REMARKS ON HYPERGEOMETRIC CAUCHY NUMBERS

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For a positive integer N, hypergeometric Cauchy numbers $c_{N,n}$ are defined by

$$\frac{1}{{}_2F_1(1,N;N+1;-x)}=\frac{(-1)^{N-1}x^N/N}{\log(1+x)-\sum_{n=1}^{N-1}(-1)^{n-1}x^n/n}=\sum_{n=0}^\infty c_{N,n}\frac{x^n}{n!}\,,$$

where ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function. When $N=1$, $c_n=c_{1,n}$ are the classical Cauchy numbers. In 1875, Glaisher gave several interesting determinant expressions of numbers, including Bernoulli, Cauchy and Euler numbers (see (4), (6) and (3) in the text). Hypergeometric numbers can be recognized as one of the most natural extensions of the classical Cauchy numbers in terms of determinants (see Section 2), though many kinds of generalizations of the Cauchy numbers have been considered by many authors. In addition, there are some relations between the hypergeometric Cauchy numbers and the classical Cauchy numbers. In this paper, we give the determinant expressions of hypergeometric Cauchy numbers and their generalizations, and show some interesting expressions of hypergeometric Cauchy numbers. As applications, we can get the inversion relations such that hypergeometric Cauchy numbers as $c_{N,n}/n!$ and the numbers $N/(N+n)$ are interchanged in terms of determinants of the so-called Hassenberg matrices.

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1. INTRODUCTION

Denote ${}_2F_1(a,b;c;z)$ be the Gauss hypergeometric function defined by

$${}_2F_1(a,b;c;z)=\sum_{n=0}^\infty\frac{(a)^{(n)}(b)^{(n)}}{(c)^{(n)}}\frac{z^n}{n!}$$

with the rising factorial $(x)^{(n)}=x(x+1)\dots(x+n-1)$ $(n\ge 1)$ and $(x)^{(0)}=1$. For $N\ge 1$, define the hypergeometric Cauchy numbers $c_{N,n}$ ([19]) by

$$\frac{1}{{}_2F_1(1,N;N+1;-x)}=\frac{(-1)^{N-1}x^N/N}{\log(1+x)-\sum_{n=1}^{N-1}(-1)^{n-1}x^n/n}=\sum_{n=0}^\infty c_{N,n}\frac{x^n}{n!}\,. \tag{1}$$

When $N=1$, $c_n=c_{1,n}$ are classical Cauchy numbers ([5]) defined by

$$\frac{x}{\log(1+x)}=\sum_{n=0}^\infty c_n\frac{x^n}{n!}\,.$$

Notice that $b_n = c_n/n!$ are sometimes called the Bernoulli numbers of the second kind. In addition, the hypergeometric Cauchy polynomials $c_{M,N,n}(z)$ ([19]) are defined by

$$\begin{aligned}\frac{1}{(1+x)^z}\frac{1}{{}_2F_1(M,N;N+1;-x)}&=\frac{1}{(1+x)^z}\sum_{n=0}^\infty c_{M,N,n}\frac{x^n}{n!}\\&=\sum_{n=0}^\infty c_{M,N,n}(z)\frac{x^n}{n!}\,.\end{aligned}$$

so that $c_{1,N,n}(0)=c_{N,n}$.

Similar hypergeometric numbers are hypergeometric Bernoulli numbers $B_{N,n}$ and hypergeometric Euler numbers. For $N\ge 1$, define hypergeometric Bernoulli numbers $B_{N,n}$ ([9, 10, 11, 12, 14, 33]) by

$$\frac{1}{{}_1F_1(1;N+1;x)}=\frac{x^N/N!}{e^x-\sum_{n=0}^{N-1}x^n/n!}=\sum_{n=0}^\infty B_{N,n}\frac{x^n}{n!}\,, \tag{2}$$

where ${}_1F_1(a;b;z)$ is the confluent hypergeometric function defined by

$${}_1F_1(a;b;z)=\sum_{n=0}^\infty\frac{(a)^{(n)}}{(b)^{(n)}}\frac{z^n}{n!}\,.$$

When $N=1$, $B_{1,n}=B_n$ are classical Bernoulli numbers, defined by

$$\frac{x}{e^x-1}=\sum_{n=0}^\infty B_n\frac{x^n}{n!}\,.$$

The hypergeometric Euler numbers $E_{N,n}$ ([30]) are defined by

$$\frac{1}{{}_1F_2(1;N+1,(2N+1)/2;x^2/4)}=\sum_{n=0}^\infty E_{N,n}\frac{x^n}{n!}\,,$$

where ${}_1F_2(a;b,c;z)$ is the hypergeometric function defined by

$${}_1F_2(a;b,c;z)=\sum_{n=0}^\infty\frac{(a)^{(n)}}{(b)^{(n)}(c)^{(n)}}\frac{z^n}{n!}\,.$$

