ON THE FOURTH POWER MEAN OF THE GENERALIZED TWO-TERM EXPONENTIAL SUMS

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The main purpose of this paper is to continue our study of a certain mixed character sum and to prove a particular exact formula of a fourth power mean. The method used in this paper involves classical manipulation of exponential sums.

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1. INTRODUCTION

Let $q \geqslant 3$ be a positive integer. For any positive integers $k > h \geqslant 1$ and any integers m and n, the generalized two-term exponential sums $G(m, n, k, h, \chi; q)$ is defined by

$$G(m, n, k, h, \chi; q) = \sum_{a=1}^{q} \chi(a) e\left(\frac{ma^k + na^h}{q}\right),$$

where $e(y) = e^{2\pi i y}$, and χ denotes a Dirichlet character mod q.

Several authors have studied various properties of $G(m, n, k, h, \chi; q)$, and the second author together with H. Liu [8] and D. Han [9] have obtained a series of interesting results. Using a consequence of the very important result of A. Weil [7], in the odd prime modulus p case, one immediately obtains the general estimate

$$|G(m, n, k, h, \chi; p)| \ll_k \sqrt{p},$$

for any integers $m > h \geqslant 1$ with $\gcd(m, p) = 1$ (The subscript in $\ll_k$ means that the involved constant depend only on k).

W. Zhang and H. Liu [8] have studied the fourth power mean of the generalized 3rd Gauss sums with $3|(p-1)$, and obtained a complex but exact closed-form expression. Precisely, they proved the identity

$$\sum_{\chi \bmod p} \left|\sum_{a=1}^{p-1} \chi(a) e\left(\frac{a^3}{p}\right)\right|^4 = 5p^3 - 18p^2 + 20p + 1 + \frac{U^5}{p} + 5pU - 5U^3 - 4U^2 + 4U,$$

where $U = \sum_{a=1}^{p} e\left(\frac{a^3}{p}\right)$ is a real constant.

Some other results related to the two-term exponential sums can also be found in the works [2]-[6], but they will not be repeated here.

Very recently, H. Zhang and W. Zhang [11] also studied the related problem and proved an identity

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p}\right)\right|^4 = \begin{cases} 2p^3-p^2 & \text{if } 3 \nmid p-1, \\ 2p^3-7p^2 & \text{if } 3 | p-1, \end{cases}$$

where p is an odd prime and $\gcd(n,p)=1$.

In this paper, we will consider the following fourth power mean of the generalized two-term exponential sums

$$\sum_{m=1}^{q-1}\left|\sum_{a=1}^{q} \chi(a) e\left(\frac{ma^3+na}{q}\right)\right|^4, \tag{1}$$

where n is any integer with $\gcd(n,q)=1$.

An interesting question that one may ask is whether there exists a closed-form expression or just an asymptotic formula of the sum in (1).

As far as we know, it seems that nobody has studied this problem yet. In this paper, we shall use analytic methods and properties of the classical Gauss sums to study this problem, and obtain an interesting closed-form formula for it, if $q=p$ is an odd prime. The result is detailed in the following theorem.

Theorem. *Let p be an odd prime with $\gcd(3,p-1)=1$, and let n be any integer with $\gcd(n,p)=1$. Then, for any character $\lambda \bmod p$, we have*

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1} \lambda(a) e\left(\frac{ma^3+na}{p}\right)\right|^4 = \begin{cases} 3p^3-8p^2 & \textit{if } \lambda = \left(\frac{*}{p}\right), \\ 2p^3-7p^2 & \textit{if } \lambda \neq \chi_0, \left(\frac{*}{p}\right), \\ 2p^3-3p^2-3p-1 & \textit{if } \lambda = \chi_0, \end{cases}$$

where $\left(\frac{*}{p}\right)$ denotes the Legendre symbol and χ_0 is the principal character mod p.

Some notes: First in our theorem, we only considered the case that p is an odd prime with $\gcd(p-1,3)=1$. If p is an odd prime with $p \equiv 1 \bmod 3$, then the situation is more complex and we can't give an exact formula for (1) at the moment. This will be our subject of further study. Secondly, it is not difficult to prove that $|G(m,n,k,h,\chi;q)|$ is a multiplicative function of q, so naturally we will ask whether there exists a closed-form expression or just an asymptotic formula of (1) with $q=p^\alpha$, a power of odd prime p, where $\alpha \geqslant 2$. This is an open problem.

