ON COHEN-MACAULAY R-PARTITE GRAPHS

REZA JAFARPOUR-GOLZARI

Communicated by Lucian Beznea

Cohen-Macaulayness of bipartite graphs has been studied by some mathematicians recently. During this researches, all Cohen-Macaulay bipartite graphs have been characterized algebraically and combinatorially. In this note, we give an algebraic necessary and sufficient condition for Cohen-Macaulayness of unmixed r-partite graphs under a certain condition named $(*)$. Also we present a combinatorial necessary condition for Cohen-Macaulayness of an r-partite graph satisfying $(*)$, and we show that this condition is not a sufficient condition.

AMS 2010 Subject Classification: Primary: 05C25; Secondary: 13F55, 05C75.

Key words: Cohen-Macaulay r-partite graph, regular sequence.

1. INTRODUCTION

In the sequel, we refer to [1], [3], [8], and [10] for backgrounds on hypergraphs, monomial ideals, commutative algebra, and graphs, respectively. Also everywhere, the graphs are finite, simple, and without isolated vertices.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For two vertices $u, v \in V(G)$ which are adjacent, we write $v \sim w$. The set of all vertices adjacent to a vertex v, is said to be the neighborhood of v and is denoted by $N(v)$. A subset C of $V(G)$ is called a vertex cover, if every edge of G intersects C in at least one element. A vertex cover C is called minimal if there is no proper subset of C which is a vertex cover. The minimum cardinality of all minimal vertex covers of G, is said to be the vertex covering number of G and is denoted by $\alpha_0(G)$. A minimum vertex cover is a vertex cover of size $\alpha_0(G)$. A graph G is called unmixed if all minimal vertex covers of G have the same size.

For a graph G, a subset T of $V(G)$ is said to be independent if no two elements of T are adjacent. A maximal independent set of G is an independent set I such that there is no other independent set T with $I \subsetneqq T$. Note that T is a maximal independent set of G if and only if $V(G) \setminus T$ is a minimal vertex cover of G. A graph G is called well-covered if all the maximal independent sets of G have the same size. Therefore a graph is unmixed if and only if it is well-covered.

For an integer $r \geq 2$, a graph G is called r-partite, if $V(G)$ can be partitioned into r disjoint parts such that for each edge $\{x, y\}$, x and y do not lie in the same part. If $r = 2, 3$, G is called bipartite and tripartite, respectively. If for every two distinct parts V_i, V_j and for every $x \in V_i$ and $y \in V_j$, $x \sim y$, G is called a complete r-partite graph.

A pure simplicial complex Δ is called completely balanced if there is a partition of its vertex set as $C_1, \ldots, C_r$ such that each facet of Δ intersects each C_i in exactly one element.

Let Δ be a simplicial complex on $[n]$, and let K be a field and $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables with coefficients in Δ. Let I_Δ be the ideal of S generated by all square-free monomials $x_{i_1} \ldots x_{i_s}$ which $\{i_1, \ldots, i_s\} \notin \Delta$. The ring $K[\Delta] := \frac{S}{I_\Delta}$ is called the Stanley-Reisner ring of Δ.

Let G be a graph with $V(G) = \{v_1, \ldots, v_n\}$ and let $S = K[x_1, \ldots, x_n]$. The ideal $I(G)$ of S, generated by all square-free monomials $x_i x_j$ which $\{v_i, v_j\} \in E(G)$, is said to be the edge ideal of G. The quotient ring $R(G) := \frac{S}{I(G)}$ is called the edge ring of G. Define the independence complex of G by

$$\Delta_G := \{F \subseteq V(G) |\ F \textit{ is an independent set of } G\}.$$

Indeed Δ_G is a simplicial complex. Clearly $K[\Delta_G] = R(G)$.

A graph G is called Cohen-Macaulay if $R(G)$ is a Cohen-Macaulay ring, for every field K.

Characterization of special classes of Cohen-Macaulay graphs have been noteworthy in recent decades. J. Herzog and T. Hibi in 2005, gave the following criterion for Cohen-Macaulayness of bipartite graphs [2].

THEOREM 1.1. *Let G be a bipartite graph with parts V_1 and V_2. Then G is Cohen-Macaulay if and only if $|V_1| = |V_2|$ and there is an order on vertices V_1 and V_2 as $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ respectively, such that:*

1) $x_i \sim y_i$, for $i = 1, \ldots, n$,

2) if $x_i \sim y_j$, then $i \leq j$,

3) for each $1 \leq i < j < k \leq n$, if $x_i \sim y_j$ and $x_j \sim y_k$, then $x_i \sim y_k$.