When $N=0$, then $E_n=E_{0,n}$ are classical Euler numbers defined by

$$\frac{1}{\cosh x}=\sum_{n=0}^\infty E_n\frac{x^n}{n!}\,.$$

Several kinds of generalizations of the Cauchy numbers (or the Bernoulli numbers of the second kind) have been considered by many authors. For example, poly-Cauchy numbers [17], multiple Cauchy numbers, shifted Cauchy numbers [27], generalized Cauchy numbers [24], incomplete Cauchy numbers [20, 22, 25], various types of q-Cauchy numbers [4, 18, 21, 28], Cauchy Carlitz numbers [15, 16]. The situations are similar and even more for Bernoulli numbers and Euler numbers. Hypergeometric numbers can be recognized as one of the most natural extensions of the classical numbers in terms of determinants, though many kinds of generalizations of the classical numbers have been considered by many authors. In [23, 30], the hypergeometric Euler numbers $E_{N,2n}$ can be expressed as

$$E_{N,2n}=(-1)^n(2n)!\begin{vmatrix} \frac{(2N)!}{(2N+2)!} & 1 & & \\ \frac{(2N)!}{(2N+4)!} & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ \frac{(2N)!}{(2N+2n)!} & \cdots & \frac{(2N)!}{(2N+4)!} & \frac{(2N)!}{(2N+2)!} \end{vmatrix}\quad (N\ge 0,\ n\ge 1)\,.$$

When $N=0$, this is reduced to a famous determinant expression of Euler numbers (*cf.* [6, p.52]):

$$E_{2n}=(-1)^n(2n)!\begin{vmatrix} \frac{1}{2!} & 1 & & & \\ \frac{1}{4!} & \frac{1}{2!} & 1 & & \\ \vdots & & \ddots & \ddots & \\ \frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & & \frac{1}{2!} & 1 \\ \frac{1}{(2n)!} & \frac{1}{(2n-2)!} & \cdots & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix}\,. \tag{3}$$

In addition, When $N=1$, $E_{1,n}$ can be expressed by Bernoulli numbers as $E_{1,n}=-(n-1)B_n$ ([30]).

In [1], the hypergeometric Bernoulli numbers $B_{N,n}$ can be expressed as

$$B_{N,n}=(-1)^n n!\begin{vmatrix} \frac{N!}{(N+1)!} & 1 & & & \\ \frac{N!}{(N+2)!} & \frac{N!}{(N+1)!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{N!}{(N+n-1)!} & \frac{N!}{(N+n-2)!} & \cdots & \frac{N!}{(N+1)!} & 1 \\ \frac{N!}{(N+n)!} & \frac{N!}{(N+n-1)!} & \cdots & \frac{N!}{(N+2)!} & \frac{N!}{(N+1)!} \end{vmatrix}\quad (N\ge 1, n\ge 1)\,.$$

When $N=1$, we have a determinant expression of Bernoulli numbers ([6, p.53]):

(4)
$$B_n = (-1)^n n! \begin{vmatrix} \frac{1}{2!} & 1 & & & \\ \frac{1}{3!} & \frac{1}{2!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \cdots & \frac{1}{2!} & 1 \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \cdots & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix}.$$

In addition, relations between $B_{N,n}$ and $B_{N-1,n}$ are shown in [1].

In this paper, we shall give similar determinant expression of hypergeometric Cauchy numbers and their generalizations. We also study some interesting relations between the hypergeometric Cauchy numbers and the classical Cauchy numbers. As applications, we can get the inversion relations such that hypergeometric Cauchy numbers as $c_{N,n}/n!$ and the numbers $N/(N+n)$ are interchanged in terms of determinants of the so-called Hassenberg matrices.

2. A DETERMINANT EXPRESSION OF THE HYPERGEOMETRIC CAUCHY NUMBERS

From the definition (1), we have

(5)
$$\sum_{i=0}^{n} \frac{(-1)^i c_{N,i}}{(N+n-i)i!} = 0 \quad (n \ge 1)$$

with $c_{N,0} = 1$ ([19, Proposition 1]).

By using this expression, the first few values of $c_{N,n}$ are given:

$$\begin{aligned} c_{N,0} &= 1\,, \\ c_{N,1} &= \frac{N}{N+1}\,, \\ c_{N,2} &= -\frac{2N}{(N+1)^2(N+2)}\,, \\ c_{N,3} &= \frac{6N(N^2+N+2)}{(N+1)^3(N+2)(N+3)}\,, \\ c_{N,4} &= -\frac{4!N(N^5+5N^4+14N^3+24N^2+20N+12)}{(N+1)^4(N+2)^2(N+3)(N+4)}\,, \\ c_{N,5} &= \frac{5!N(N^7+8N^6+35N^5+96N^4+160N^3+184N^2+116N+48)}{(N+1)^5(N+2)^2(N+3)(N+4)(N+5)}\,. \end{aligned}$$

In [19, Theorem 1], an explicit expression of hypergeometric Cauchy numbers is given.

Lemma 2.1. *For $N, n \ge 1$, we have*

$$c_{N,n} = (-1)^n n! \sum_{r=1}^{n} (-N)^r \sum_{\substack{i_1+\cdots+i_r=n \\ i_1,\dots,i_r \ge 1}} \frac{1}{(N+i_1)\cdots(N+i_r)}\,.$$

Such values of $c_{N,n}$ can be expressed in terms of the determinant.

