FIBONACCI AND LUCAS NUMBERS ASSOCIATED WITH BROCARD-RAMANUJAN EQUATION II

PRAPANPONG PONGSRIIAM

Communicated by Alexandru Zaharescu

We explicitly solve the Diophantine equations of the form

$$A_1A_2A_3\cdots A_n \pm 1 = B_m^{\ell}$$

where $(A_n)_{n\geq 1}$ and $(B_m)_{m\geq 1}$ are the Fibonacci or Lucas sequences and m, n, ℓ are positive integers. This extends some results concerning a Fibonacci version of Brocard-Ramanujan equation.

AMS 2010 Subject Classification: Primary 11B39; Secondary 11D99.

Key words: Fibonacci number, Lucas number, Brocard-Ramanujan equation, Diophantine equation.

1. INTRODUCTION AND STATEMENTS OF MAIN RESULTS

Let $(F_n)_{n\geq 1}$ be the Fibonacci sequence given by $F_1 = F_2 = 1$, and $F_n = F_{n-1}+F_{n-2}$ for $n \geq 3$, and let $(L_n)_{n\geq 1}$ be the Lucas sequence given by the same recursive pattern as the Fibonacci sequence but with the initial values $L_1 = 1$ and $L_2 = 3$. The problem of finding all integral solutions to the Diophantine equation

$$n! + 1 = m^2 \tag{1.1}$$

is known as Brocard-Ramanujan problem. Berndt and Galway [1] use computer programming to check that the only solutions to (1.1) in the range $1 \leq n \leq 10^9$ are $(n, m) = (4, 5), (5, 11)$, and $(7, 71)$. It is still open whether the Brocard-Ramanujan equation has a solution when $n > 10^9$. Some variations of (1.1) have been considered by various authors and we refer the reader to [1], [6], [7], [11] and references therein for additional information and history.

Improving the results of Marques [14] and Szalay [20], Pongsriiam [17], [18] solves the Diophantine equations of the form

$$A_{n_1}A_{n_2}A_{n_3}\cdots A_{n_k} + a = B_m^{\ell}$$

where $(A_n)_{n\geq 1}$ and $(B_m)_{m\geq 1}$ are the Fibonacci or Lucas sequences, $a = \pm 1$, $\ell \in \{1, 2\}$, $1 \leq n_1 \leq n_2 \leq \cdots \leq n_k$, and m is any positive integer. Similar equations where $a = 0$ and ℓ is any positive integer are also solved in [16]. In this article, we continue this kind of investigation. Recall that Marques [15] considers a variant of (1.1) by replacing $n!$ by a Fibotorial number $F_1F_2F_3 \cdots F_n$ and m^2 by a power of Fibonacci number. He claims [15, Theorem 2] that the Diophantine equation

$$F_1F_2F_3 \cdots F_n + 1 = F_m^\ell \tag{1.2}$$

has no solutions in positive integers m, n, and $1 \le \ell \le 10$. But this is wrong, for instance, $F_1 + 1 = F_3$, $F_1F_2 + 1 = F_3$, and $F_1F_2F_3 + 1 = F_4$ give solutions to the above equation. In fact, we can find all solutions to (1.2) in positive integers m, n, and ℓ by applying the result of Bravo, Komatsu, and Luca [3, Corollary 1] which is obtained by referring to lower bounds for linear forms in p-adic logarithms. Nevertheless, it is possible to explicitly solve (1.2) by using only elementary methods. Furthermore, they can also be used to solve the following Diophantine equations:

$$F_1F_2F_3 \cdots F_n + a = L_m^\ell \tag{1.3}$$

$$L_1L_2L_3 \cdots L_n + a = F_m^\ell \tag{1.4}$$

$$L_1L_2L_3 \cdots L_n + a = L_m^\ell \tag{1.5}$$

where $a \in \{1, -1\}$, m, n, and ℓ are arbitrary positive integers. Our main tools are the primitive divisor theorem of Carmichael [5], the distribution of Fibonacci numbers modulo 2^k by Jacobson [8], and the distribution of Lucas numbers modulo 2^k by P. Bundschuh and R. Bundschuh [4]. The solutions to (1.3), (1.4), and (1.5) are as follows.