Although the above theorem characterizes all Cohen-Macaulay bipartite graphs, if one wants to prove the Cohen-Macaulayness of a bipartite graph G by means of it, then needs to find an appropriate order on vertices of G, a difficult thing in practice.

R. Zaare-Nahandi in 2015 presented the following combinatorial criterion for Cohen-Macaulayness of a bipartite graph.

THEOREM 1.2 ([11, Theorem 1]). *Let G be a bipartite graph with parts V_1 and V_2. Then, G is Cohen-Macaulay if and only if there is a perfect matching in G as $\{x_1, y_1\}, \ldots, \{x_n, y_n\}$, such that $x_i \in V_1$ and $y_i \in V_2$, for $i = 1, \ldots, n$,* *and two following conditions hold.*

1) *The induced subgraph on* $N(x_i) \cup N(y_i)$ *is a complete bipartite graph, for* $i = 1, \ldots, n$.

2) *If* $x_i \sim y_j$, *then* $x_j \nsim y_i$.

Using the above criterion, there is not difficulty of finding an appropriate order on vertices.

In Theorem 1.2, the condition 1 is equal to unmixedness of the graph G (see [6]).

For proving Theorem 1.2, R. Zaare-Nahandi, first proves the following algebraic criterion for Cohen-Macaulayness of a bipartite graph G.

LEMMA 1.3 ([11, Lemma 2]). *Let* G *be an unmixed bipartite graph with a perfect matching* $\{x_1, y_1\}, \ldots, \{x_n, y_n\}$. *Then* G *is Cohen-Macaulay if and only if the sequence* $\overline{x}_1 + \overline{y}_1, \ldots, \overline{x}_n + \overline{y}_n$ *is a regular sequence in* $R(G)$.

We intend to investigate the Cohen-Macaulayness of r-partite graphs.

In the proof of Theorems 1.1 and 1.2, and also Lemma 1.3, the existence of a perfect matching plays an essential role. According to this and for ease of argument, we restrict ourselves to the class of r-partite graphs which satisfy the following condition.

We say a graph G satisfies the condition $(*)$ for an integer $r \geq 2$, if G can be partitioned into r parts $V_i = \{x_{1i}, \ldots, x_{ni}\}$,$(1 \leq i \leq r)$, such that for all $1 \leq j \leq n$, $\{x_{j1}, x_{j2}, \ldots, x_{jr}\}$ is a clique.

Let G be an r-partite graph which satisfies $(*)$ for $r \geq 2$. By Theorem 4.2 in [5] (where $d = 2$), G is unmixed if and only if no term of sequence $\sum_{i=1}^{r} \overline{x}_{1i}, \ldots, \sum_{i=1}^{r} \overline{x}_{ni}$ is zero-divisor in the ring $R(G)$. This is an algebraic criterion for unmixedness of an r-partite graph satisfying the condition $(*)$.

In this paper, we give first an algebraic criterion, and then a combinatorial necessary condition for Cohen-Macaulayness of an r-partite graph satisfying $(*)$. Also by an appropriate counterexample, we show that this condition is not a sufficient condition.

2. THE MAIN RESULTS

As an algebraic criterion for Cohen-Macaulayness of an r-partite graph G satisfying the condition $(*)$ for $r \geq 2$, we prove the following criterion.

THEOREM 2.1. *Let* G *be an unmixed* r*-partite graph satisfying the condition* $(*)$ *for* $r \geq 2$. *Then* G *is Cohen-Macaulay if and only if the sequence*

$$\sum_{i=1}^{r} \overline{x}_{1i}, \ldots, \sum_{i=1}^{r} \overline{x}_{ni}$$

is a regular sequence in $R(G)$.

Proof. Let G be Cohen-Macaulay. We prove that the sequence

$$\sum_{i=1}^{r}\overline{x}_{1i},\ldots,\sum_{i=1}^{r}\overline{x}_{ni}$$

is a regular sequence in $R(G)$. The cliques $\{x_{j1},x_{j2},\ldots,x_{jr}\}, 1\leq j\leq n$, form a partition of $V(G)$, and every maximal independent set intersects any one of these cliques in exactly one element (because a maximal independent set can not intersect a clique in more than one element and if there is a maximal independent set M which dose not intersect one of the cliques, then $|M|$ is at most $n-1$, a contradiction. Note that the size of all maximal independent sets is the same and equals n, since V_1 is a maximal independent set and G is well-covered). Therefore, the simplicial complex Δ_G is completely balanced. Now by Corollary 4.2 and its Remark in [7],

$$\{\sum_{i=1}^{r}\overline{x}_{1i},\ldots,\sum_{i=1}^{r}\overline{x}_{ni}\}$$

is a homogeneous system of parameters in $K[\Delta_G]$. But $K[\Delta_G]=R(G)$ is Cohen-Macaulay. Then by Theorem 5.9 in [8], the sequence

$$\sum_{i=1}^{r}\overline{x}_{1i},\ldots,\sum_{i=1}^{r}\overline{x}_{ni}$$

is a regular sequence in $R(G)$.