Theorem 2.2. *For $N, n \ge 1$, we have*

$$c_{N,n} = n! \begin{vmatrix} \frac{N}{N+1} & 1 & & & \\ \frac{N}{N+2} & \frac{N}{N+1} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{N}{N+n-1} & \frac{N}{N+n-2} & \cdots & \frac{N}{N+1} & 1 \\ \frac{N}{N+n} & \frac{N}{N+n-1} & \cdots & \frac{N}{N+2} & \frac{N}{N+1} \end{vmatrix}\,.$$

Remark. When $N = 1$, we have a determinant expression of Cauchy numbers ([6, p.50]):

$$c_n = n! \begin{vmatrix} \frac{1}{2} & 1 & & & \\ \frac{1}{3} & \frac{1}{2} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{1}{n} & \frac{1}{n-1} & \cdots & \frac{1}{2} & 1 \\ \frac{1}{n+1} & \frac{1}{n} & \cdots & \frac{1}{3} & \frac{1}{2} \end{vmatrix}\,. \tag{6}$$

The value of this determinant, that is, $b_n = c_n/n!$ are called *Bernoulli numbers of the second kind.*

Proof of Theorem 2.2. Put $b_{N,n} = c_{N,n}/n!$. Then, we shall prove that

$$b_{N,n} = \begin{vmatrix} \frac{N}{N+1} & 1 & & & \\ \frac{N}{N+2} & \frac{N}{N+1} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{N}{N+n-1} & \frac{N}{N+n-2} & \cdots & \frac{N}{N+1} & 1 \\ \frac{N}{N+n} & \frac{N}{N+n-1} & \cdots & \frac{N}{N+2} & \frac{N}{N+1} \end{vmatrix}\,. \tag{7}$$

Since by (5)

$$c_{N,n} = \sum_{i=0}^{n-1} (-1)^{n-i-1} \frac{n!}{i!} \frac{N}{N+n-i} c_{N,i}\,,$$

we have

$$b_{N,n} = N \sum_{i=0}^{n-1} \frac{(-1)^{n-i-1}}{N+n-i} b_{N,i}\,.$$

When $n = 1$, (7) is true as $b_{N,1} = N/(N+1)$. Assume that the result (7) is true up to $n-1$. Then, by expanding the last column of the right-hand side of (7), it is equal to

$$
\frac{N}{N+1} b_{N,n-1} - \begin{vmatrix} \frac{N}{N+1} & 1 & & & \\ \frac{N}{N+2} & \frac{N}{N+1} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{N}{N+n-2} & \frac{N}{N+n-3} & \cdots & \frac{N}{N+1} & 1 \\ \frac{N}{N+n} & \frac{N}{N+n-1} & \cdots & \frac{N}{N+3} & \frac{N}{N+2} \end{vmatrix}
$$

$$
= \frac{N}{N+1} b_{N,n-1} - \frac{N}{N+2} b_{N,n-2} + \begin{vmatrix} \frac{N}{N+1} & 1 & & & \\ \frac{N}{N+2} & \frac{N}{N+1} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{N}{N+n-3} & \frac{N}{N+n-4} & \cdots & \frac{N}{N+1} & 1 \\ \frac{N}{N+n} & \frac{N}{N+n-1} & \cdots & \frac{N}{N+4} & \frac{N}{N+3} \end{vmatrix}
$$

$$
= \frac{N}{N+1} b_{N,n-1} - \frac{N}{N+2} b_{N,n-2} + \frac{N}{N+3} b_{N,n-3}
$$

$$
- \cdots + (-1)^{n-2} \begin{vmatrix} \frac{N}{N+1} & 1 \\ \frac{N}{N+n} & \frac{N}{N+n-1} \end{vmatrix}
$$

$$
= N \sum_{i=0}^{n-1} \frac{(-1)^{n-i-1}}{N+n-i} b_{N,i} = b_{N,n} \,.
$$

Note that

$$
b_{N,1} = \frac{N}{N+1} \quad \text{and} \quad b_{N,0} = 1 \,.
$$

□

3. A RELATION BETWEEN $C_{N,N}$ AND $C_{N-1,N}$

In this section, we show the following relation between $c_{N,n}$ and $c_{N-1,n}$.

Proposition 3.1. *For $N \geq 2$ and $n \geq 1$, we have*

$$
c_{N,n} = \sum_{m=0}^{n} \left(\frac{N}{1-N} \right)^m \sum_{0 \leq i_m < \cdots < i_1 < i_0 = n} \frac{n!}{i_m!} \, c_{N-1,i_m} \prod_{k=1}^{m} \frac{c_{N-1,i_{k-1}-i_k+1}}{(i_{k-1}-i_k+1)!},
$$

where $i_0 = n$.

Examples

(i) $c_{N,1} = c_{N-1,1} + \dfrac{N}{1-N} \times \dfrac{c_{N-1,0} c_{N-1,2}}{2}$

(ii) $$c_{N,2}=c_{N-1,2}+\frac{N}{1-N}\left(\frac{c_{N-1,3}}{3}+c_{N-1,1}c_{N-1,2}\right)+\left(\frac{N}{1-N}\right)^2\times\frac{c_{N-1,2}^2}{2}$$

LEMMA 3.2. *For $N\ge 2$ and $n\ge 0$, we have*

$$c_{N,n}=c_{N-1,n}-\frac{N}{(n+1)(N-1)}\sum_{m=0}^{n-1}\binom{n+1}{m}c_{N,m}c_{N-1,n-m+1}.$$

Proof. From (1), we have

$$(-1)^{N-1}\times\frac{x^N}{N}=\left(\sum_{n=0}^\infty c_{N,n}\frac{x^n}{n!}\right)\left\{\log(1+x)-\sum_{n=1}^{N-2}(-1)^{n-1}\frac{x^n}{n}\right.$$
$$\left.-(-1)^{N-2}\frac{x^{N-1}}{N-1}\right\}.$$

