A NOTE ON GENERALIZED COHEN-MACAULAY RINGS

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Let $(R, \mathfrak{m})$ be a complete local generalized Cohen-Macaulay ring of dimension d. In this paper it is shown that $\mathrm{Hom}_R(H^d_{\mathfrak{m}}(R), H^d_{\mathfrak{m}}(R))$ is a generalized Cohen-Macaulay R-module.

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1. INTRODUCTION

Let $(R, \mathfrak{m})$ denote a commutative Noetherian local ring of dimension d and assume that M is a non-zero finitely generated R-module of dimension n. Then M is called a *generalized Cohen-Macaulay R-module* precisely when $H^i_{\mathfrak{m}}(M)$ is finitely generated for all $i \neq n$. This family of modules for first time were introduced in [7]. In this paper we will define the R-modules ω_R and Ω_R as $\omega_R = D(H^d_{\mathfrak{m}}(R))$ and $\Omega_R := D(H^d_{\mathfrak{m}}(D(H^d_{\mathfrak{m}}(R))))$, where $D(-) := \mathrm{Hom}_R(-, E)$ denotes the Matlis dual functor and $E := E_R(R/\mathfrak{m})$ is the injective hull of the residue field $R/\mathfrak{m}$. We shall see in Lemma 2.4 that there is an isomorphism of R-modules as $\Omega_R \simeq \mathrm{Hom}_R(H^d_{\mathfrak{m}}(R), H^d_{\mathfrak{m}}(R))$, when R is complete.

The main goal of this paper is to prove that if $(R, \mathfrak{m})$ is a Noetherian complete local generalized Cohen-Macaulay ring then the R-module $\Omega_R = \mathrm{Hom}_R(H^d_{\mathfrak{m}}(R), H^d_{\mathfrak{m}}(R))$ is generalized Cohen-Macaulay too.

Recall that for an R-module M, the ith local cohomology module $H^i_I(M)$ with respect to the ideal I of R is defined as

$$H^i_I(M) = \varinjlim_{n \geq 1} \mathrm{Ext}^i_R(R/I^n, M).$$

We refer the reader to [2] or [1] for more details about local cohomology.

Throughout this paper, for each R-module M, we denote by $E_R(M)$ the injective envelope (or injective hull) of M. For any Noetherian local ring $(R, \mathfrak{m})$,

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we denote the Matlis dual functor $\mathrm{Hom}_R(-, E_R(R/\mathfrak{m}))$ by $D(-)$. Also, for any ideal $\mathfrak{a}$ of R, we denote $\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq \mathfrak{a}\}$ by $V(\mathfrak{a})$. For each R-module L, we denote by $\operatorname{Assh}_R L$ the set $\{\mathfrak{p} \in \operatorname{Ass}_R L : \dim R/\mathfrak{p} = \dim L\}$. Also, for any R-module M we denote the projective dimension of M by $\mathrm{pd}_R M$. For any R-module L and any ideal I of R, the submodule $\bigcup_{n\geq 1}(0 :_L I^n)$ of L is denoted by $\Gamma_I(L)$. Finally, for any ideal $\mathfrak{b}$ of R, *the radical of* $\mathfrak{b}$, denoted by $\mathrm{Rad}(\mathfrak{b})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b}$ for some $n \in \mathbb{N}\}$. For any unexplained notation and terminology we refer the reader to [1] and [4].

2. THE RESULTS

Recall that, for any ideal I of a Noetherian ring R, we say a sequence $x_1, \dots, x_n$ of elements R is an I-filter regular sequence for a finitely generated R-module M, if $x_1, \dots, x_n \in I$ and

$$x_i \notin \mathfrak{p} \text{ for all } \mathfrak{p} \in \operatorname{Ass}_R M/(x_1, \dots, x_{i-1})M/\Gamma_I(M/(x_1, \dots, x_{i-1})M),$$

for all $i = 1, \dots, n$. The concept of an I-filter regular sequence for M is a generalization of the filter regular sequence which has been studied in references [7, 8, 9] and has led to some interesting results. Note that both concepts coincide if I is the maximal ideal of a local ring. (For some applications of the filter regular sequences see for example [5]). We start this section with the following two well known lemmas.

Lemma 2.1 (See [3, Proposition 1.2]). *Let R be a Noetherian ring and I an ideal of R. Let M be a finitely generated non-zero R-module and let $x_1, ..., x_n \in I$ $(n > 0)$ be an I-filter regular sequence on M. Then*

$$H^i_I(M) \cong \begin{cases} H^i_{(x_1,\dots,x_n)}(M) & \text{if } 0 \leq i < n \\ H^{i-n}_I\big(H^n_{(x_1,\dots,x_n)}(M)\big) & \text{if } i \geq n. \end{cases}$$

Recall that, the *arithmetic rank* of the ideal I, denoted by $\mathrm{ara}(I)$, is the least number of elements of I required to generate an ideal which has the same radical as I.

