A CLASSIFICATION OF UNIONS OF THE FREE SEMIGROUP IN TWO GENERATORS IN THE IDEAL CASE

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We prove that in the ideal case, up to isomorphism, there are only one type of semigroups which are the union of two copies of the free monogenic semigroup. Similarly, there are only five types of semigroups which are the union of three copies of the free monogenic semigroup. And there are only two types of semigroups which are the union of two copies of the free semigroup in two generators. We provide finite presentations for each of these semigroups

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INTRODUCTION AND PRELIMINARIES

There are several well-known examples of structural theorems for semigroups, which involve decomposing a semigroup into a disjoint union of subsemigroups. For example, up to isomorphism, the Rees Theorem states that every completely simple semigroup is a Rees matrix semigroup over a group G, and is thus a disjoint union of copies of G, see [10, Theorem 3.3.1]; every Clifford semigroup is a strong semilattice of groups and as such it is a disjoint union of its maximal subgroups, see [10, Theorem 4.2.1]; every commutative semigroup is a semilattice of archimedean semigroups, see [9, Theorem 2.2].

If S is a semigroup which can be decomposed into a disjoint union of subsemigroups, then it is natural to ask how the properties of the subsemigroups influence S. For example, if the subsemigroups are finitely generated, then so is S. There are several further examples in the literature where such questions are addressed: Araújo et al. [6] consider the finite presentability of semigroups which are the disjoint union of finitely presented subsemigroups; Golubov [7] showed that a semigroup which is the disjoint union of residually finite subsemigroups is residually finite; in [5] the authors proved that every semigroup which is a disjoint union of finitely many copies of $\mathbb{N}$ is finitely presented, and such a semigroup has linear growth which implies that the corresponding semigroup algebra is a PI algebra, see [3, Theorem 2.10 and Corollary 2.11]; Abughazalah gave concrete algorithms for subsemigroup problem and word problem, as in [1, Theorem 3.2 and Theorem 3.13]; further references are [8, 12]; in [4] we completely classify those semigroups which are the disjoint union of two or three copies of the free monogenic semigroup in general.

In this paper we classify those semigroups but in a special case when it has an ideal, which is one of the copies or is a disjoint union of two copies. And we classify also the semigroup which is a disjoint union of two copies of the free semigroup in two generators in which one of the two copies is an ideal.

The main theorems of this paper are the following.

THEOREM 0.1. *Let S be a semigroup. Let $\langle a\rangle$ and $\langle b\rangle$ be two copies of the free monogenic semigroup. Then S is a disjoint union of $\langle a\rangle$ and $\langle b\rangle$, where $\langle a\rangle$ is an ideal in S, if and only if S is isomorphic to the semigroup defined by the following presentation:*

$$\langle\ c, d \mid cd = dc = c^k\ \rangle \ \textit{for some } k \geq 1$$

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THEOREM 0.2. *Let S be a semigroup. Let $\langle a\rangle$, $\langle b\rangle$ and $\langle c\rangle$ be three copies of the free monogenic semigroup. Then S is a disjoint union of $\langle a\rangle$, $\langle b\rangle$ and $\langle c\rangle$, where the subsemigroup $\langle a\rangle$ or $\langle a\rangle \cup \langle b\rangle$ is an ideal in S, if and only if S is isomorphic to the semigroup defined by one of the following presentations:*

(i) $\langle\ d, f, g \mid df = d^i, fd = d^i, dg = d^j, gd = d^j, fg = d^k, gf = d^k\ \rangle$ *where* $i + j = k + 2$ *and* $i, j, k \in \mathbb{N}$;

(ii) $\langle\ d, f, g \mid df = d^i, fd = d^i, dg = d^j, gd = d^j, fg = f^k, gf = f^k\ \rangle$ *where* $i + j + k - ik = 2$ *and* $i, j, k \in \mathbb{N}$;