By dividing $\log(1+x)-\sum_{n=1}^{N-2}(-1)^{n-1}x^n/n$, we have

$$\frac{1-N}{N}\,x\times\sum_{n=0}^\infty c_{N-1,n}\frac{x^n}{n!}=\left(\sum_{n=0}^\infty c_{N,n}\frac{x^n}{n!}\right)\left(1-\sum_{n=0}^\infty c_{N-1,n}\frac{x^n}{n!}\right),$$

and hence

$$\begin{aligned}\sum_{n=1}^\infty\frac{(1-N)n}{N}\,c_{N-1,n-1}\frac{x^n}{n!}&=\sum_{n=0}^\infty c_{N,n}\frac{x^n}{n!}-\sum_{m=0}^\infty\sum_{k=0}^\infty c_{N,m}c_{N-1,k}\frac{x^{m+k}}{m!k!}\\&=\sum_{n=0}^\infty c_{N,n}\frac{x^n}{n!}-\sum_{n=0}^\infty\sum_{m=0}^n c_{N,m}c_{N-1,n-m}\frac{x^n}{m!(n-m)!}\\&=\sum_{n=0}^\infty\left\{c_{N,n}-\sum_{m=0}^n\binom{n}{m}c_{N,m}c_{N-1,n-m}\right\}\frac{x^n}{n!}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}c_{N-1,n-1}&=\frac{N}{(1-N)n}\left\{c_{N,n}-\sum_{m=0}^n\binom{n}{m}c_{N,m}c_{N-1,n-m}\right\}\\&=\frac{N}{(1-N)n}\left\{-\binom{n}{n-1}c_{N,n-1}c_{N-1,1}-\sum_{m=0}^{n-2}\binom{n}{m}c_{N,m}c_{N-1,n-m}\right\}\\&=\frac{N}{(1-N)n}\left\{\frac{(1-N)n}{N}\,c_{N,n-1}-\sum_{m=0}^{n-2}\binom{n}{m}c_{N,m}c_{N-1,n-m}\right\}\\&=c_{N,n-1}-\frac{N}{(1-N)n}\sum_{m=0}^{n-2}\binom{n}{m}c_{N,m}c_{N-1,n-m},\end{aligned}$$

for $n\ge 1$, and the proof is complete. $\square$

Proof of Proposition 3.1. We give the proof by induction for n. In the case $n=1$, the assertion means

$$c_{N,1}=c_{N-1,1}+\frac{N}{1-N}\times\frac{c_{N-1,0}c_{N-1,2}}{2},$$

and this equality follows from $c_{N,1}=N/(N+1)$, $c_{N-1,1}=(N-1)/N$, $c_{N-1,0}=1$ and $c_{N-1,2}=-2(N-1)/N^2(N+1)$. Assume that the assertion holds up to $n-1$. By Lemma 3.2, we have

$$\begin{aligned}
c_{N,n}&=c_{N-1,n}-\frac{N}{(n+1)(N-1)}\sum_{i_1=0}^{n-1}\binom{n+1}{i_1}c_{N,i_1}c_{N-1,n-i_1+1}\\
&=c_{N-1,n}-\frac{N}{(n+1)(N-1)}\sum_{i_1=0}^{n-1}\binom{n+1}{i_1}c_{N-1,n-i_1+1}\\
&\quad\times\sum_{m=1}^{i_1}\left(\frac{N}{1-N}\right)^{m-1}\sum_{0\le i_m<\cdots<i_1}\frac{i_1!}{i_m!}c_{N-1,i_m}\prod_{k=2}^{m}\frac{c_{N-1,i_{k-1}-i_k+1}}{(i_{k-1}-i_k+1)!}\\
&=c_{N-1,n}+\sum_{i_1=0}^{n-1}\sum_{m=1}^{i_1}\sum_{0\le i_m<\cdots<i_1}\left(\frac{N}{1-N}\right)^{m}\frac{n!}{i_m!}c_{N-1,i_m}\prod_{k=1}^{m}\frac{c_{N-1,i_{k-1}-i_k+1}}{(i_{k-1}-i_k+1)!}\\
&=c_{N-1,n}+\sum_{m=1}^{n}\left(\frac{N}{1-N}\right)^{m}\sum_{0\le i_m<\cdots<i_1<i_0=n}\frac{n!}{i_m!}c_{N-1,i_m}\prod_{k=1}^{m}\frac{c_{N-1,i_{k-1}-i_k+1}}{(i_{k-1}-i_k+1)!}\\
&=\sum_{m=0}^{n}\left(\frac{N}{1-N}\right)^{m}\sum_{0\le i_m<\cdots<i_1<i_0=n}\frac{n!}{i_m!}c_{N-1,i_m}\prod_{k=1}^{m}\frac{c_{N-1,i_{k-1}-i_k+1}}{(i_{k-1}-i_k+1)!}.
\end{aligned}$$

$\square$

4. MULTIPLE HYPERGEOMETRIC CAUCHY NUMBERS

For positive integers N and r, define the *hypergeometric Cauchy numbers* $c_{N,n}^{(r)}$ by the generating function

$$\begin{aligned}
\frac{1}{\left({}_2F_1(1,N;N+1;-x)\right)^r}&=\left(\frac{(-1)^{N-1}x^N/N}{\log(1+x)-\sum_{n=1}^{N-1}(-1)^{n-1}x^n/n}\right)^r\\
&=\sum_{n=0}^{\infty}c_{N,n}^{(r)}\frac{x^n}{n!}\,. \qquad (8)
\end{aligned}$$