Lemma 2.2 (See [5, Proposition 2.1]). *Let I be a non-nilpotent proper ideal of the Noetherian ring R with* $\mathrm{ara}(I) = n$. *Then there exists an I-filter regular sequence $y_1, \dots, y_n$ for R such that* $\mathrm{Rad}(I) = \mathrm{Rad}(y_1, \dots, y_n)$.

Corollary 2.3. *Let $(R, \mathfrak{m})$ be a Noetherian ring of dimension $d \geq 1$. Then there exists a filter-regular sequence $x_1, ..., x_d \in \mathfrak{m}$ such that $\mathfrak{m} =$* $\mathrm{Rad}(x_1, \dots, x_d)$. *In particular, $x_1, ..., x_d$ is a system of parameters for R.*

Proof. Since $\operatorname{ara}(\mathfrak{m}) = d$, the assertion follows from Lemma 2.2. $\square$

PROPOSITION 2.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. Then the following statements hold:*

i) *There is an isomorphism of R-modules $\operatorname{Hom}_R(\omega_R, \omega_R) \simeq \Omega_R$.*

ii) *There is an isomorphism of R-modules $\operatorname{Hom}_R(H^d_{\mathfrak{m}}(R), H^d_{\mathfrak{m}}(R)) \simeq \Omega_R$, whenever R is a complete local ring with respect to $\mathfrak{m}$-adic topology.*

Proof. (i) By using [1, Exercise 6.1.9] one sees that

$$\begin{aligned}
\Omega_R &= D(H^d_{\mathfrak{m}}(D(H^d_{\mathfrak{m}}(R)))) \\
&= \operatorname{Hom}_R(H^d_{\mathfrak{m}}(D(H^d_{\mathfrak{m}}(R))), E_R(R/\mathfrak{m})) \\
&\simeq \operatorname{Hom}_R(H^d_{\mathfrak{m}}(R) \otimes_R D(H^d_{\mathfrak{m}}(R)), E_R(R/\mathfrak{m})) \\
&\simeq \operatorname{Hom}_R(D(H^d_{\mathfrak{m}}(R)), \operatorname{Hom}_R(H^d_{\mathfrak{m}}(R), E_R(R/\mathfrak{m}))) \\
&= \operatorname{Hom}_R(\omega_R, \omega_R).
\end{aligned}$$

(ii) Since by the hypothesis R is a complete local ring and in view of [1, Theorems 7.1.3] the R-module $H^d_{\mathfrak{m}}(R)$ is Artinian, it follows that $D(D(H^d_{\mathfrak{m}}(R))) \simeq H^d_{\mathfrak{m}}(R)$ and hence

$$\begin{aligned}
\Omega_R &= D(H^d_{\mathfrak{m}}(D(H^d_{\mathfrak{m}}(R)))) \\
&= \operatorname{Hom}_R(H^d_{\mathfrak{m}}(D(H^d_{\mathfrak{m}}(R))), E_R(R/\mathfrak{m})) \\
&\simeq \operatorname{Hom}_R(H^d_{\mathfrak{m}}(R) \otimes_R D(H^d_{\mathfrak{m}}(R)), E_R(R/\mathfrak{m})) \\
&\simeq \operatorname{Hom}_R(H^d_{\mathfrak{m}}(R), \operatorname{Hom}_R(D(H^d_{\mathfrak{m}}(R)), E_R(R/\mathfrak{m}))) \\
&= \operatorname{Hom}_R(H^d_{\mathfrak{m}}(R), D(D(H^d_{\mathfrak{m}}(R)))) \\
&\simeq \operatorname{Hom}_R(H^d_{\mathfrak{m}}(R), H^d_{\mathfrak{m}}(R)). \qquad \square
\end{aligned}$$

The following proposition plays a key role in this paper.

PROPOSITION 2.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$. Then for every R-module T the following statements hold:*

i) $(\operatorname{Ann} \bigoplus_{i=0}^{d-1} H^i_{\mathfrak{m}}(R))^{2d-1} \subseteq \operatorname{Ann} \bigoplus_{i=1}^{\infty} \operatorname{Ext}^i_R(T, H^d_{\mathfrak{m}}(D(H^d_{\mathfrak{m}}(R))))$,

ii) $(\operatorname{Ann} \bigoplus_{i=0}^{d-1} H^i_{\mathfrak{m}}(R))^{2d-1} \subseteq \operatorname{Ann} \bigoplus_{i=1}^{\infty} \operatorname{Tor}^R_i(T, \Omega_R)$.