Theorem 1.1. *The following statements hold.*

(i) *For $a = 1$, the only solutions to* (1.3) *are* $(n, m, \ell) = (3, 2, 1), (4, 4, 1)$.

(ii) *For $a = -1$, the only solutions to* (1.3) *are* $(n, m, \ell) = (5, 7, 1)$ *or* $(n, m, \ell) = (3, 1, \ell)$ *where ℓ is any positive integer.*

(iii) *For $a = 1$, the only solutions to* (1.4) *are* $(n, m, \ell) = (1, 3, 1), (3, 7, 1), (2, 3, 2)$.

(iv) *For $a = -1$, the only solution to* (1.4) *is* $(n, m, \ell) = (2, 3, 1)$.

(v) *For $a = 1$, the only solution to* (1.5) *is* $(n, m, \ell) = (2, 3, 1)$.

(vi) *For $a = -1$, the only solution to* (1.5) *is* $(n, m, \ell) = (3, 5, 1)$.

Finally, we remark that Luca and Shorey [12], [13] obtain a general result which implies that if $t = 0$ or t is not a perfect square is fixed, then the Diophantine equation

$$F_1F_2F_3 \cdots F_n + t = y^\ell$$

with $n \geq 1$, $y \geq 2$, and $\ell \geq 2$ has only finitely many solutions. Nevertheless, this does not give the explicit solutions and it cannot be applied in our situation because 1 is a perfect square.

2. PRELIMINARIES AND LEMMAS

Since one of the main tools in solving the above equations is the primitive divisor theorem of Carmichael [5], we first recall some facts about it. Let α and β be algebraic numbers such that $\alpha + \beta$ and $\alpha\beta$ are nonzero coprime integers and $\alpha\beta^{-1}$ is not a root of unity. Let $(u_n)_{n\geq 0}$ be the sequence given by

$$u_0 = 0,\ u_1 = 1, \text{ and } u_n = (\alpha+\beta)u_{n-1} - (\alpha\beta)u_{n-2} \text{ for } n \geq 2.$$

Then we have Binet's formula for u_n given by

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{for } n \geq 0.$$

So if $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, then (u_n) is the Fibonacci sequence.

A prime p is said to be a primitive divisor of u_n if $p \mid u_n$ but p does not divide $u_1 u_2 \cdots u_{n-1}$. Then the primitive divisor theorem of Carmichael can be stated as follows.

THEOREM 2.1 (Primitive Divisor Theorem of Carmichael [5])). *If α and β are real numbers and $n \neq 1, 2, 6$, then u_n has a primitive divisor except when $n = 12$, $\alpha + \beta = 1$ and $\alpha\beta = -1$.*

There is a long history about primitive divisors and the most remarkable results in this topic are given by Bilu, Hanrot, and Voutier [2], by Stewart [19], and by Kunrui [9], but Theorem 2.1 is good enough in our situation.

Recall that we can define F_n and L_n for a negative integer n by the formula

$$F_{-k} = (-1)^{k+1}F_k \text{ and } L_{-k} = (-1)^k L_k \text{ for } k \geq 0.$$

Then we have the following identities which valid for all integers m, k.

$$F_{m-k}F_{m+k} = F_m^2 + (-1)^{m-k+1}F_k^2. \tag{2.1}$$

$$F_m L_k = F_{m+k} + (-1)^k F_{m-k}. \tag{2.2}$$

The identities (2.1) and (2.2) can be proved using Binet's formula and straightforward algebraic manipulation, see for example in [17], [18]. We will particularly apply (2.1) and (2.2) in the following form.

LEMMA 2.2. *For every $m \geq 1$, we have*

(i) $F_m^2 - 1 = \begin{cases} F_{m-1}F_{m+1}, & \textit{if } m \textit{ is odd;} \\ F_{m-2}F_{m+2}, & \textit{if } m \textit{ is even.} \end{cases}$

(ii) $F_m^2 + 1 = \begin{cases} F_{m-1}F_{m+1}, & \textit{if } m \textit{ is even;} \\ F_{m-2}F_{m+2}, & \textit{if } m \textit{ is odd.} \end{cases}$