From the definition (8),

$$
\begin{aligned}
&\left(\frac{(-1)^{N-1}x^N}{N}\right)^r\\
&=\left(\sum_{i=0}^\infty\frac{(-1)^{i+N-1}x^{N+i}}{N+i}\right)^r\left(\sum_{n=0}^\infty c_{N,n}^{(r)}\frac{x^n}{n!}\right)\\
&=x^{rN}(-1)^{r(N-1)}\left(\sum_{l=0}^\infty\sum_{\substack{i_1+\cdots+i_r=l\\ i_1,\dots,i_r\ge 0}}\frac{(-1)^l l!}{(N+i_1)\cdots(N+i_r)}\frac{x^l}{l!}\right)\left(\sum_{n=0}^\infty c_{N,n}^{(r)}\frac{x^n}{n!}\right)\\
&=x^{rN}(-1)^{r(N-1)}\sum_{n=0}^\infty\sum_{m=0}^n\sum_{\substack{i_1+\cdots+i_r=n-m\\ i_1,\dots,i_r\ge 0}}\binom{n}{m}\frac{(-1)^{n-m}(n-m)!}{(N+i_1)\cdots(N+i_r)}c_{N,m}^{(r)}\frac{x^n}{n!}\,.
\end{aligned}
$$

Hence, as a generalization of Proposition (5), for $n\ge 1$, we have the following.

Proposition 4.1.

$$
\sum_{m=0}^n\sum_{\substack{i_1+\cdots+i_r=n-m\\ i_1,\dots,i_r\ge 0}}\frac{(-1)^{n-m}c_{N,m}^{(r)}}{m!(N+i_1)\cdots(N+i_r)}=0\,.
$$

By using Proposition 4.1 or

$$
c_{N,n}^{(r)}=-n!N^r\sum_{m=0}^{n-1}\sum_{\substack{i_1+\cdots+i_r=n-m\\ i_1,\dots,i_r\ge 0}}\frac{(-1)^{n-m}c_{N,m}^{(r)}}{m!(N+i_1)\cdots(N+i_r)}\tag{9}
$$

with $c_{N,0}^{(r)}=1$ $(N\ge 1)$, some values of $c_{N,n}^{(r)}$ $(0\le n\le 4)$ are explicitly given by the following.

$$
\begin{aligned}
c_{N,0}^{(r)}&=1\,,\\
c_{N,1}^{(r)}&=\frac{rN}{N+1}\,,\\
c_{N,2}^{(r)}&=\frac{r(r+1)N^2}{(N+1)^2}-\frac{2rN}{N+2}\,,\\
c_{N,3}^{(r)}&=\frac{r(r+1)(r+2)N^3}{(N+1)^3}-\frac{6r(r+1)N^2}{(N+1)(N+2)}+\frac{6rN}{N+3}\,,\\
c_{N,4}^{(r)}&=\frac{r(r+1)(r+2)(r+3)N^4}{(N+1)^4}-\frac{12r(r+1)(r+2)N^3}{(N+1)^2(N+2)}
\end{aligned}
$$

$$+\frac{24r(r+1)N^2}{(N+1)(N+3)}+\frac{12r(r+1)N^2}{(N+2)^2}-\frac{24rN}{N+4}\,.$$

As a generalization of Lemma 2.1, we have an explicit expression of $c_{N,n}^{(r)}$.

PROPOSITION 4.2. *For $N,n\ge 1$, we have*

$$c_{N,n}^{(r)}=n!\sum_{k=1}^{n}(-1)^{n-k}\sum_{\substack{e_1+\cdots+e_k=n\\ e_1,\dots,e_k\ge 1}}D_r(e_1)\cdots D_r(e_k)\,,$$

where

$$D_r(e)=\sum_{\substack{i_1+\cdots+i_r=e\\ i_1,\dots,i_r\ge 0}}\frac{N^r}{(N+i_1)\cdots(N+i_r)}\,. \tag{10}$$

The first few values of $D_r(e)$ are given by the following.

$$\begin{aligned}
D_r(1)&=\frac{rN}{N+1}\,,\\
D_r(2)&=\frac{rN}{N+2}+\frac{r(r-1)N^2}{2(N+1)^2}\,,\\
D_r(3)&=\frac{rN}{N+3}+\frac{r(r-1)N^2}{(N+1)(N+2)}+\binom{r}{3}\frac{N^3}{(N+1)^3}\,,\\
D_r(4)&=\frac{rN}{N+4}+\frac{r(r-1)N^2}{(N+1)(N+3)}+\binom{r}{2}\frac{N^2}{(N+1)^2}\\
&\quad+r\binom{r-1}{2}\frac{N^3}{(N+1)^2(N+2)}+\binom{r}{4}\frac{N^4}{(N+1)^4}\,.
\end{aligned}$$

We shall introduce the Hasse-Teichmüller derivative in order to prove Proposition 4.2 easily. Let $\mathbb{F}$ be a field of any characteristic, $\mathbb{F}[[z]]$ the ring of formal power series in one variable z, and $\mathbb{F}((z))$ the field of Laurent series in z. Let n be a nonnegative integer. We define the Hasse-Teichmüller derivative $H^{(n)}$ of order n by

$$H^{(n)}\left(\sum_{m=R}^{\infty}c_m z^m\right)=\sum_{m=R}^{\infty}c_m\binom{m}{n}z^{m-n}$$

for $\sum_{m=R}^{\infty}c_m z^m\in\mathbb{F}((z))$, where R is an integer and $c_m\in\mathbb{F}$ for any $m\ge R$. Note that $\binom{m}{n}=0$ if $m<n$.