Proof. (i) Let $J_i := \operatorname{Ann} H^i_{\mathfrak{m}}(R)$, for $i = 0, 1, ..., d-1$. In view of Corollary 2.3 there is a system of parameters $x_1, ..., x_d$ for R such that $x_1, ..., x_d$ is a filter-regular sequence on R. Then by the definition we have

$$\Gamma_{Rx_1}(R) = \Gamma_{\mathfrak{m}}(R) \simeq H^0_{\mathfrak{m}}(R).$$

So, by using [1, Remark 2.2.7 and Theorem 2.2.16] we get the following exact sequence
$$0 \to R/\Gamma_{\mathfrak{m}}(R) \to R_{x_1} \to H^1_{Rx_1}(R) \to 0,$$
which induces the following exact sequence
$$0 \to D(H^1_{Rx_1}(R)) \to D(R_{x_1}) \to D(R/\Gamma_{\mathfrak{m}}(R)) \to 0. \quad (2.5.1)$$

Since, the map $R_{x_1} \xrightarrow{x_1} R_{x_1}$ is an isomorphism it follows that the map
$$D(R_{x_1}) \xrightarrow{x_1} D(R_{x_1})$$
is an isomorphism. Therefore, for each $j \geq 0$, the map
$$H^j_{\mathfrak{m}}(D(R_{x_1})) \xrightarrow{x_1} H^j_{\mathfrak{m}}(D(R_{x_1}))$$
is an isomorphism. But the R-module $H^j_{\mathfrak{m}}(D(R_{x_1}))$ is $\mathfrak{m}$-torsion and hence is Rx_1-torsion. So, we get $H^j_{\mathfrak{m}}(D(R_{x_1})) = 0$, for each $j \geq 0$. Hence, by using the exact sequence (2.5.1) we achieve the isomorphisms
$$E_{R/\Gamma_{\mathfrak{m}}(R)}(R/\mathfrak{m}) \simeq D(R/\Gamma_{\mathfrak{m}}(R)) \simeq H^0_{\mathfrak{m}}(D(R/\Gamma_{\mathfrak{m}}(R))) \simeq H^1_{\mathfrak{m}}(D(H^1_{Rx_1}(R))).$$
Therefore, for each $i \geq 0$ we get the isomorphism
$$\operatorname{Ext}^i_R(T, H^1_{\mathfrak{m}}(D(H^1_{Rx_1}(R)))) \simeq \operatorname{Ext}^i_R(T, E_{R/\Gamma_{\mathfrak{m}}(R)}(R/\mathfrak{m})).$$

On the other hand, from the exact sequence
$$0 \to H^0_{\mathfrak{m}}(R) \to R \to R/\Gamma_{\mathfrak{m}}(R) \to 0,$$
we achieve the exact sequence
$$0 \to E_{R/\Gamma_{\mathfrak{m}}(R)}(R/\mathfrak{m}) \to E_R(R/\mathfrak{m}) \to D(H^0_{\mathfrak{m}}(R)) \to 0,$$
which yields the isomorphism of R-modules
$$\operatorname{Ext}^{i+1}_R(T, E_{R/\Gamma_{\mathfrak{m}}(R)}(R/\mathfrak{m})) \simeq \operatorname{Ext}^i_R(T, D(H^0_{\mathfrak{m}}(R))), \text{ for each } i \geq 1,$$
and the exact sequence
$$\operatorname{Hom}_R(T, D(H^0_{\mathfrak{m}}(R))) \to \operatorname{Ext}^1_R(T, E_{R/\Gamma_{\mathfrak{m}}(R)}(R/\mathfrak{m})) \to 0.$$
Now, it is clear that
$$J_0 \subseteq \operatorname{Ann} \bigoplus_{i=1}^{\infty} \operatorname{Ext}^i_R(T, E_{R/\Gamma_{\mathfrak{m}}(R)}(R/\mathfrak{m}))$$
and therefore one has
$$J_0 \subseteq \operatorname{Ann} \bigoplus_{i=1}^{\infty} \operatorname{Ext}^i_R(T, H^1_{\mathfrak{m}}(D(H^1_{Rx_1}(R)))).$$

Moreover, using [1, Exercise 2.1.9] and Lemma 2.1 we have
$$\Gamma_{Rx_2}(H^1_{Rx_1}(R)) \simeq H^0_{Rx_2}(H^1_{Rx_1}(R)) \simeq$$

$$H^0_{(x_1,x_2)}(H^1_{Rx_1}(R)) \simeq H^0_{\mathfrak{m}}(H^1_{Rx_1}(R)) \simeq H^1_{\mathfrak{m}}(R)$$

and hence $\Gamma_{Rx_2}(H^1_{Rx_1}(R)) = \Gamma_{\mathfrak{m}}(H^1_{Rx_1}(R)) \simeq H^1_{\mathfrak{m}}(R)$. Therefore, we obtain the exact sequence

$$0 \to H^1_{\mathfrak{m}}(R) \to H^1_{Rx_1}(R) \to H^1_{Rx_1}(R)/\Gamma_{Rx_2}(H^1_{Rx_1}(R)) \to 0,$$

which induces the exact sequence

$$0 \to D(H^1_{Rx_1}(R)/\Gamma_{Rx_2}(H^1_{Rx_1}(R))) \to D(H^1_{Rx_1}(R)) \to D(H^1_{\mathfrak{m}}(R)) \to 0. \quad (2.5.2)$$