(iii) $F_m - 1 = \begin{cases} F_{\frac{m+2}{2}}L_{\frac{m-2}{2}}, & \textit{if } m \equiv 0 \pmod 4; \\ F_{\frac{m-1}{2}}L_{\frac{m+1}{2}}, & \textit{if } m \equiv 1 \pmod 4; \\ F_{\frac{m-2}{2}}L_{\frac{m+2}{2}}, & \textit{if } m \equiv 2 \pmod 4; \\ F_{\frac{m+1}{2}}L_{\frac{m-1}{2}}, & \textit{if } m \equiv 3 \pmod 4. \end{cases}$

(iv) $F_m + 1 = \begin{cases} F_{\frac{m-2}{2}}L_{\frac{m+2}{2}}, & \textit{if } m \equiv 0 \pmod 4; \\ F_{\frac{m+1}{2}}L_{\frac{m-1}{2}}, & \textit{if } m \equiv 1 \pmod 4; \\ F_{\frac{m+2}{2}}L_{\frac{m-2}{2}}, & \textit{if } m \equiv 2 \pmod 4; \\ F_{\frac{m-1}{2}}L_{\frac{m+1}{2}}, & \textit{if } m \equiv 3 \pmod 4. \end{cases}$

Proof. This follows immediately from (2.2). For example, if m is even, replacing m by $\frac{m+2}{2}$ and k by $\frac{m-2}{2}$ in (2.2), we obtain

$$F_{\frac{m+2}{2}}L_{\frac{m-2}{2}} = F_m + (-1)^{\frac{m-2}{2}}F_2,$$

which is equal to $F_m - 1$ if $m \equiv 0 \pmod 4$ and is equal to $F_m + 1$ if $m \equiv 2 \pmod 4$. $\square$

We also need a factorization of $L_m \pm 1$ and $L_m^2 \pm 1$ as follows.

LEMMA 2.3. *For every $m \geq 1$, we have*

(i) $L_m^2 - 1 = \begin{cases} F_{3m}/F_m, & \textit{if } m \textit{ is even;} \\ 5F_{m-1}F_{m+1}, & \textit{if } m \textit{ is odd.} \end{cases}$

(ii) $L_m^2 + 1 = \begin{cases} F_{3m}/F_m, & \textit{if } m \textit{ is odd;} \\ 5F_{m-1}F_{m+1}, & \textit{if } m \textit{ is even.} \end{cases}$

(iii) $L_m - 1 = \begin{cases} L_{\frac{3m}{2}}/L_{\frac{m}{2}}, & \textit{if } m \equiv 0 \pmod 4; \\ 5F_{\frac{m+1}{2}}F_{\frac{m-1}{2}}, & \textit{if } m \equiv 1 \pmod 4; \\ F_{\frac{3m}{2}}/F_{\frac{m}{2}}, & \textit{if } m \equiv 2 \pmod 4; \\ L_{\frac{m+1}{2}}L_{\frac{m-1}{2}}, & \textit{if } m \equiv 3 \pmod 4. \end{cases}$

(iv) $$L_m + 1 = \begin{cases} F_{\frac{3m}{2}}/F_{\frac{m}{2}}, & \text{if } m \equiv 0 \pmod 4; \\ L_{\frac{m+1}{2}} L_{\frac{m-1}{2}}, & \text{if } m \equiv 1 \pmod 4; \\ L_{\frac{3m}{2}}/L_{\frac{m}{2}}, & \text{if } m \equiv 2 \pmod 4; \\ 5F_{\frac{m+1}{2}} F_{\frac{m-1}{2}}, & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

Proof. Similar to (2.1), this can be checked easily using Binet's formula. □

Recall that for each $m \in \mathbb{N}$, the p-adic valuation of m, denoted by $v_p(m)$, is the exponent of p in the prime factorization of m. In addition, the order of appearance of m in the Fibonacci sequence, denoted by $z(m)$, is the smallest positive integer k such that $m \mid F_k$. Then we have the following result.

