbana-Champaign, 2012.

[2] A. Connes, *Classification of injective factors. Cases* II_1, II_∞, III_λ, $\lambda \neq 1$. Ann. of Math. **2** (1976), *1*, 73–115.

[3] J. Dixmier, *Quelques proprietes des suites centrales dans les facteurs de type* II_1. Invent. Math. **7** (1969), 211–225.

[4] J. Dixmier and E.C. Lance, *Deux nouveaux facteurs de type* II_1. Invent. Math. **7** (1969), 226–234.

[5] M. Junge and B. Udrea, *Generalized q-gaussian von Neumann algebras with coefficients, I. Relative strong solidity.* Preprint arXiv 2015.

[6] I. Krolak, *Wick product for commutation relations connected with Yang-Baxter operators and new constructions of factors.* Comm. Math. Phys., **210** (2000), *3*, 685–701.

[7] I. Krolak, *Factoriality of von Neumann algebras connected with general commutation relations–finite dimensional case.* In: *Quantum probability* pp. 277–284, Banach Center Publ., **73**, Polish Acad. Sci. Inst. Math., Warsaw, 2006.

[8] D. McDuff, *Central sequences and the hyperfinite factor.* Proc. London Math. Soc. **21** (1970), 443–461.

[9] F. J. Murray and J. von Neumann, *On rings of operators. IV.* Ann. of Math. **44** (1943), 716–808.

[10] N. Ozawa, *Solid von Neumann algebras.* Acta Math. **192** (2004), *1*, 111–117.

[11] S. Popa, *On Ozawa's property for free group factors.* Int. Math. Res. Notes **2007** (2007), *11.*

[12] S. Popa, *On the superrigidity of malleable actions with spectral gap.* J. Amer. Math. Soc. **21** (2008), 981–1000.

[13] E. Ricard, *Factoriality of q-Gaussian von Neumann algebras.* Comm. Math. Phys. **257** (2005), *3*, 659–665.

[14] P. Sniady, *Factoriality of Bożejko-Speicher von Neumann algebras.* Comm. Math. Phys. **246** (2004), *3*, 561–567.

Received August 23, 2017

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