The exact sequence (2.5.2) induces the exact sequence

$$H^0_{\mathfrak{m}}(D(H^1_{\mathfrak{m}}(R))) \overset{f_1}{\to} H^1_{\mathfrak{m}}(D(H^1_{Rx_1}(R)/\Gamma_{Rx_2}(H^1_{Rx_1}(R))))$$
$$\overset{f_2}{\to} H^1_{\mathfrak{m}}(D(H^1_{Rx_1}(R))) \overset{f_3}{\to} H^1_{\mathfrak{m}}(D(H^1_{\mathfrak{m}}(R))). \quad (2.5.3)$$

Let

$$A_1 := H^0_{\mathfrak{m}}(D(H^1_{\mathfrak{m}}(R))), \quad B_1 := H^1_{\mathfrak{m}}(D(H^1_{Rx_1}(R)/\Gamma_{Rx_2}(H^1_{Rx_1}(R)))),$$
$$C_1 := H^1_{\mathfrak{m}}(D(H^1_{Rx_1}(R))) \quad \text{and} \quad D_1 := H^1_{\mathfrak{m}}(D(H^1_{\mathfrak{m}}(R))).$$

Then, the exact sequence (2.5.3) yields the exact sequences

$$0 \to \ker(f_3) \to C_1 \to \operatorname{im}(f_3) \to 0, \quad (2.5.4)$$

and

$$0 \to \operatorname{im}(f_1) \to B_1 \to \operatorname{im}(f_2) \to 0. \quad (2.5.5)$$

Since, $J_1\operatorname{im}(f_3) = 0$, it follows that $(J_0 \cap J_1)\operatorname{im}(f_3) = 0$ and hence one sees that

$$(J_0 \cap J_1) \operatorname{Ext}^i_R(T, \operatorname{im}(f_3)) = 0, \quad \text{for each } i \geq 0.$$

Moreover, from the fact that

$$J_0 \subseteq \operatorname{Ann} \bigoplus_{i=1}^{\infty} \operatorname{Ext}^i_R(T, C_1),$$

we get

$$(J_0 \cap J_1) \subseteq \operatorname{Ann} \bigoplus_{i=1}^{\infty} \operatorname{Ext}^i_R(T, C_1).$$

The exact sequence (2.5.4) yields the long exact sequence

$$0 \to \operatorname{Hom}_R(T, \ker(f_3)) \to \operatorname{Hom}_R(T, C_1) \to \operatorname{Hom}_R(T, \operatorname{im}(f_3))$$
$$\to \operatorname{Ext}^1_R(T, \ker(f_3)) \to \operatorname{Ext}^1_R(T, C_1) \to \operatorname{Ext}^1_R(T, \operatorname{im}(f_3)) \to \cdots,$$

which implies that

$$(J_0 \cap J_1)^2 \subseteq \operatorname{Ann} \bigoplus_{i=1}^{\infty} \operatorname{Ext}^i_R(T, \ker(f_3)).$$

Hence, one has

$$(J_0 \cap J_1)^2 \subseteq \operatorname{Ann} \bigoplus_{i=1}^{\infty} \operatorname{Ext}^i_R(T, \operatorname{im}(f_2)).$$

Since $J_1 \operatorname{im}(f_1) = 0$ it is clear that $(J_0 \cap J_1)\operatorname{im}(f_1) = 0$ and hence one sees that

$$(J_0 \cap J_1) \operatorname{Ext}^i_R(T, \operatorname{im}(f_1)) = 0, \quad \text{for each } i \geq 0.$$

So, the exact sequence (2.5.5) yields the exact sequence

$$0 \to \operatorname{Hom}_R(T, \operatorname{im}(f_1)) \to \operatorname{Hom}_R(T, B_1) \to \operatorname{Hom}_R(T, \operatorname{im}(f_2))$$
$$\to \operatorname{Ext}^1_R(T, \operatorname{im}(f_1)) \to \operatorname{Ext}^1_R(T, B_1) \to \operatorname{Ext}^1_R(T, \operatorname{im}(f_2)) \to \cdots,$$

which implies that

$$(J_0 \cap J_1)^3 \subseteq \operatorname{Ann} \bigoplus_{i=1}^{\infty} \operatorname{Ext}^i_R(T, B_1) =$$

$$\operatorname{Ann} \bigoplus_{i=1}^{\infty} H^1_{\mathfrak{m}}(D(H^1_{Rx_1}(R)/\Gamma_{Rx_2}(H^1_{Rx_1}(R)))).$$