Lemma 2.4 (Lengyel [10]). *For every $n \geq 1$, we have*

$$v_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 3; \\ 1, & \text{if } n \equiv 3 \pmod 6; \\ v_2(n) + 2, & \text{if } n \equiv 0 \pmod 6, \end{cases}$$

$$v_2(L_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 3; \\ 2, & \text{if } n \equiv 3 \pmod 6; \\ 1, & \text{if } n \equiv 0 \pmod 6, \end{cases}$$

$v_5(F_n) = v_5(n)$, $v_5(L_n) = 0$, *and if p is a prime, $p \neq 2$ and $p \neq 5$, then*

$$v_p(F_n) = \begin{cases} v_p(n) + v_p(F_{z(p)}), & \text{if } n \equiv 0 \pmod{z(p)}; \\ 0, & \text{otherwise.} \end{cases}$$

$$v_p(L_n) = \begin{cases} v_p(n) + v_p(F_{z(p)}), & \text{if } z(p) \text{ is even and } n \equiv \frac{z(p)}{2} \pmod{z(p)}; \\ 0, & \text{otherwise.} \end{cases}$$

We also refer the reader to Ward's articles such as [22] for a general result concerning with prime divisors of the Fibonacci and Lucas numbers. Next we apply Lemma 2.4 to obtain the 2-adic valuations of the products $F_1F_2F_3 \cdots F_n$ and $L_1L_2L_3 \cdots L_n$.

Lemma 2.5. *For every $n \geq 1$, we have*

(i) $$v_2(F_1F_2F_3 \cdots F_n) = \left\lfloor \frac{n+3}{6} \right\rfloor + 3\left\lfloor \frac{n}{6} \right\rfloor + v_2\left(\left\lfloor \frac{n}{6} \right\rfloor !\right),$$

(ii) $$v_2(L_1L_2L_3 \cdots L_n) = 2\left\lfloor \frac{n+3}{6} \right\rfloor + \left\lfloor \frac{n}{6} \right\rfloor.$$

Proof. By Lemma 2.4, we obtain

$$\begin{aligned} v_2(F_1F_2F_3\cdots F_n) &= \sum_{k\le\frac{n+3}{6}} v_2\left(F_{6k-3}\right) + \sum_{k\le\frac{n}{6}} v_2\left(F_{6k}\right) \\ &= \left\lfloor \frac{n+3}{6} \right\rfloor + \sum_{k\le\frac{n}{6}} (v_2(6k)+2) \\ &= \left\lfloor \frac{n+3}{6} \right\rfloor + 3\left\lfloor \frac{n}{6} \right\rfloor + \sum_{k\le\frac{n}{6}} v_2(k) \\ &= \left\lfloor \frac{n+3}{6} \right\rfloor + 3\left\lfloor \frac{n}{6} \right\rfloor + v_2\left(\left\lfloor \frac{n}{6} \right\rfloor !\right). \end{aligned}$$

Similarly, part (ii) can be proved by using Lemma 2.4. □

Wall [21] proves that any recurrence sequence (a_n) given by $a_n = a_{n-1} + a_{n-2}$ with initial values $a_1, a_2 \in \mathbb{Z}$ is simply periodic if reduced modulo any $m \in \mathbb{N}$. So in particular, the sequences $(F_n \bmod m)_{n\ge1}$ and $(L_n \bmod m)_{n\ge1}$ are simply periodic. For each $m \in \mathbb{N}$, let $s_F(m)$ and $s_L(m)$ be the (shortest) period of $(F_n \bmod m)_{n\ge1}$ and $(L_n \bmod m)_{n\ge1}$, respectively. In addition, for a modulus $m \ge 2$ and a residue $b \bmod m$, let $v_F(m,b)$ and $v_L(m,b)$ be the number of occurrences of b as a residue in a period of the Fibonacci sequence modulo m and in a period of the Lucas sequence modulo m, respectively.

For example, the sequence $(F_n \bmod 4)_{n\ge1}$ is $1,1,2,3,1,0$, and repeated. So $s_F(4) = 6$, $v_p(4,1) = 3$, $v_p(4,2) = v_p(4,3) = v_p(4,0) = 1$. It is well-known that $s_F(2^k) = s_L(2^k) = 3\cdot 2^{k-1}$ (see for instance, a general result obtained by Wall [21, Theorems 5 and 9]). Furthermore, Jacobson [8] obtains the result concerning with the distribution of $F_n \bmod 2^k$ as follows.

LEMMA 2.6 (Jacobson [8]). *For $k \ge 5$, we have*

$$v_F(2^k,b) = \begin{cases} 1, & \text{if } b \equiv 3 \pmod 4; \\ 2, & \text{if } b \equiv 0 \pmod 8; \\ 3, & \text{if } b \equiv 1 \pmod 4; \\ 8, & \text{if } b \equiv 2 \pmod{32}; \\ 0, & \text{otherwise.} \end{cases}$$

By using Jacobson's result, it is easy to solve the congruences $F_m \equiv 1 \pmod{2^k}$ and $F_m \equiv -1 \pmod{2^k}$ as follows.