(iii) $\langle\ d, f, g \mid df = d^i, fd = d^i, dg = d^i, gd = d^i, fg = g^2, gf = f^2\ \rangle$ *where* $i \in \mathbb{N}$;

(iv) $\langle\ d, f, g \mid df = d^i, fd = d^i, dg = g^2, gd = d^2, fg = g^i, gf = d^i\ \rangle$ *where* $i \in \mathbb{N}$;

(v) $\langle\ d, f, g \mid df = d^i, fd = d^i, dg = g^2, gd = d^2, fg = g^i, gf = g^i\ \rangle$ *where* $i \in \mathbb{N}$;

THEOREM 0.3. *Let S be* an Equitable semigroup. *Let $\langle a_1, a_2\rangle$ and $\langle b_1, b_2\rangle$ be two copies of the free semigroup in two generators. Then S is a disjoint union of $\langle a_1, a_2\rangle$ and $\langle b_1, b_2\rangle$, where the subsemigroup $\langle b_1, b_2\rangle$ is an ideal in S, if and only if S is isomorphic to the semigroup defined by one of the following presentations:*

(i) $$\left\langle\ c_1, c_2, d_1, d_2 \;\middle|\; \begin{array}{l} c_1d_1 = d_1^2,\ d_1c_1 = d_1^2,\ c_2d_1 = d_2d_1,\ d_1c_2 = d_1d_2, \\ c_1d_2 = d_1d_2,\ d_2c_1 = d_2d_1,\ c_2d_2 = d_2^2,\ d_2c_2 = d_2^2 \end{array} \right\rangle;$$

$$\text{(ii)}\quad \begin{array}{ll}\langle\ c_1, c_2, d_1, d_2\ | & c_1d_1 = d_1^2,\ d_1c_1 = d_1^2,\ c_2d_1 = d_1^2,\ d_1c_2 = d_1^2, \\ & c_1d_2 = d_1d_2,\ d_2c_1 = d_2d_1,\ c_2d_2 = d_1d_2, d_2c_2 = d_2d_1\rangle.\end{array}$$

We prove Theorems 0.1 and 0.2 in Section 1, and Theorem 0.3 in Section 2 respectively.

Let A be a set, and let S be any semigroup. Then we denote by A^+ the *free semigroup* on A, which consists of the non-empty words over A. Any mapping $\psi : A \to S$ can be extended in a unique way to a homomorphism $\phi : A^+ \to S$, and A^+ is determined up to isomorphism by these properties. If A is a generating set for S, then the identity mapping on A induces an epimorphism $\pi : A^+ \to S$. The kernel $\ker(\pi)$ is a congruence on S; if $R \subseteq A^+ \times A^+$ generates this congruence we say that $\langle A \,|\, R\rangle$ is a presentation for S. We say that S *satisfies a relation* $(u, v) \in A^+ \times A^+$ if $\pi(u) = \pi(v)$; we write $u = v$ in this case: see [2]. Suppose we are given a set $R \subseteq A^+ \times A^+$ and two words $u, v \in A^+$. We write $u \equiv v$ if u and v are equal as elements of A^+. We say that the relation $u = v$ is a *consequence* of R if there exist words $u \equiv w_1, w_2, \ldots, w_{k-1}, w_k \equiv v$ $(k \geq 1)$ such that for each $i = 1, \ldots, k-1$ we can write $w_i \equiv \alpha_i u_i \beta_i$ and $w_{i+1} \equiv \alpha_i v_i \beta_i$ where $(u_i, v_i) \in R$ or $(v_i, u_i) \in R$. We say that $\langle A\,|\,R\rangle$ is a presentation for S if and only if S satisfies all relations from R, and every relation that S satisfies is a consequence of R: see [11, Proposition 1.4.2]. If A and R are finite, then S is finitely presented.

Let ρ be a congruence on a semigroup S, and let $\phi : S \to T$ be a homomorphism such that $\rho \subseteq \ker\phi$. Then there is a unique homomorphism $\beta : S/\rho \to T$ defined by $s/\rho \mapsto \phi(s)$ and such that $\operatorname{im}\beta = \operatorname{im}\phi$; [10, Theorem 1.5.3].