Let $\Gamma_2 := \Gamma_{Rx_2}(H^1_{Rx_1}(R))$. Then, by using [1, Remark 2.2.7 and Theorem 2.2.16] we achieve the exact sequence

$$0 \to H^1_{Rx_1}(R)/\Gamma_2 \to (H^1_{Rx_1}(R)/\Gamma_2)_{x_2} \to H^1_{Rx_2}(H^1_{Rx_1}(R)) \to 0,$$

which by using [6, Corollary 3.5], yields the isomorphic short exact sequence

$$0 \to H^1_{Rx_1}(R)/\Gamma_2 \to (H^1_{Rx_1}(R)/\Gamma_2)_{x_2} \to H^2_{(x_1,x_2)}(R) \to 0. \quad (2.5.6)$$

From the exact sequence (2.5.6), we get the exact sequence

$$0 \to D(H^2_{(x_1,x_2)}(R)) \to D((H^1_{Rx_1}(R)/\Gamma_2)_{x_2}) \to D(H^1_{Rx_1}(R)/\Gamma_2) \to 0 \quad (2.5.7).$$

By, the same argument as in the first lines of the proof, the exact sequence (2.5.7) yields the isomorphism

$$H^2_{\mathfrak{m}}(D(H^2_{(x_1,x_2)}(R))) \simeq H^1_{\mathfrak{m}}(D(H^1_{Rx_1}(R)/\Gamma_2)),$$

which implies that

$$(J_0 \cap J_1)^3 \subseteq \operatorname{Ann} \bigoplus_{i=1}^{\infty} \operatorname{Ext}^i_R(T, H^2_{\mathfrak{m}}(D(H^2_{(x_1,x_2)}(R)))).$$

Furthermore, by using [1, Exercise 2.1.9] and Lemma 2.1 we get

$$\Gamma_{Rx_3}(H^2_{(x_1,x_2)}(R)) \simeq H^0_{Rx_3}(H^2_{(x_1,x_2)}(R)) \simeq H^0_{(x_1,x_2,x_3)}(H^2_{(x_1,x_2)}(R)) \simeq H^2_{\mathfrak{m}}(R)$$

and hence $\Gamma_{Rx_3}(H^2_{(x_1,x_2)}(R)) = \Gamma_{\mathfrak{m}}(H^2_{(x_1,x_2)}(R)) \simeq H^2_{\mathfrak{m}}(R)$. Therefore, we get the exact sequence

$$0 \to H^2_{\mathfrak{m}}(R) \to H^2_{(x_1,x_2)}(R) \to H^2_{(x_1,x_2)}(R)/\Gamma_{Rx_3}(H^2_{(x_1,x_2)}(R)) \to 0,$$

which induces the exact sequence

$$0 \to D(H^2_{(x_1,x_2)}(R)/\Gamma_{Rx_3}(H^2_{(x_1,x_2)}(R))) \to D(H^2_{(x_1,x_2)}(R)) \to D(H^2_{\mathfrak{m}}(R)) \to 0. \quad (2.5.8)$$

Let $\Gamma_3 := \Gamma_{Rx_3}(H^2_{(x_1,x_2)}(R))$. Then, from the exact sequence (2.5.8) we obtain the exact sequence

$$H^1_{\mathfrak{m}}(D(H^2_{\mathfrak{m}}(R))) \xrightarrow{g_1} H^2_{\mathfrak{m}}(D(H^2_{(x_1,x_2)}(R)/\Gamma_3)) \xrightarrow{g_2} H^2_{\mathfrak{m}}(D(H^2_{(x_1,x_2)}(R))) \xrightarrow{g_3} H^2_{\mathfrak{m}}(D(H^2_{\mathfrak{m}}(R))). \quad (2.5.9)$$

Let

$$A_2 := H^1_{\mathfrak{m}}(D(H^2_{\mathfrak{m}}(R))), \quad B_2 := H^2_{\mathfrak{m}}(D(H^2_{(x_1,x_2)}(R)/\Gamma_3)),$$
$$C_2 := H^2_{\mathfrak{m}}(D(H^2_{(x_1,x_2)}(R))), \quad D_2 := H^2_{\mathfrak{m}}(D(H^2_{\mathfrak{m}}(R))).$$

Then, the exact sequence (2.5.9) yields the exact sequences

$$0 \to \ker(g_3) \to C_2 \to \operatorname{im}(g_3) \to 0, \quad (2.5.10)$$

and

$$0 \to \operatorname{im}(g_1) \to B_2 \to \operatorname{im}(g_2) \to 0. \quad (2.5.11)$$