LEMMA 2.7. *For $m \ge 1$, we have*

(i) $F_m \equiv 1 \pmod{2^k}$ *if and only if* $m \equiv 1, 2, 3\cdot 2^{k-1} - 1 \pmod{3\cdot 2^{k-1}}$,

(ii) $F_m \equiv -1 \pmod{2^k}$ *if and only if* $m \equiv 3 \cdot 2^{k-1} - 2 \pmod{3 \cdot 2^{k-1}}$.

Proof. As mentioned above, $s_F(2^k) = 3 \cdot 2^{k-1}$. Therefore $F_m \equiv F_\ell \pmod{2^k}$ if and only if $m \equiv \ell \pmod{3 \cdot 2^{k-1}}$. By Lemma 2.6, we have $v(2^k, 1) = 3$. So there are exactly three values of m such that $1 \leq m \leq 3 \cdot 2^{k-1}$ and $F_m \equiv 1 \pmod{2^k}$. Two obvious values of m are $m = 1$ and $m = 2$. By the property of the period, we have

$$F_{3 \cdot 2^{k-1}} \equiv F_0 \equiv 0 \pmod{2^k} \text{ and } F_{3 \cdot 2^{k-1}+1} \equiv F_1 \equiv 1 \pmod{2^k},$$

which implies $F_{3 \cdot 2^{k-1}-1} = F_{3 \cdot 2^{k-1}+1} - F_{3 \cdot 2^{k-1}} \equiv 1 \pmod{2^k}$. Therefore

$$F_m \equiv 1 \pmod{2^k} \text{ if and only if } m \equiv 1, 2, 3 \cdot 2^{k-1} - 1 \pmod{3 \cdot 2^{k-1}}.$$

In addition, we have

$$F_{3 \cdot 2^{k-1}-2} = F_{3 \cdot 2^{k-1}} - F_{3 \cdot 2^{k-1}-1} \equiv -1 \pmod{2^k}.$$

By Lemma 2.6, $v(2^k, -1) = 1$. So we see that

$$F_m \equiv -1 \pmod{2^k} \text{ if and only if } m \equiv 3 \cdot 2^{k-1} - 2 \pmod{3 \cdot 2^{k-1}}.$$

This completes the proof. $\square$

The distribution of L_n mod 2^k is obtained by P. Bundschuh and R. Bundschuh [4]. In fact, they obtain more but we only need Theorem 2 in [4].

Lemma 2.8 (P. Bundschuh and R. Bundschuh [4, Theorem 2]). *Suppose $k \in \mathbb{N}$ and $k \geq 3$. Then for each b in the least nonnegative residue system mod 2^k, one has*

$$v_L(2^k, b) = \begin{cases} 1, & \text{if } b \equiv 1 \pmod{4}; \\ 3, & \text{if } b \equiv 3 \pmod{4}; \\ 2, & \text{if } b \equiv 4 \pmod{8}; \\ 2^{\lfloor \frac{k}{2} \rfloor}, & \text{if } b = 2; \\ 2^{\lfloor \frac{k}{2} \rfloor}, & \text{if } b \equiv 2^{2\lfloor \frac{k-1}{2} \rfloor} + 2 \text{ and } k \geq 5; \\ 16, & \text{if } b \equiv 18 \pmod{128} \text{ and } k \geq 7; \\ 2^\ell, & \text{if } b \equiv 5 \cdot 2^{2\ell-4} + 2 \pmod{2^{2\ell-1}} \text{ for some } \ell \in \left\{5, \ldots, \left\lfloor \frac{k+1}{2} \right\rfloor\right\}; \\ 0, & \text{otherwise.} \end{cases}$$

By the above lemma, we can easily solve the congruence $L_m \equiv 1 \pmod{2^k}$ and $L_m \equiv -1 \pmod{2^k}$ as follows.