2. SEVERAL LEMMAS

In this section, we shall present two simple lemmas that are necessary in the proof of our theorem. Hereinafter, we shall use a few basic results from elementary number theory, including the properties of the classical Gauss sums, for which we refer the reader to the introductory books by Apostol [1] or W. Zhang and H. Li [10]. Here we only list a few of them. The classical Gauss sums $\tau(\chi)$ is defined by

$$\tau(\chi) = \sum_{a=1}^{p-1} \chi(a) e\left(\frac{a}{p}\right),$$

and they satisfy the identity

$$\sum_{a=1}^{p-1} \chi(a) e\left(\frac{na}{p}\right) = \overline{\chi}(n)\tau(\chi). \tag{2}$$

If χ is a primitive character mod p, then we have $|\tau(\chi)| = \sqrt{p}$. Now we have:

LEMMA 1. *Let p be an odd prime with* $\gcd(p-1, 3) = 1$, *and let λ be any Dirichlet character* mod p. *Then, for any integer n with* $\gcd(n, p) = 1$, *we have*

$$\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} \lambda(a) e\left(\frac{ma^3 + na}{p}\right) \right|^2 = \begin{cases} p(p-2) & \text{if } \lambda \neq \chi_0; \\ p^2 - p - 1 & \text{if } \lambda = \chi_0. \end{cases}$$

Proof. Since p is an odd prime, so if $\lambda \neq \chi_0$, then λ must be a primitive character mod p. Note that $\gcd(n, p) = 1$ and the identity $|\tau(\lambda)| = \sqrt{p}$, applying the trigonometric identity

$$\sum_{m=1}^{q} e\left(\frac{nm}{q}\right) = \begin{cases} q & \text{if } q \mid n, \\ 0 & \text{if } q \nmid n \end{cases} \tag{3}$$

we have

$$\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} \lambda(a) e\left(\frac{ma^3 + na}{p}\right) \right|^2$$

$$= \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \lambda(a) e\left(\frac{ma^3 + na}{p}\right) \right|^2 - \left| \sum_{a=1}^{p-1} \lambda(a) e\left(\frac{na}{p}\right) \right|^2$$

$$= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \lambda\left(a\bar{b}\right) \sum_{m=0}^{p-1} e\left(\frac{m\left(a^3 - b^3\right) + n(a-b)}{p}\right) - \left| \sum_{a=1}^{p-1} \lambda(a) e\left(\frac{a}{p}\right) \right|^2$$

$$(4) \quad = \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \lambda(a) \sum_{m=0}^{p-1} e\left(\frac{mb^3\left(a^3-1\right)+nb(a-1)}{p}\right) - p.$$

Since $\gcd(3, p-1) = 1$, then the congruence $x^3 \equiv 1 \bmod p$ has only one solution $x \equiv 1 \bmod p$. In fact, since $\gcd(3, p-1) = 1$, then there exists an integer k such that $3k \equiv 1 \bmod (p-1)$. So from Euler theorem we know that for any integer x, we have $x^{3k} \equiv x \bmod p$. If x_1 and x_2 satisfy congruence $x^3 \equiv 1 \bmod p$, then $x_1^3 \equiv x_2^3 \equiv 1 \bmod p$, and $x_1^{3k} \equiv x_2^{3k} \equiv 1 \bmod p$, which implies that $x_1 \equiv x_2 \equiv 1 \bmod p$. So the congruence $x^3 \equiv 1 \bmod p$ has only one solution $x \equiv 1 \bmod p$.

Since $\lambda(1) = 1$, by (3) and (4) it follows that

$$(5) \qquad \sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\lambda(a)e\left(\frac{ma^3+na}{p}\right)\right|^2 = p(p-1)-p = p(p-2).$$

If $\lambda = \chi_0$ is the principal character mod p, then from (3) we have the identity

$$\left|\sum_{a=1}^{p-1}\chi_0(a)e\left(\frac{na}{p}\right)\right|^2 = 1.$$

Applying the same method used to demonstrate relation (5), we obtain

$$(6) \qquad \sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\lambda(a)e\left(\frac{ma^3+na}{p}\right)\right|^2 = p(p-1)-1 = p^2-p-1.$$

Now Lemma 1 follows from identities (5) and (6). □

Lemma 2. *Let p be an odd prime with* $\gcd(p-1, 3) = 1$*, and let λ be any fixed character* mod p*. Then, for any integer n with* $\gcd(n, p) = 1$ *and any non-principal character χ* mod p*, we have*

(7)

$$\left|\sum_{m=1}^{p-1}\chi(m)\left|\sum_{a=1}^{p-1}\lambda(a)e\left(\frac{ma^3+na}{p}\right)\right|^2\right|^2 = p^2\left|\sum_{a=1}^{p-2}\lambda\left(\overline{a}+1\right)\overline{\chi}\left(3a^2+3a+1\right)\right|^2.$$