Since $J_2\operatorname{im}(g_3) = 0$ it is clear that $(J_0 \cap J_1 \cap J_2)\operatorname{im}(g_3) = 0$ and hence

$$(J_0 \cap J_1 \cap J_2)\operatorname{Ext}^i_R(T, \operatorname{im}(g_3)) = 0, \quad \text{for each } i \geq 0.$$

Moreover, from the fact that

$$(J_0 \cap J_1)^3 \subseteq \operatorname{Ann} \bigoplus_{i=1}^{\infty} \operatorname{Ext}^i_R(T, C_2),$$

we get

$$(J_0 \cap J_1 \cap J_2)^3 \subseteq \operatorname{Ann} \bigoplus_{i=1}^{\infty} \operatorname{Ext}^i_R(T, C_2).$$

The exact sequence (2.5.10) yields the long exact sequence

$$0 \to \operatorname{Hom}_R(T, \ker(g_3)) \to \operatorname{Hom}_R(T, C_2) \to \operatorname{Hom}_R(T, \operatorname{im}(g_3)) \to \operatorname{Ext}^1_R(T, \ker(g_3)) \to \operatorname{Ext}^1_R(T, C_2) \to \operatorname{Ext}^1_R(T, \operatorname{im}(g_3)) \to \cdots,$$

which implies that

$$(J_0 \cap J_1 \cap J_2)^4 \subseteq \operatorname{Ann} \bigoplus_{i=1}^{\infty} \operatorname{Ext}^i_R(T, \ker(g_3)).$$

Hence, one has

$$(J_0 \cap J_1 \cap J_2)^4 \subseteq \operatorname{Ann} \bigoplus_{i=1}^{\infty} \operatorname{Ext}^i_R(T, \operatorname{im}(g_2)).$$

Since $J_2 \operatorname{im}(g_1) = 0$ we see that $(J_0 \cap J_1 \cap J_2)\operatorname{im}(g_1) = 0$ and hence

$$(J_0 \cap J_1 \cap J_2) \operatorname{Ext}^i_R(T, \operatorname{im}(g_1)) = 0, \quad \text{for each } i \geq 0.$$

The exact sequence (2.5.11) yields the exact sequence

$$\begin{gathered} 0 \to \operatorname{Hom}_R(T, \operatorname{im}(g_1)) \to \operatorname{Hom}_R(T, B_2) \to \operatorname{Hom}_R(T, \operatorname{im}(g_2)) \\ \to \operatorname{Ext}^1_R(T, \operatorname{im}(g_1)) \to \operatorname{Ext}^1_R(T, B_2) \to \operatorname{Ext}^1_R(T, \operatorname{im}(g_2)) \to \cdots, \end{gathered}$$

which implies that

$$(J_0 \cap J_1 \cap J_2)^5 \subseteq \operatorname{Ann} \bigoplus_{i=1}^{\infty} \operatorname{Ext}^i_R(T, B_2) = \operatorname{Ann} \bigoplus_{i=1}^{\infty} H^2_{\mathfrak{m}}(D(H^2_{(x_1,x_2)}(R)/\Gamma_3)).$$

Let $\Gamma_3 := \Gamma_{Rx_3}(H^2_{(x_1,x_2)}(R))$. Then, by using [1, Remark 2.2.7 and Theorem 2.2.16] we get the exact sequence

$$0 \to H^2_{(x_1,x_2)}(R)/\Gamma_3 \to (H^2_{(x_1,x_2)}(R)/\Gamma_3)_{x_3} \to H^1_{Rx_3}(H^2_{(x_1,x_2)}(R)) \to 0,$$

which by using [6, Corollary 3.5], yields the isomorphic short exact sequence

$$0 \to H^2_{(x_1,x_2)}(R)/\Gamma_3 \to (H^2_{(x_1,x_2)}(R)\Gamma_3)_{x_3} \to H^3_{(x_1,x_2,x_3)}(R) \to 0. \quad (2.5.12)$$

From the exact sequence (2.5.12) we get the exact sequence

$$\begin{gathered} 0 \to D(H^3_{(x_1,x_2,x_3)}(R)) \to D((H^2_{(x_1,x_2)}(R)\Gamma_3)_{x_3}) \to \\ D(H^2_{(x_1,x_2)}(R)/\Gamma_3) \to 0 \quad (2.5.13). \end{gathered}$$

By the same argument as in the first lines of the proof, the exact sequence (2.5.13) yields the isomorphism

$$H^3_{\mathfrak{m}}(D(H^3_{(x_1,x_2,x_3)}(R))) \simeq H^2_{\mathfrak{m}}(D(H^2_{(x_1,x_2)}(R)/\Gamma_3)),$$

which implies that

$$(J_0 \cap J_1 \cap J_2)^5 \subseteq \operatorname{Ann} \bigoplus_{i=1}^{\infty} \operatorname{Ext}^i_R(T, H^3_{\mathfrak{m}}(D(H^3_{(x_1,x_2,x_3)}(R)))).$$