The Hasse-Teichmüller derivatives satisfy the product rule [35], the quotient rule [7] and the chain rule [8]. One of the product rules can be described as follows.

LEMMA 4.3. *For $f_i \in \mathbb{F}[[z]]$ $(i = 1, \dots, k)$ with $k \ge 2$ and for $n \ge 1$, we have*

$$H^{(n)}(f_1 \cdots f_k) = \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 0}} H^{(i_1)}(f_1)\cdots H^{(i_k)}(f_k)\,.$$

The quotient rules can be described as follows.

LEMMA 4.4. *For $f \in \mathbb{F}[[z]]\backslash\{0\}$ and $n \ge 1$, we have*

$$H^{(n)}\left(\frac{1}{f}\right) = \sum_{k=1}^{n} \frac{(-1)^k}{f^{k+1}} \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 1}} H^{(i_1)}(f)\cdots H^{(i_k)}(f) \tag{11}$$

$$= \sum_{k=1}^{n} \binom{n+1}{k+1} \frac{(-1)^k}{f^{k+1}} \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 0}} H^{(i_1)}(f)\cdots H^{(i_k)}(f)\,. \tag{12}$$

Proof of Proposition 4.2. Put $h(x) = \big(f(x)\big)^r$, where

$$f(x) = \frac{\sum_{i=N}^{\infty}(-1)^{i-1}\frac{x^i}{i}}{(-1)^{N-1}\frac{x^N}{N}} = \sum_{j=0}^{\infty} \frac{(-1)^j N}{N+j} x^j\,.$$

Since

$$\begin{aligned} H^{(i)}(f)\Big|_{x=0} &= \sum_{j=i}^{\infty} \frac{(-1)^j N}{N+j}\binom{j}{i} x^{j-i}\Bigg|_{x=0} \\ &= \frac{(-1)^i N}{N+i} \end{aligned}$$

by the product rule of the Hasse-Teichmüller derivative in Lemma 4.3, we get

$$\begin{aligned} H^{(e)}(h)\Big|_{x=0} &= \sum_{\substack{i_1+\cdots+i_r=e\\ i_1,\dots,i_r\ge 0}} H^{(i_1)}(f)\Big|_{x=0} \cdots H^{(i_r)}(f)\Big|_{x=0} \\ &= \sum_{\substack{i_1+\cdots+i_r=e\\ i_1,\dots,i_r\ge 0}} \frac{(-1)^{i_1}N}{N+i_1}\cdots\frac{(-1)^{i_r}N}{N+i_r} \\ &= (-1)^e \sum_{\substack{i_1+\cdots+i_r=e\\ i_1,\dots,i_r\ge 0}} \frac{N^r}{(N+i_1)\cdots(N+i_r)} := (-1)^e D_r(e)\,. \end{aligned}$$

Hence, by the quotient rule of the Hasse-Teichmüller derivative in Lemma 4.4 (11), we have

$$\frac{c_{N,n}^{(r)}}{n!} = \sum_{k=1}^{n} \frac{(-1)^k}{h^{k+1}}\Bigg|_{x=0} \sum_{\substack{e_1+\cdots+e_k=n\\ e_1,\dots,e_k\ge 1}} H^{(e_1)}(h)\Big|_{x=0} \cdots H^{(e_k)}(h)\Big|_{x=0}$$

$$= \sum_{k=1}^{n} (-1)^k \sum_{\substack{e_1+\cdots+e_k=n \\ e_1,\dots,e_k \ge 1}} (-1)^n D_r(e_1)\cdots D_r(e_k)\,.$$

□

Now, we can also show a determinant expression of $c_{N,n}^{(r)}$.

THEOREM 4.5. *For $N, n \ge 1$, we have*

$$c_{N,n}^{(r)} = n! \begin{vmatrix} D_r(1) & 1 & & & \\ D_r(2) & D_r(1) & & & \\ \vdots & \vdots & \ddots & 1 & \\ D_r(n-1) & D_r(n-2) & \cdots & D_r(1) & 1 \\ D_r(n) & D_r(n-1) & \cdots & D_r(2) & D_r(1) \end{vmatrix}.$$

where $D_r(e)$ are given in (10)*.*

Remark. When $r = 1$ in Theorem 4.5, we have the result in Theorem 2.2.