Corollary 2.9. *For every $k \geq 3$, we have*

(i) $L_m \equiv 1 \pmod{2^k}$ *if and only if* $m \equiv 1 \pmod{3 \cdot 2^{k-1}}$,

(ii) $L_m \equiv -1 \pmod{2^k}$ *if and only if* $m \equiv 2^{k-1}, 2^k, 3\cdot 2^{k-1}-1 \pmod{3\cdot 2^{k-1}}$.

Proof. As mentioned earlier, we have $s_L(2^k) = 3 \cdot 2^{k-1}$. So we see that $L_m \equiv L_r \pmod{2^k}$ if and only if $m \equiv r \pmod{3 \cdot 2^{k-1}}$. So if $m \equiv 1 \pmod{3 \cdot 2^{k-1}}$, then $L_m \equiv L_1 \equiv 1 \pmod{2^k}$. By Lemma 2.8, there is exactly one $m \pmod{3 \cdot 2^{k-1}}$ with $L_m \equiv 1 \pmod{2^k}$. This proves (i). Again, by Lemma 2.8, there are exactly three $m \pmod{3\cdot 2^{k-1}}$ with $L_m \equiv -1 \pmod{2^k}$. So to prove (ii), it is enough to show that

$$L_{2^{k-1}} \equiv L_{2^k} \equiv L_{3\cdot 2^{k-1}-1} \equiv -1 \pmod{3 \cdot 2^{k-1}}.$$

By the property of period, we have

$$L_{3\cdot 2^{k-1}} \equiv L_0 \equiv 2 \pmod{2^k} \quad \text{and} \quad L_{3\cdot 2^{k-1}+1} \equiv L_1 \equiv 1 \pmod{2^k},$$

which implies

$$L_{3\cdot 2^{k-1}-1} = L_{3\cdot 2^{k-1}+1} - L_{3\cdot 2^{k-1}} \equiv -1 \pmod{2^k}.$$

So it remains to show that

$$L_{2^{k-1}} \equiv L_{2^k} \equiv -1 \pmod{2^k}. \tag{2.3}$$

We will prove (2.3) by induction on $k \geq 3$. It is easy to check that (2.3) holds when $k = 3$. So assume that $k \geq 3$ and (2.3) holds for k. Then $L_{2^{k-1}} = -1+2^k\ell$ for some $\ell \in \mathbb{Z}$. Recall the identity that $L_n^2 = L_{2n} + 2(-1)^n$, which can be verified by using Binet's formula. Therefore

$$L_{2^k} = (L_{2^{k-1}})^2 - 2 = (-1 + 2^k\ell)^2 - 2 \equiv -1 \pmod{2^{k+1}}, \quad \text{and}$$
$$L_{2^{k+1}} = (L_{2^k})^2 - 2 \equiv (-1)^2 - 2 \equiv -1 \pmod{2^{k+1}}.$$

This completes the proof. □

3. PROOF OF MAIN RESULTS

We divide the proof of Theorem 1.1 into three steps and call them Theorems 3.1, 3.2, and 3.3, respectively. Then we combine Theorems 3.1, 3.2, and 3.3 with a general result in [17], [18] to obtain Theorem 1.1.

THEOREM 3.1. *Let* $a \in \{1, -1\}$, $m \geq 2$, *and* $\ell \geq 3$. *Then the Diophantine equation*

$$F_1F_2F_3\cdots F_n + a = L_m^\ell \tag{3.1}$$

has no solution in positive integers $n \geq 1$.

Proof. Using a basic command in MAPLE, it is easy to check that the left hand side of (3.1) is squarefree for $a = 1$ and $1 \le n \le 25$. For $a = -1$, the left hand side of (3.1) is also squarefree for every $3 \le n \le 25$ except when $n = 13$, where we have

$$F_1F_2F_3 \cdots F_{13} - 1 = (23)(43^2)(8603183137).$$

In any case, they are not perfect ℓ power. Suppose for a contradiction that there exists $n \ge 26$ satisfying (3.1). Recall the well-known identity that $F_{2k} = F_kL_k$. So if $m \le \frac{n}{2}$, then

$$L_m \mid F_{2m} \mid F_1F_2F_3 \cdots F_n,$$

and so $3 \le L_m \mid L_m^\ell - F_1F_2F_3 \cdots F_n = a$, which is a contradiction. Therefore