Proof. From (2) and the properties of reduced residue system mod p we have

$$\sum_{m=1}^{p-1}\chi(m)\left|\sum_{a=1}^{p-1}\lambda(a)e\left(\frac{ma^3+na}{p}\right)\right|^2$$
$$= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\lambda\left(a\overline{b}\right)\sum_{m=1}^{p-1}\chi(m)e\left(\frac{m(a^3-b^3)+n(a-b)}{p}\right)$$

$$= \tau(\chi)\sum_{a=1}^{p-1}\lambda(a)\sum_{b=1}^{p-1}\overline{\chi}\left(b^3(a^3-1)\right)e\left(\frac{nb(a-1)}{p}\right)$$

$$= \tau(\chi)\tau\left(\overline{\chi}^3\right)\sum_{a=1}^{p-1}\lambda(a)\overline{\chi}\left(a^3-1\right)\chi^3\left(a-1\right)\chi^3(n)$$

$$= \tau(\chi)\tau\left(\overline{\chi}^3\right)\chi^3(n)\sum_{a=1}^{p-2}\lambda(a+1)\overline{\chi}\left(a^3+3a^2+3a\right)\chi^3\left(a\right)$$

$$= \tau(\chi)\tau\left(\overline{\chi}^3\right)\chi^3(n)\sum_{a=1}^{p-2}\lambda(a+1)\overline{\chi}\left(1+3\overline{a}+3\overline{a}^2\right)$$

$$(8)\qquad = \tau(\chi)\tau\left(\overline{\chi}^3\right)\chi^3(n)\sum_{a=1}^{p-2}\lambda\left(\overline{a}+1\right)\overline{\chi}\left(3a^2+3a+1\right).$$

Since, by hypothesis $\chi \neq \chi_0$ and $\gcd(p-1,3)=1$, it follows that the order of $\chi \bmod p$ is not equal to three, and also that $\overline{\chi}^3$ is not equal to the principal character mod p. Then, by (8) and $|\tau(\chi)| = \left|\tau\left(\overline{\chi}^3\right)\right| = \sqrt{p}$, we immediately deduce the identity (7). This proves Lemma 2. □

3. PROOF OF THE THEOREM

Now we will complete the proof of our main result. First, for any odd prime p with $\gcd(3,p-1)=1$ and any integer n with $\gcd(n,p)=1$, by the orthogonality of characters mod p, we see that

$$\sum_{\chi \bmod p}\left|\sum_{m=1}^{p-1}\chi(m)\left|\sum_{a=1}^{p-1}\lambda(a)e\left(\frac{ma^3+na}{p}\right)\right|^2\right|^2$$

$$(9)\qquad = (p-1)\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\lambda(a)e\left(\frac{ma^3+na}{p}\right)\right|^4.$$

Then, on the other hand, if $\lambda \neq \chi_0$, note that

$$\sum_{a=1}^{p-2}\lambda\left(\overline{a}+1\right)\chi_0\left(3a^2+3a+1\right) = \sum_{a=1}^{p-2}\lambda\left(\overline{a}+1\right) = \sum_{a=1}^{p-1}\lambda(a)-1 = -1.$$

By Lemmas 1 and 2 we also have

$$\sum_{\chi \bmod p}\left|\sum_{m=1}^{p-1}\chi(m)\left|\sum_{a=1}^{p-1}\lambda(a)e\left(\frac{ma^3+na}{p}\right)\right|^2\right|^2$$

$$= \sum_{\chi\neq\chi_0}\left|\sum_{m=1}^{p-1}\chi(m)\left|\sum_{a=1}^{p-1}\lambda(a)e\left(\frac{ma^3+na}{p}\right)\right|^2\right|^2$$

$$+\left|\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\lambda(a)e\left(\frac{ma^3+na}{p}\right)\right|^2\right|^2$$

$$= p^2\sum_{\chi\neq\chi_0}\left|\sum_{a=1}^{p-2}\lambda\left(\overline{a}+1\right)\overline{\chi}\left(3a^2+3a+1\right)\right|^2+p^2(p-2)^2. \tag{10}$$

$$\sum_{\chi\neq\chi_0}\left|\sum_{a=1}^{p-2}\lambda\left(\overline{a}+1\right)\overline{\chi}\left(3a^2+3a+1\right)\right|^2$$

$$= \sum_{\chi}\left|\sum_{a=1}^{p-2}\lambda\left(\overline{a}+1\right)\overline{\chi}\left(3a^2+3a+1\right)\right|^2-\left|\sum_{a=1}^{p-2}\lambda\left(\overline{a}+1\right)\right|^2$$

$$= (p-1)\sum_{\substack{a=1\\ a^2+a\equiv b^2+b \bmod p}}^{p-2}\sum_{b=1}^{p-2}\lambda\left(\overline{a}+1\right)\overline{\lambda}\left(\overline{b}+1\right)-1$$