Conversely, let the mentioned sequence be regular. We have

$$\dim(R(G))=\dim(S)-\mathrm{ht}(I(G))=rn-(r-1)n=n,$$

for $S=K[x_{11},\ldots,x_{n1},x_{12},\ldots,x_{n2},\ldots,x_{1r},\ldots,x_{nr}]$, where K is a field and $R(G)=\frac{S}{I(G)}$. Note that by Corollary 7.2.4 in [9], $\mathrm{ht}(I(G))$ is equal to the cardinality of a minimum vertex cover of G and by unmixedness of G and the fact that $\bigcup_{i=1}^{r-1}V_i$ is a minimal vertex cover, this cardinality is $(r-1)n$. Therefore

$$\dim(R(G))\leq \mathrm{depth}(R(G)).$$

Then $\dim(R(G))=\mathrm{depth}(R(G))$, and therefore G is Cohen-Macaulay. □

Now we consider another class of r-partite graphs; We say that a graph G satisfies the condition (▲) for an integer $r\geq 2$, if G is an r- partite graph with parts $V_1,V_2,\ldots,V_r$ such that $|V_1|=|V_2|=\ldots=|V_r|=n$ and:

1) every maximal clique in G is of size r,

2) we can order the vertices of V_i $(1 \leq i \leq r)$ in the form $x_{1i}, x_{2i}, \ldots, x_{ni}$ such that the sequence

$$\sum_{i=1}^{r} \overline{x}_{1i}, \ldots, \sum_{i=1}^{r} \overline{x}_{ni}$$

is a regular sequence in $R(G)$.

The following theorem can be proved similarly to one part of Theorem 2.1. Note that if G satisfies ($\blacktriangle$) for $r \geq 2$, then for every $1 \leq i, i' \leq r$ whith $i \neq i'$, every vertex in V_i is adjacent with at least one vertex in $V_{i'}$, because it lies in a maximal clique.

THEOREM 2.2. *Let G be an unmixed graph satisfying the condition ($\blacktriangle$) for $r \geq 2$. Then G is Cohen-Macaulay.*

Now, we give a combinatorial necessary condition for Cohen-Macaulayness of an r-partite graph satisfying the condition $(*)$ for $r \geq 2$.

THEOREM 2.3. *Let G be an r-partite graph satisfying $(*)$ for $r \geq 2$. If G is Cohen-Macaulay, then for every $1 \leq q, q' \leq n$ with $q \neq q'$, and for every $1 \leq i \leq r$, if for every $1 \leq i'(\neq i) \leq r$, we have $x_{qi} \sim x_{q'i'}$, then there exists some $1 \leq i'(\neq i) \leq r$ such that $x_{q'i} \nsim x_{qi'}$.*

Proof. Suppose the contrary. Then there are distinct integers q and q' and integer $1 \leq i \leq r$ such that for every $1 \leq i'(\neq i) \leq r$,

$$x_{qi} \sim x_{q'i'}, \quad x_{q'i} \sim x_{qi'}.$$

Without loss of generality, we assume that $q < q'$. Now in the ring

$$R' = \frac{R(G)}{(\sum_{t=1}^{r} \overline{x}_{1t}, \ldots, \sum_{t=1}^{r} \overline{x}_{(q'-1)t})}$$

the element $\overline{\overline{x}}_{qi}$ is not zero (here $\overline{\overline{x}}_{qi}$ is the image of $\overline{x}_{qi}$ in R'), because otherwise

$$\overline{x}_{qi} \in (\sum_{t=1}^{r} \overline{x}_{1t}, \ldots, \sum_{t=1}^{r} \overline{x}_{(q'-1)t}),$$

and this means that there are $f_k + I(G)$ in $R(G)$ $(1 \leq k \leq q'-1)$ such that

$$x_{qi} - f_1 \sum_{t=1}^{r} x_{1t} - \cdots - f_{q'-1} \sum_{t=1}^{r} x_{(q'-1)t} \in I(G),$$

and this is impossible. Note that $I(G)$ is a monomial ideal generated by monomials of degree 2.