Proposition 0.4 ([11]). *Let S be the semigroup defined by the presentation $\langle A \mid R\rangle$. If T is any semigroup satisfying the relations R, then T is a homomorphic image of S.*

Let $\langle a\rangle$ be the free monogenic semigroup. Then any two non-empty subsemigroups S and T of $\langle a\rangle$ have non-empty intersection, since $a^i \in S$ and $a^j \in T$ implies $a^{ij} \in S \cap T$. Let S be a semigroup which is the disjoint union of $m \in \mathbb{N}$ copies of the free monogenic semigroup, and let $a_1, \ldots, a_m \in S$ be the generators of these copies. Suppose that S is also the disjoint union of $n \in \mathbb{N}$ copies of the free monogenic semigroup. Then there exist $b_1, \ldots, b_n \in S$ such that $\langle b_1\rangle, \ldots, \langle b_n\rangle$ are free, disjoint, and

$$S = \langle a_1\rangle \cup \cdots \cup \langle a_m\rangle = \langle b_1\rangle \cup \cdots \cup \langle b_n\rangle.$$

If $n > m$, say, then there exist i, j such that $b_i, b_j \in \langle a_k\rangle$ for some k. But then $\langle b_i\rangle \cap \langle b_j\rangle \neq \varnothing$, a contradiction. Hence a semigroup cannot be the disjoint union of m and n copies of the free monogenic semigroup when $n \neq m$.

Lemma 0.5 ([4, Lemma 1.4]). *Let S and T be semigroups which are the disjoint union of $m \in \mathbb{N}$ copies of the free monogenic semigroup, and let $A =$*

$\{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_m\}$ be the generators of these copies in S and T, respectively. Then every homomorphism $\varphi : T \to S$ such that $\varphi(a_i) = b_i$ for all i is an isomorphism.

Proof. Since φ is surjective, it follows that the function $f : \{1, \ldots, m\} \to \{1, \ldots, m\}$ defined by $\varphi(a_i) \in \langle b_{f(i)} \rangle$ is a bijection.

Suppose that there exist $x, y \in S$ such that $\varphi(x) = \varphi(y)$. Then there exist $a, b \in A$ such that $x = a^i$ and $y = b^j$ for some $i, j \in \mathbb{N}$. It follows that $\varphi(a)^i = \varphi(b)^j$, which implies that $\varphi(a), \varphi(b) \in \langle c \rangle$ for some $c \in B$. Hence $a = b$, since f is a bijection, and so $x = y$. $\square$

Since the free monogenic semigroup is anti-isomorphic to itself, it follows that a semigroup S is the disjoint union of m copies of the free monogenic semigroups if and only if any semigroup anti-isomorphic to S has this property.

LEMMA 0.6. *Let A be a set, let ρ be a congruence on A^+, let $\varphi : A \to T \cup \{0\}$ be any mapping where T is a free semigroup in two generators, and let $\psi : A^+ \to T \cup \{0\}$ be the unique homomorphism extending φ. If $\rho \subseteq \ker \psi$ and $a, b \in A$ such that $\varphi(a) \neq 0$ and $\varphi(b) \neq 0$, then $\langle a/\rho \rangle \cup \langle b/\rho \rangle$ is infinite subsemigroups of A^+/ρ.*

Proof. Since $\rho \subseteq \ker \psi$, it follows that $\overline{\psi} : A^+/\rho \to T \cup \{0\}$ defined by $\overline{\psi}(w/\rho) = \psi(w)$ is a homomorphism. Homomorphisms map elements of finite order to elements of finite order, and since $\varphi(a) \neq 0$ and $\varphi(b) \neq 0$ do not have finite order, $a/\rho \cup b/\rho$ must have infinite order in A^+/ρ. $\square$

1. TWO AND THREE COPIES OF THE FREE MONOGENIC SEMIGROUP

In this section we prove Theorems 0.1 and 0.2.