Proceeding in the same way, after finitely many steps we get

$$(\operatorname{Ann} \bigoplus_{i=0}^{d-1} H^i_{\mathfrak{m}}(R))^{2d-1} \quad = \quad (\bigcap_{i=0}^{d-1} J_i)^{2d-1}$$

$$\begin{aligned} &\subseteq \operatorname{Ann} \bigoplus_{i=1}^{\infty} \operatorname{Ext}^i_R(T, H^d_{\mathfrak{m}}(D(H^d_{(x_1,\cdots,x_d)}(R)))) \\ &= \operatorname{Ann} \bigoplus_{i=1}^{\infty} \operatorname{Ext}^i_R(T, H^d_{\mathfrak{m}}(D(H^d_{\mathfrak{m}}(R)))). \end{aligned}$$

(ii) Since T can be viewed as the direct limit of its finitely generated submodules, we have

$$T = \varinjlim_{\lambda\in\Lambda} T_\lambda,$$

with finitely generated R-modules T_λ. Then for each integer $i \geq 0$ and each $\lambda \in \Lambda$, by the adjointness we have

$$D(\operatorname{Ext}^i_R(T_\lambda, H^d_{\mathfrak{m}}(D(H^d_{\mathfrak{m}}(R))))) \simeq \operatorname{Tor}^R_i(T_\lambda, \Omega_R).$$

Therefore, by (i), for each integer $i \geq 1$ we have

$$\begin{aligned} (\operatorname{Ann} \bigoplus_{i=0}^{d-1} H^i_{\mathfrak{m}}(R))^{2d-1} &\subseteq \operatorname{Ann} \operatorname{Ext}^i_R(T_\lambda, H^d_{\mathfrak{m}}(D(H^d_{\mathfrak{m}}(R)))) \\ &\subseteq \operatorname{Ann} D(\operatorname{Ext}^i_R(T_\lambda, H^d_{\mathfrak{m}}(D(H^d_{\mathfrak{m}}(R))))) \\ &= \operatorname{Ann} \operatorname{Tor}^R_i(T_\lambda, \Omega_R). \end{aligned}$$

So, as the torsion functor $\operatorname{Tor}^R_i(-, \Omega_R)$ commutes with direct limits, it follows that

$$\begin{aligned} (\operatorname{Ann} \bigoplus_{i=0}^{d-1} H^i_{\mathfrak{m}}(R))^{2d-1} &\subseteq \bigcap_{\lambda\in\Lambda} \operatorname{Ann} \operatorname{Tor}^R_i(T_\lambda, \Omega_R) \\ &\subseteq \operatorname{Ann} \varinjlim_{\lambda\in\Lambda} \operatorname{Tor}^R_i(T_\lambda, \Omega_R) \\ &= \operatorname{Ann} \operatorname{Tor}^R_i(\varinjlim_{\lambda\in\Lambda} T_\lambda, \Omega_R) \\ &= \operatorname{Ann} \operatorname{Tor}^R_i(T, \Omega_R). \end{aligned}$$

Thus, it is clear that

$$(\operatorname{Ann} \bigoplus_{i=0}^{d-1} H^i_{\mathfrak{m}}(R))^{2d-1} \subseteq \operatorname{Ann} \bigoplus_{i=1}^{\infty} \operatorname{Tor}^R_i(T, \Omega_R).$$

□

The following well known result is needed in the proof of Corollary 2.7.

Lemma 2.6 (See [4, §19 Lemma 1]). *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a non-zero finitely generated R-module. Then*

$$\operatorname{pd}_R(M) = \sup\{n \in \mathbb{N}_0 \ : \ \operatorname{Tor}^R_n(R/\mathfrak{m}, M) \neq 0\}.$$

The following consequence of Proposition 2.5 is needed in the proof of Theorem 2.8. Recall that for an R-module M the *free locus of* M is defined as

$$\operatorname{frl}(M) := \{\mathfrak{p} \in \operatorname{Spec} R \ : \ M_{\mathfrak{p}} \text{ is a free } R_{\mathfrak{p}}\text{-module}\}.$$

COROLLARY 2.7. *Let $(R, \mathfrak{m})$ be a Noetherian complete local ring of dimension $d \geq 1$. Then Ω_R is a finitely generated R-module of dimension d such that $\operatorname{Ass}_R \Omega_R = \operatorname{Assh}_R R$. Moreover, we have*

$$\operatorname{Spec} R \backslash V(\operatorname{Ann} \bigoplus_{i=1}^{d-1} H^i_{\mathfrak{m}}(R)) \subseteq \operatorname{frl}(\Omega_R).$$