Now

$$\overline{\overline{x}}_{qi} \sum_{t=1}^{r} \overline{\overline{x}}_{q't} = \overline{\overline{x}}_{qi} \overline{\overline{x}}_{q'i} = - \sum_{t(\neq i)=1}^{r} \overline{\overline{x}}_{qt} \overline{\overline{x}}_{q'i} = 0_{R'}.$$

Therefore the sequence $\sum_{i=1}^{r} \overline{x}_{1t}, \ldots, \sum_{i=1}^{r} \overline{x}_{nt}$ is not regular in $R(G)$, a contradiction (with Theorem 2.1). $\square$

The following example shows that the above necessary condition for Cohen-Macaulayness, is not a sufficient condition.

Example 2.4. Consider the graph G presented in Figure 1.

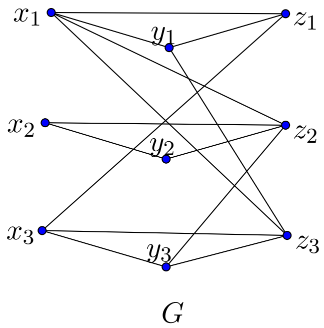

Figure 1

The graph G is 3-partite with parts:

$$V_1 = \{x_1, x_2, x_3\}, \quad V_2 = \{y_1, y_2, y_3\}, \quad V_3 = \{z_1, z_2, z_3\}.$$

Since $\{x_1, y_1, z_1\}, \{x_2, y_2, z_2\}, \{x_3, y_3, z_3\}$ are cliques, G satisfies the condition $(*)$ for $r = 3$.

By Theorem 2.3 in [4], G is unmixed. Also G satisfies 2 in Theorem 2.3. We show that G is not Cohen-Macaulay (Of course this can be check by a suitable mathematical software, too). By Theorem 2.1 it is enough to show that $\overline{\overline{x}}_3 + \overline{\overline{y}}_3 + \overline{\overline{z}}_3$ is a zero-divisor in the ring $R' = \frac{R(G)}{(\overline{x}_1+\overline{y}_1+\overline{z}_1, \overline{x}_2+\overline{y}_2+\overline{z}_2)}$. But we have

$$(\overline{\overline{y}}_1 \overline{\overline{z}}_2)(\overline{\overline{x}}_3 + \overline{\overline{y}}_3 + \overline{\overline{z}}_3) = 0_{R'},$$

because

$$y_1 z_2 (x_3 + y_3 + z_3) - z_2 x_3 (x_1 + y_1 + z_1) \in I(G).$$

Note that $\overline{y}_1 \overline{z}_2 \notin (\overline{x}_1 + \overline{y}_1 + \overline{z}_1, \overline{x}_2 + \overline{y}_2 + \overline{z}_2)$ and therefore $\overline{\overline{y}}_1 \overline{\overline{z}}_2$ is not zero in R'.

Acknowledgments. I would like to thank Rashid Zaare-Nahandi who motivated me to this work. I would also like to thank the referee for his or her useful and valuable comments.

REFERENCES

[1] C. Berge, *Hypergraphs*. Elsevier Science Publishers B. V., 1989.

[2] J. Herzog and T. Hibi, *Distributive lattices, bipartite graphs, and Alexander duality*. J. Algebraic Combin., **22** (2005), *3*, 289–302.

[3] J. Herzog and T. Hibi, *Monomial Ideals*. Graduate Texts in Mathematics 260, Springer, 2011.

[4] R. Jafarpour-Golzari and R. Zaare-Nahandi, *Unmixed r-partite graphs*. Bull. Iran. Math. Soc., **43** (2017), *3*, 781–787.

[5] R. Jafarpour-Golzari and R. Zaare-Nahandi, *Unmixed d-uniform r-partite hypergraphs*. To appear in Ars Combin.

[6] G. Ravindra, *Well-covered graphs*. J. Combin. Inform., System Sci., **2** (1977), *1*, 20–21.

[7] R. P. Stanley, *Balanced Cohen-Macaulay complexes*. Trans. Amer. Math. Soc., **249** (1979), 139–157.

[8] R. P. Stanley, *Combinatorics and Commutative Algebra*. Birkhäuser, Boston, 1983.

[9] R. H. Villarreal, *Monomial Algebras, Second Edition*. Monographs and Research Notes in Mathematics, CRC Press, 2015.

[10] D. West, *Introduction to graph theory*. 2nd Ed., Prentic Hall, Upper Saddle River, NJ, 2001.

[11] R. Zaree-Nahandi, *Cohen-Macaulayness of bipartite graphs, revisited*. Bull. Malays. Math. Sci. Soc., **38** (2015), *4*, 1601–1607.

Received November 25, 2017

Institute for Advanced Studies in Basic Science (IASBS)
Department of Mathematics
P.O.Box 45195-1159, Zanjan, Iran
r.golzary@iasbs.ac.ir