Proof of Theorem 0.1. ($\Leftarrow$) We proved in [4, Theorem1.1] that the semigroup S' which defined by the presentation

$$\langle\ c, d \mid cd = dc = c^k\ \rangle \quad \text{for some } k \geq 1,$$

is a disjoint union of two copies of the free monogenic semigroup $\langle c \rangle$ and $\langle d \rangle$. It is clear from the presentation that $\langle c \rangle \cdot S' \subseteq \langle c \rangle$ and $S' \cdot \langle c \rangle \subseteq \langle c \rangle$, which implies that $\langle c \rangle$ is an ideal in S'.

($\Rightarrow$) Let S be a semigroup which is the disjoint union of the free semigroups $\langle a \rangle$ and $\langle b \rangle$. Clearly one of the following must hold:

(a) $ab, ba \in \langle a \rangle$,

(b) $ab, ba \in \langle b \rangle$,

(c) $ab \in \langle a \rangle$ and $ba \in \langle b \rangle$,

(d) $ab \in \langle b \rangle$ and $ba \in \langle a \rangle$.

In case (b), S is isomorphic to a semigroup satisfying (a) and in case (c) and (d) neither $\langle a \rangle$ nor $\langle b \rangle$ is an ideal which is a contradiction. Hence we may assume without loss of generality that just (a) holds.

Case (a) There exist $m, n \in \mathbb{N}$ such that $ab = a^m$ and $ba = a^n$. Hence

$$a^{m+1} = a^m a = (ab)a = a(ba) = aa^n = a^{n+1}$$

and so $m = n$. So, in this case, S is a homomorphic image of the semigroup T defined by the presentation $\langle\ a, b \mid ab = ba = a^m\ \rangle$. It follows from Lemma 0.5 that S is isomorphic to T. □

Proof of Theorem 0.2. (⇐) We proved that the semigroup defined by one of the presentations in [4, Theorem1.2] is a disjoint union of three copies of the free monogenic semigroup.
Case (i): $\langle d \rangle$ is an ideal in S' which implies that $\langle a \rangle$ is an ideal in S.
Case (ii):$\langle d \rangle$ is an ideal in S' which implies that $\langle a \rangle$ is an ideal in S.
Case (iii): $\langle d \rangle$ is an ideal in S' which implies that $\langle a \rangle$ is an ideal in S.
Case (iv): $\langle d \rangle \cup \langle g \rangle$ is an ideal in S' which implies that $\langle a \rangle \cup \langle c \rangle$ is an ideal in S.
Case (v): $\langle d \rangle \cup \langle g \rangle$ is an ideal in S' which implies that $\langle a \rangle \cup \langle c \rangle$ is an ideal in S.
Cases (vi,vii,viii,ix)): There is no ideal which is a disjoint union of copies of the free monogenic semigroup in these cases.
Thus, we just have 5 types of semigroups satisfy the ideal condition.

(⇒) As we have seen in [4, Theorem 1.2] that the semigroup which is a disjoint union of three copies of the free monogenic semigroup is isomorphic to a semigroup defined by one of the presentations in that theorem . Therefore, the semigroup which is a disjoint union of three copies of the free monogenic semigroup which contains an ideal $\langle a \rangle$ or $\langle a \rangle \cup \langle b \rangle$ must be defined by one of the presentations (i,ii,iii,iv,v) in [4, Theorem 1.2]. □

2. TWO COPIES OF THE FREE SEMIGROUP IN TWO GENERATORS

Definition 2.1. An *Equitable semigroup* defines by a presentation of the following form

$$\langle a,b,c,d \mid \quad ac = w_{ac},\ ca = w_{ca},\ bc = w_{bc},\ cb = w_{cb},\ ad = w_{ad},\ da = w_{da},\ bd = w_{bd},\ db = w_{db}\ \rangle$$

where $w_{xy} \in \{a,b,c,d\}^+,\ |w_{xy}| = 2$.