Proof. For simplicity, put $b_{N,n}^{(r)} = c_{N,n}^{(r)}/n!$. Then, we shall prove that for any $n \ge 1$

$$b_{N,n}^{(r)} = \begin{vmatrix} D_r(1) & 1 & & & \\ D_r(2) & D_r(1) & & & \\ \vdots & \vdots & \ddots & 1 & \\ D_r(n-1) & D_r(n-2) & \cdots & D_r(1) & 1 \\ D_r(n) & D_r(n-1) & \cdots & D_r(2) & D_r(1) \end{vmatrix}. \tag{13}$$

When $n = 1$, (13) is valid because

$$D_r(1) = \frac{rN^r}{N^{r-1}(N+1)} = \frac{rN}{N+1} = b_{N,1}^{(r)}\,.$$

Assume that (13) is valid up to $n-1$. Notice that by (9), we have

$$b_{N,n}^{(r)} = \sum_{l=1}^{n} (-1)^{l-1} b_{N,n-l}^{(r)} D_r(l)\,.$$

Thus, by expanding the first row of the right-hand side (13), it is equal to

$$D_r(1) b_{N,n-1}^{(r)} - \begin{vmatrix} D_r(2) & 1 & & & \\ D_r(3) & D_r(1) & & & \\ \vdots & \vdots & \ddots & 1 & \\ D_r(n-1) & D_r(n-3) & \cdots & D_r(1) & 1 \\ D_r(n) & D_r(n-2) & \cdots & D_r(2) & D_r(1) \end{vmatrix}$$

$$
\begin{aligned}
&= D_r(1)b_{N,n-1}^{(r)} - D_r(2)b_{N,n-2}^{(r)} \\
&\quad + \begin{vmatrix} D_r(3) & 1 & & & \\ D_r(4) & D_r(1) & & & \\ \vdots & \vdots & \ddots & 1 & \\ D_r(n-1) & D_r(n-4) & \cdots & D_r(1) & 1 \\ D_r(n) & D_r(n-3) & \cdots & D_r(2) & D_r(1) \end{vmatrix} \\
&= D_r(1)b_{N,n-1}^{(r)} - D_r(2)b_{N,n-2}^{(r)} + \cdots + (-1)^{n-2} \begin{vmatrix} D_r(n-1) & 1 \\ D_r(n) & D_r(1) \end{vmatrix} \\
&= \sum_{l=1}^{n} (-1)^{l-1} D_r(l) b_{N,n-l}^{(r)} = b_{N,n}^{(r)} .
\end{aligned}
$$

Note that $b_{N,1}^{(r)} = D_r(1)$ and $b_{N,0}^{(r)} = 1$. □

5. A RELATION BETWEEN $C_{N,N}^{(R)}$ AND $C_{N,N}$

In this section, we show the following relation between $c_{N,n}^{(r)}$ and $c_{N,n}$.

LEMMA 5.1. *For $r \ge 2$ and $N, n \ge 0$, we have*

$$
c_{N,n}^{(r)} = \sum_{\substack{n_1,\dots,n_r \ge 0 \\ n_1+\cdots+n_r=n}} \frac{n!}{n_1! \cdots n_r!} \, C_{N,n_1} \cdots C_{N,n_r},
$$

Proof. From the definition (8), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} C_{N,n}^{(r)} \frac{n!}{x^n} &= \left(\sum_{n=0}^{\infty} C_{N,n} \frac{x^n}{n!} \right)^r \\
&= \sum_{n=0}^{\infty} \sum_{\substack{n_1,\dots,n_r \ge 0 \\ n_1+\cdots+n_r=n}} \frac{n!}{n_1! \cdots n_r!} \, C_{N,n_1} \cdots C_{N,n_r} \frac{x^n}{n!},
\end{aligned}
$$

and we get the assertion. □

Examples

(i) $c_{N,0}^{(r)} = c_{N,0}^{r+1}$

(ii) $c_{N,1}^{(r)} = r c_{N,1}$

(iii) $c_{N,2}^{(r)} = r c_{N,2} c_{N,0}^{N-1} + r(r-1) c_{N,1}^{2} c_{N,0}^{r-2}$

6. APPLICATIONS BY THE TRUDI'S FORMULA AND INVERSION EXPRESSIONS

The expressions in Theorem 2.2 and Theorem 4.5 are useful for several applications too. We can obtain different explicit expressions for the numbers $c_{N,n}^{(r)}$, $c_{N,n}$ and c_n by using the Trudi's formula. We also show some inversion formulas. The following relation is known as Trudi's formula [31, Vol.3, p.214],[36] and the case $a_0 = 1$ of this formula is known as Brioschi's formula [2],[31, Vol.3, pp.208–209].

Lemma 6.1. *For a positive integer m, we have*

$$\begin{vmatrix} a_1 & a_2 & \cdots & \cdots & a_m \\ a_0 & a_1 & \ddots & & \vdots \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & a_1 & a_2 \\ & & & a_0 & a_1 \end{vmatrix}$$
$$= \sum_{t_1+2t_2+\cdots+mt_m=m} \binom{t_1+\cdots+t_m}{t_1,\dots,t_m} (-a_0)^{m-t_1-\cdots-t_m} a_1^{t_1} a_2^{t_2} \cdots a_m^{t_m},$$

where $\binom{t_1+\cdots+t_m}{t_1,\dots,t_m} = \frac{(t_1+\cdots+t_m)!}{t_1!\cdots t_m!}$ are the multinomial coefficients.