$$= (p-1)\sum_{\substack{a=1\\ (a-b)(a+b+1)\equiv 0 \bmod p}}^{p-2}\sum_{b=1}^{p-2}\lambda\left(\overline{a}+1\right)\overline{\lambda}\left(\overline{b}+1\right)-1$$

$$= (p-1)(p-2)+(p-1)\sum_{a=1}^{p-2}\lambda\left(\overline{a}+1\right)\overline{\lambda}\left(-\overline{(a+1)}+1\right)$$

$$-(p-1)\lambda\left(\overline{\left(\frac{p-1}{2}\right)}+1\right)\overline{\lambda}\left(\overline{\left(\frac{p-1}{2}\right)}+1\right)-1, \tag{11}$$

where we have used the identity

$$\sum_{\substack{a=1\\ (a-b)(a+b+1)\equiv 0 \bmod p}}^{p-2}\sum_{b=1}^{p-2}f(a,b)=\sum_{a=1}^{p-2}f(a,a)+\sum_{\substack{a=1\\ a+b+1=p}}^{p-2}\sum_{b=1}^{p-2}f(a,b)-f\left(\frac{p-1}{2},\frac{p-1}{2}\right).$$

Using the properties of multiplicative characters and suitable operations in the multiplicative group of residue classes mod p we also have

$$\sum_{a=1}^{p-2}\lambda\left(\overline{a}+1\right)\overline{\lambda}\left(-\overline{(a+1)}+1\right)=\sum_{a=1}^{p-2}\overline{\lambda}(a)\lambda(a+1)\lambda(a+1)\overline{\lambda}(-1+a+1)$$

$$=\sum_{a=1}^{p-2}\overline{\lambda}^2(a)\lambda^2(a+1)=\sum_{a=1}^{p-2}\lambda^2\left(1+\overline{a}\right)=\sum_{a=1}^{p-2}\lambda^2\left(1+a\right)$$

$$= -1 + \sum_{a=1}^{p-1} \lambda^2(a) = \begin{cases} p-2 & \text{if } \lambda = \left(\frac{*}{p}\right), \\ -1 & \text{if } \lambda \neq \chi_0, \left(\frac{*}{p}\right) \end{cases} \tag{12}$$

and

$$\lambda\left(\overline{\left(\frac{p-1}{2}\right)}+1\right)\overline{\lambda}\left(\overline{\left(\frac{p-1}{2}\right)}+1\right) = 1. \tag{13}$$

Combining (10), (11), (12) and (13) we immediately deduce that

$$\sum_{\chi \bmod p} \left| \sum_{m=1}^{p-1} \chi(m) \left| \sum_{a=1}^{p-1} \lambda(a) e\left(\frac{ma^3+na}{p}\right)\right|^2 \right|^2$$

$$= \begin{cases} p^2(p-1)(3p-8) & \text{if } \lambda = \left(\frac{*}{p}\right), \\ p^2(p-1)(2p-7) & \text{if } \lambda \neq \chi_0, \left(\frac{*}{p}\right), \end{cases} \tag{14}$$

where $\left(\frac{*}{p}\right)$ denotes the Legendre symbol mod p.

Suppose now that $\lambda = \chi_0$. Then, by following the similar steps as in the previous case, applying again Lemmas 1 and 2 , and using the orthogonality of multiplicative characters mod p, we obtain

$$\begin{aligned}
&\sum\nolimits_{\chi \bmod p} \left| \sum_{m=1}^{p-1} \chi(m) \left| \sum_{a=1}^{p-1} \lambda(a) e\left(\frac{ma^3+na}{p}\right)\right|^2 \right|^2 \\
&= \sum_{\chi \neq \chi_0} \left| \sum_{m=1}^{p-1} \chi(m) \left| \sum_{a=1}^{p-1} \lambda(a) e\left(\frac{ma^3+na}{p}\right)\right|^2 \right|^2 \\
&\qquad + \left| \sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} \lambda(a) e\left(\frac{ma^3+na}{p}\right)\right|^2 \right|^2 = \\
&= p^2 \sum_{\chi \neq \chi_0} \left| \sum_{a=1}^{p-2} \lambda\left(\overline{a}+1\right) \overline{\chi}\left(3a^2+3a+1\right)\right|^2 + \left(p^2-p-1\right)^2 \\
&= p^2 \sum_{\chi \bmod p} \left| \sum_{a=1}^{p-2} \overline{\chi}\left(3a^2+3a+1\right)\right|^2 + \left(p^2-p-1\right)^2 - p^2(p-2)^2 \\
&= 2p^2(p-1)(p-2) - p^2(p-1) + \left(p^2-p-1\right)^2 - p^2(p-2)^2 \\
&= (p-1)\left(2p^3-3p^2-3p-1\right).
\end{aligned} \tag{15}$$

The theorem now follows by putting together relations (9), (14) and (15). This completes the proof of our theorem. □

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