Classifying all possible Equitable semigroups which are disjoint unions of two copies of the free semigroup in two generators is still quite complicated.

In this paper, our aim is to classify Equitable semigroups in the case that one of the two copies is an ideal in S.

LEMMA 2.2. *Let S be an Equitable semigroup which is the disjoint union of the free semigroup $\langle a,b\rangle$, $\langle c,d\rangle$ where $\langle c,d\rangle$ is an ideal in S. Then one of the following must hold:*

(i) $ac = c^2$ *and then* $ca = c^2,\ bc = cb = c^2,\ ad = cd,\ da = dc,\ bd = cd, db = dc.$

(ii) $ac = dc$ *and then* $ca = cd,\ bc = dc,\ cb = cd,\ ad = da = bd = db = d^2.$

Proof. As $\langle c,d\rangle$ is an ideal then $ac \in \{c^2, dc, d^2, cd\}$. But $ac \neq d^2$ and $ac \neq cd$ because if $ac = d^2$ or $ac = cd$ then $c(ac) \neq (ca)c$ in the both cases. Thus $ac \in \{c^2, dc\}$.

Now if $ac = c^2$ then by the associativity on $c(ac) = c^3$ and $(ca)c = w_{ca}c$ which implies that $w_{ca} = ca = c^2$. And $c(ad) = cw_{a,d}$, $(ca)d = c^2d$ which implies $w_{ad} = ad = cd$. Therefore $da = dc$ by the associativity on (dad). Also, $d(bc) = dw_{bc}$ and $(db)c = w_{db}c$, which means that the word w_{bc} ends with the letter c and the word w_{db} starts with the letter d. So there are two possibilities, $w_{bc} = c^2$ or dc and $w_{db} = db = d^2$ or dc and then if $w_{bc} = c^2$ that gives us $w_{db} = dc$ by associativity, and if $w_{bc} = dc$ that implies $w_{db} = d^2$ and after that we can continue to get cb and bd. So when $w_{bc} = c^2$ this means $cb = c^2$ and $bd = cd$ and when $w_{bc} = dc$ that means $cb = cd$ and $bd = d^2$. Therefore, if $ac = c^2$ then we have two types of semigroups:

(1) The semigroup with relations $ac = ca = bc = cb = c^2, ad = bd = cd, da = db = dc$;

(2) The semigroup with relations $ac = ca = c^2, ad = cb = cd, da = bc = dc, db = bd = d^2$.

The second value is when $ac = dc$ and then by the associativity on (cac) we get $ca = cd$. Then $c(ad) = cw_{a,d}$ and $(ca)d = cd^2$ which implies that $w_{a,d} = ad = d^2$ and similarly $da = d^2$ by the associativity on (dad). Also by the associativity on (cbc) we have two possibilities $bc \in \{c^2, dc\}$ and $cb \in \{c^2, cd\}$ and we get that:

(i) $cb = bc = c^2$ and,

(ii) $cb = cd,\ bc = dc$.

In case (i) we have $db = dc,\ bd = cd$ by the associativity on (dbc) and (cbd) respectively.
In case (ii) we have $bd = d^2,\ db = d^2$ by the associativity on (cbd) and (dbc) respectively.

Therefore there are two types of semigroups.

(1′) The semigroup with the relations: $ac = bc = dc, ca = cb = cd, ad = da = bd = db = d^2$;

(2′) The semigroup with the relations: $ac = db = dc, ca = bd = cd, ad = da = d^2, bc = cb = c^2$;

Notice that the semigroup in (1) is isomorphic to the semigroup in (1′) by just replacing c by d and d by c in (1), and similarly the semigroup in (2) is isomorphic to the semigroup in (2′) by, as above, replacing c by d and d by c in (2). As a result, up to isomorphism, we just have two types of semigroups with relations (1) and (2). □

Proof of Theorem 0.3. (⇐) To prove the converse implication, it suffices to show that the semigroups mentioned in Theorem 0.3 are disjoint unions of two copies of the free semigroup in two generators.