In addition, there exists the following inversion formula (see, e.g., [26]), which is based upon the relation:

$$\sum_{k=0}^{n} (-1)^{n-k} \alpha_k R(n-k) = 0 \quad (n \ge 1)\,.$$

Lemma 6.2. *If $\{\alpha_n\}_{n\ge 0}$ is a sequence defined by $\alpha_0 = 1$ and*

$$\alpha_n = \begin{vmatrix} R(1) & 1 & & \\ R(2) & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ R(n) & \cdots & R(2) & R(1) \end{vmatrix}, \text{ then } R(n) = \begin{vmatrix} \alpha_1 & 1 & & \\ \alpha_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \alpha_n & \cdots & \alpha_2 & \alpha_1 \end{vmatrix}.$$

Moreover, if

$$A = \begin{pmatrix} 1 & & & \\ \alpha_1 & 1 & & \\ \vdots & \ddots & \ddots & \\ \alpha_n & \cdots & \alpha_1 & 1 \end{pmatrix}, \text{ then } A^{-1} = \begin{pmatrix} 1 & & & \\ R(1) & 1 & & \\ \vdots & \ddots & \ddots & \\ R(n) & \cdots & R(1) & 1 \end{pmatrix}.$$

From Trudi's formula, it is possible to give the combinatorial expression

$$\alpha_n = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n}(-1)^{n-t_1-\cdots-t_n} R(1)^{t_1} R(2)^{t_2} \cdots R(n)^{t_n}\,.$$

By applying these lemmata to Theorem 4.5, we obtain an explicit expression for the generalized hypergeometric Cauchy numbers $c_{N,n}^{(r)}$. A different version can be seen in [29].

Theorem 6.3. *For $n \ge 1$*

$$c_{N,n}^{(r)} = n! \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n}(-1)^{n-t_1-\cdots-t_n} D_r(1)^{t_1} D_r(2)^{t_2} \cdots D_r(n)^{t_n}\,,$$

where $D_r(e)$ are given in (10)*. Moreover,*

$$D_r(n) = \begin{vmatrix} \frac{c_{N,1}^{(r)}}{1!} & 1 & & \\ \frac{c_{N,2}^{(r)}}{2!} & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \frac{c_{N,n}^{(r)}}{n!} & \cdots & \frac{c_{N,2}^{(r)}}{2!} & \frac{c_{N,1}^{(r)}}{1!} \end{vmatrix},$$

and

$$\begin{pmatrix} 1 & & & & \\ \frac{c_{N,1}^{(r)}}{1!} & 1 & & & \\ \frac{c_{N,2}^{(r)}}{2!} & \frac{c_{N,1}^{(r)}}{1!} & 1 & & \\ \vdots & & \ddots & \ddots & \\ \frac{c_{N,n}^{(r)}}{n!} & \cdots & \frac{c_{N,2}^{(r)}}{2!} & \frac{c_{N,1}^{(r)}}{1!} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & & & & \\ D_r(1) & 1 & & & \\ D_r(2) & D_r(1) & 1 & & \\ \vdots & & \ddots & \ddots & \\ D_r(n) & \cdots & D_r(2) & D_r(1) & 1 \end{pmatrix}.$$

When $r = 1$ in Theorem 6.3, we have an explicit expression for the numbers $c_{N,n}$.

Corollary 6.4. *For $n \ge 1$*

$$c_{N,n} = n! \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{n-t_1-\cdots-t_n}{t_1,\dots,t_n}(-1)^{t_1+\cdots+t_n} \times \left(\frac{N}{N+1}\right)^{t_1} \left(\frac{N}{N+2}\right)^{t_2} \cdots \left(\frac{N}{N+n}\right)^{t_n}$$

and

$$\frac{N}{N+n} = \begin{vmatrix} \frac{c_{N,1}}{1!} & 1 & & \\ \frac{c_{N,2}}{2!} & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \frac{c_{N,n}}{n!} & \cdots & \frac{c_{N,2}}{2!} & \frac{c_{N,1}}{1!} \end{vmatrix},$$

When $r = N = 1$ in Theorem 6.3, we have a different expression of the classical Cauchy numbers.

COROLLARY 6.5. *We have for* $n \geq 1$

$$c_n = n! \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n} (-1)^{n-t_1-\cdots-t_n} \times \left(\frac{1}{2}\right)^{t_1} \left(\frac{1}{3}\right)^{t_2} \cdots \left(\frac{1}{n+1}\right)^{t_n}$$

and

$$\frac{1}{n+1} = \begin{vmatrix} \frac{c_1}{1!} & 1 & & \\ \frac{c_2}{2!} & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \frac{c_n}{n!} & \cdots & \frac{c_2}{2!} & \frac{c_1}{1!} \end{vmatrix}.$$

7. ADDITIONAL COMMENTS

Hypergeometric Cauchy numbers are not integers, but fractions. Hence, combinatorial interpretations of the above results or congruent relations seem to be difficult to obtain. Nevertheless, definition (1) is not obvious or artificial, but has motivations from Combinatorics, in particular, graph theory. In 1989, Cameron [3] considered the operator A defined on the set of sequences of non-negative integers as follows. For $x = \{x_n\}_{n\geq 1}$ and $z = \{z_n\}_{n\geq 1}$, let $Ax = z$, where

$$1 + \sum_{n=1}^{\infty} z_n t^n = \left(1 - \sum_{n=1}^{\infty} x_n t^n\right)^{-1}. \tag{14}$$

For hypergeometric Cauchy numbers, we have

$$x_n = \frac{c_{N,n}}{n!} \quad \text{and} \quad z_n = \frac{(-1)^{n-1}N}{N+n}$$

and vice versa. If x enumerates a class C, then Ax enumerates the class of disjoint unions of members. More concrete examples can be seen in [3].

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