Proof. In view of [1, Theorems 7.1.3 and 7.3.2] the R-module $H^d_{\mathfrak{m}}(R)$ is Artinian with the set of attached primes $\operatorname{Assh}_R R$. As by the hypothesis R is complete, one sees that $D(H^d_{\mathfrak{m}}(R))$ is a finitely generated R-module of dimension d with $\operatorname{Ass}_R D(H^d_{\mathfrak{m}}(R)) = \operatorname{Assh}_R R$ and hence by [1, Theorems 7.1.3 and 7.3.2] the R-module $H^d_{\mathfrak{m}}(D(H^d_{\mathfrak{m}}(R)))$ is Artinian with $\operatorname{Att}_R H^d_{\mathfrak{m}}(D(H^d_{\mathfrak{m}}(R))) = \operatorname{Assh}_R R$. Since by the hypothesis R is complete we see that $D(H^d_{\mathfrak{m}}(D(H^d_{\mathfrak{m}}(R)))) = \Omega_R$ is a finitely generated R-module of dimension d with $\operatorname{Ass}_R \Omega_R = \operatorname{Assh}_R R$.

Now let $\mathfrak{p} \in \operatorname{Spec} R \backslash V(\operatorname{Ann} \bigoplus_{i=1}^{d-1} H^i_{\mathfrak{m}}(R))$ and let $j \geq 1$ be an integer. Since by Proposition 2.5 one has

$$(\operatorname{Ann} \bigoplus_{i=1}^{d-1} H^i_{\mathfrak{m}}(R))^{2d-1} \subseteq \operatorname{Ann} \operatorname{Tor}^R_j(R/\mathfrak{p}, \Omega_R),$$

we get $\mathfrak{p} \notin \operatorname{Supp} \operatorname{Tor}^R_j(R/\mathfrak{p}, \Omega_R)$ and hence

$$\operatorname{Tor}^{R_{\mathfrak{p}}}_j(R_{\mathfrak{p}}/\mathfrak{p} R_{\mathfrak{p}}, (\Omega_R)_{\mathfrak{p}}) \simeq (\operatorname{Tor}^R_j(R/\mathfrak{p}, \Omega_R))_{\mathfrak{p}} = 0.$$

So, if $(\Omega_R)_{\mathfrak{p}} = 0$ then it is clear that $\mathfrak{p} \in \operatorname{frl}(\Omega_R)$. But, if $(\Omega_R)_{\mathfrak{p}} \neq 0$ then by Lemma 2.6 we deduce that $\operatorname{pd}_{R_{\mathfrak{p}}}((\Omega_R)_{\mathfrak{p}}) = 0$ and hence as $R_{\mathfrak{p}}$ is a local Noetherian ring and $(\Omega_R)_{\mathfrak{p}}$ is a finitely generated projective $R_{\mathfrak{p}}$-module we see that $(\Omega_R)_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module and so $\mathfrak{p} \in \operatorname{frl}(\Omega_R)$. □

Now we are ready to state and prove the main result of this paper.

THEOREM 2.8. *Let $(R, \mathfrak{m})$ be a complete local generalized Cohen-Macaulay ring of dimension $d \geq 1$. Then the R-module Ω_R is generalized Cohen-Macaulay.*

Proof. Since by the hypothesis R is a generalized Cohen-Macaulay ring, from [1, Exercise 9.5.7(i)] it follows that $\operatorname{Ass}_R R \backslash \{\mathfrak{m}\} = \operatorname{Assh}_R R$ and $R_{\mathfrak{q}}$ is Cohen-Macaulay ring for all $\mathfrak{q} \in \operatorname{Spec} R \backslash \{\mathfrak{m}\}$.

On the other hand, by the definition we have $V(\operatorname{Ann} \bigoplus_{i=1}^{d-1} H^i_{\mathfrak{m}}(R)) \subseteq \{\mathfrak{m}\}$ and so for all $\mathfrak{q} \in \operatorname{Supp} \Omega_R \backslash \{\mathfrak{m}\}$ by Corollary 2.7, $(\Omega_R)_{\mathfrak{q}}$ is a free $R_{\mathfrak{q}}$-module.

Now, as $R_{\mathfrak{q}}$ is a Cohen-Macaulay ring for all $\mathfrak{q} \in \operatorname{Spec} R \backslash \{\mathfrak{m}\}$, it follows that $(\Omega_R)_{\mathfrak{q}}$ is a Cohen-Macaulay $R_{\mathfrak{q}}$-module for all $\mathfrak{q} \in \operatorname{Spec} R \backslash \{\mathfrak{m}\}$. Moreover, by Corollary 2.7 we have $\operatorname{Ass}_R \Omega_R = \operatorname{Assh}_R \Omega_R$.

Finally, since R is a complete local ring it follows from Cohen's Structure Theorem that R is a homomorphic image of a regular ring.

Now, it follows from [1, Exercise 9.5.7(ii)] that Ω_R is generalized Cohen-Macaulay R-module. $\square$

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