Let S be the semigroup defined by the presentation

$$\langle\, c_1, c_2, d_1, d_2 \mid c_1d_1 = d_1^2, d_1c_1 = d_1^2, c_2d_1 = d_2d_1, d_1c_2 = d_1d_2, c_1d_2 = d_1d_2, d_2c_1 = d_2d_1, c_2d_2 = d_2^2, d_2c_2 = d_2^2 \,\rangle.$$

It is clear that every word from S is a product of letters in $\{c_1, c_2\}$ or in $\{d_1, d_2\}$, and so $S = \langle c_1, c_2\rangle \cup \langle d_1, d_2\rangle$. Since there is no relation in the presentation that can be applied to a word from $\{c_1, c_2\}^+$, it follows that $\langle c_1, c_2\rangle \cap \langle d_1, d_2\rangle = \varnothing$ and $\langle c_1, c_2\rangle$ is infinite. We show that $\langle d_1, d_2\rangle$ is infinite using Lemma 0.6. Let ρ be the congruence on $\{c_1, c_2, d_1, d_2\}^+$ generated by the relations

$$c_1d_1 = d_1^2, d_1c_1 = d_1^2, c_2d_1 = d_2d_1, d_1c_2 = d_1d_2, c_1d_2 = d_1d_2, d_2c_1 = d_2d_1, c_2d_2 = d_2^2, d_2c_2 = d_2^2,$$

let $\varphi : \{c_1, c_2, d_1, d_2\} \to T$ be defined by $\varphi(c_1) = f$, $\varphi(c_2) = g$, $\varphi(d_1) = f$, $\varphi(d_2) = g$, where T is a free semigroup in two generators f and g and let $\psi : \{c_1, c_2, d_1, d_2\}^+ \to T$ be the unique homomorphism extending φ. Then

$$\psi(c_1 d_1) = \psi(c_1)\psi(d_1) = ff = f^2 = \psi(d_1^2)$$

and, similarly,

$$\begin{aligned}
\psi(d_1 c_1) &= \psi(d_1)\psi(c_1) = ff = f^2 = \psi(d_1^2);\\
\psi(c_2 d_1) &= \psi(c_2)\psi(d_1) = gf = \psi(d_2 d_1);\\
\psi(d_1 c_2) &= \psi(d_1)\psi(c_2) = fg = \psi(d_1 d_2);\\
\psi(c_1 d_2) &= \psi(c_1)\psi(d_2) = fg = \psi(d_1 d_2);\\
\psi(d_2 c_1) &= \psi(d_2)\psi(c_1) = gf = \psi(d_2 d_1);\\
\psi(c_2 d_2) &= \psi(c_2)\psi(d_2) = gg = g^2 = \psi(d_2^2);\\
\psi(d_2 c_2) &= \psi(d_2)\psi(c_2) = gg = g^2 = \psi(d_2^2).
\end{aligned}$$

Hence $\rho \subseteq \ker\psi$ and so $\langle d_1, d_2\rangle$ is infinite in S, by Lemma 0.6.

Let G be the semigroup defined by the presentation

$$\begin{aligned}
\langle\, c_1, c_2, d_1, d_2 \mid\ & c_1 d_1 = d_1^2,\ d_1 c_1 = d_1^2,\ c_2 d_1 = d_1^2,\ d_1 c_2 = d_1^2,\\
& c_1 d_2 = d_1 d_2,\ d_2 c_1 = d_2 d_1,\ c_2 d_2 = d_1 d_2,\ d_2 c_2 = d_2 d_1\rangle.
\end{aligned}$$

Then as above $G = \langle c_1, c_2\rangle \cup \langle d_1, d_2\rangle$. It is clear that every word from S is a product of letters in $\{c_1, c_2\}$ or in $\{d_1, d_2\}$, and so $S = \langle c_1, c_2\rangle \cup \langle d_1, d_2\rangle$. Since there is no relation in the presentation that can be applied to a word from $\{c_1, c_2\}^+$, it follows that $\langle c_1, c_2\rangle \cap \langle d_1, d_2\rangle = \varnothing$. The proof that $\langle c_1, c_2\rangle$ and $\langle d_1, d_2\rangle$ are infinite follows using a similar argument as above but where $\varphi : \{c_1, c_2, d_1, d_2\} \to T$ is defined by $\varphi(c_1) = \varphi(c_2) = \varphi(d_1) = f$ and $\varphi(d_2) = g$, where T is a free semigroup in two generators f and g. Let $\psi : \{c_1, c_2, d_1, d_2\}^+ \to T$ be the unique homomorphism extending φ. Then

$$\psi(c_1 d_1) = \psi(c_1)\psi(d_1) = ff = f^2 = \psi(d_1^2)$$

and, similarly,

$$\begin{aligned}
\psi(d_1 c_1) &= \psi(d_1)\psi(c_1) = ff = f^2 = \psi(d_1^2);\\
\psi(c_2 d_1) &= \psi(c_2)\psi(d_1) = ff = f^2 = \psi(d_1^2);\\
\psi(d_1 c_2) &= \psi(d_1)\psi(c_2) = ff = f^2 = \psi(d_1^2);\\
\psi(c_1 d_2) &= \psi(c_1)\psi(d_2) = fg = \psi(d_1 d_2);\\
\psi(d_2 c_1) &= \psi(d_2)\psi(c_1) = gf = \psi(d_2 d_1);\\
\psi(c_2 d_2) &= \psi(c_2)\psi(d_2) = fg = \psi(d_1 d_2);\\
\psi(d_2 c_2) &= \psi(d_2)\psi(c_2) = gf = \psi(d_2 d_1).
\end{aligned}$$

Hence $\rho \subseteq \ker\psi$ and so $\langle d_1, d_2\rangle$ is infinite in S, by Lemma 0.6.

($\Rightarrow$) Let S' be a semigroup defined by one of the following presentations:

(a) $\langle\ c_1, c_2, d_1, d_2 \mid \ c_1d_1 = d_1^2,\ d_1c_1 = d_1^2,\ c_2d_1 = d_2d_1,\ d_1c_2 = d_1d_2,$
$c_1d_2 = d_1d_2,\ d_2c_1 = d_2d_1,\ c_2d_2 = d_2^2,\ d_2c_2 = d_2^2\ \rangle;$

(b) $\langle\ c_1, c_2, d_1, d_2 \mid \ c_1d_1 = d_1^2,\ d_1c_1 = d_1^2,\ c_2d_1 = d_1^2,\ d_1c_2 = d_1^2,$
$c_1d_2 = d_1d_2,\ d_2c_1 = d_2d_1,\ c_2d_2 = d_1d_2, d_2c_2 = d_2d_1\rangle.$

Let S be a semigroup which is the disjoint union of the free semigroups $\langle a, b\rangle$ and $\langle c, d\rangle$. Thus one of the following must hold by Lemma 2.2:

(i) $ac = ca = bc = cb = c^2, ad = bd = cd, da = db = dc;$

(ii) $ac = ca = c^2, ad = cb = cd, da = bc = dc, db = bd = d^2.$

Clearly S with the relations in case (i) and case (ii) satisfies the relations in the presentation (a) and (b) respectively and then S is a homomorphic image of S' by Proposition 0.4. If it is a proper homomorphic image then there is without loss of generality u_1 and $u_2 \in \langle c_1, c_2\rangle$ or u and v in $\langle c_1, c_2\rangle$ and $\langle d_1, d_2\rangle$ respectively such that $u_1 = u_2$ or $u = v$ which contradicts with the fact that there is no element in the free semigroup of rank two is of finite order or contradicts with $\langle c_1, c_2\rangle \cap \langle d_1, d_2\rangle = \varnothing$. Thus $S' \cong S$. □

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