$$m > \frac{n}{2}. \tag{3.2}$$

Since $n \ge 26$, we see that $m > 13$. Next, let $A = \frac{L_m^\ell - a}{L_m - a}$. Then (3.1) can be written as

$$F_1F_2F_3 \cdots F_n = (L_m - a)A. \tag{3.3}$$

Case 1. $a = 1$. Then $A \in \mathbb{Z}$ and by Lemma 2.3(iii) and the identity $F_{2k} = F_kL_k$, we can rewrite (3.3) according to the residue class of m modulo 4 as follows:

$$\text{for } m \equiv 0 \pmod 4,\ F_1F_2 \cdots F_nF_{\frac{3m}{2}}F_m = F_{3m}F_{\frac{m}{2}}A \tag{3.4}$$

$$\text{for } m \equiv 1 \pmod 4,\ F_1F_2 \cdots F_n = 5F_{\frac{m+1}{2}}F_{\frac{m-1}{2}}A \tag{3.5}$$

$$\text{for } m \equiv 2 \pmod 4,\ F_1F_2 \cdots F_nF_{\frac{m}{2}} = F_{\frac{3m}{2}}A \tag{3.6}$$

$$\text{for } m \equiv 3 \pmod 4,\ F_1F_2 \cdots F_nF_{\frac{m+1}{2}}F_{\frac{m-1}{2}} = F_{m+1}F_{m-1}A. \tag{3.7}$$

For (3.4), if $3m > n$, then by Theorem 2.1, there exists a prime p dividing F_{3m} but p does not divide the left hand side of (3.4), which is not the case. So $3m \le n$. Similarly $\frac{m+1}{2} \le n$ in (3.5), $\frac{3m}{2} \le n$ in (3.6), and $m + 1 \le n$ in (3.7). In any case,

$$m \le 2n - 1. \tag{3.8}$$

Next we consider (3.1) in the following 3 subcases.

Case 1.1. $a = 1$, ℓ is even, and m is even. By Lemma 2.3(i), we have

$$\frac{F_{3m}}{F_m} = L_m^2 - 1 \mid L_m^\ell - 1 = F_1F_2F_3 \cdots F_n.$$

So $F_{3m} \mid F_1F_2F_3 \cdots F_nF_m$. But by (3.2), we have $3m > n$ and we obtain by Theorem 2.1 that there exists a prime p dividing F_{3m} but p does not divide $F_1F_2F_3 \cdots F_nF_m$, a contradiction. So there is no solution in this case.

Case 1.2. $a = 1$, ℓ is even, and m is odd. If $4 \mid \ell$, then by Lemma 2.3(ii), we obtain

$$\frac{F_{3m}}{F_m} = L_m^2 + 1 \mid L_m^4 - 1 \mid L_m^\ell - 1 = F_1 F_2 \cdots F_n,$$

which leads to a contradiction in the same way as in Case 1.1. So assume that $\ell = 2\ell_0$ and ℓ_0 is odd. Since $\ell \geq 3$, we have $\ell_0 \geq 3$. By Lemma 2.3(i), we obtain

$$5F_{m-1}F_{m+1} = L_m^2 - 1 \mid L_m^\ell - 1 = F_1 F_2 F_3 \cdots F_n. \tag{3.9}$$

If $m+1 > n$, then by Theorem 2.1, there is a prime p dividing F_{m+1} but p does not divide $F_1 F_2 \cdots F_n$, which contradicts (3.9). So $m + 1 \leq n$. Let $x = L_m^2$. Then we have

$$L_m^\ell - 1 = x^{\ell_0} - 1 = (x - 1)\left(x^{\ell_0 - 1} + x^{\ell_0 - 2} + \cdots + x + 1\right),$$

and the factor $x^{\ell_0-1} + x^{\ell_0-2} + \cdots + x + 1$ is odd because ℓ_0 is odd. This and Lemma 2.3(i) lead to

$$v_2(L_m^\ell - 1) = v_2(x - 1) = v_2(L_m^2 - 1) = v_2(F_{m-1}F_{m+1}). \tag{3.10}$$

Since m is odd, 2 divides both $m-1$ and $m+1$, and because $(m+1)-(m-1) = 2$, 3 divides at most one of them. By Lemma 2.4, we see that

$$v_2(F_{m-1}F_{m+1}) \leq \max\{v_2(m-1) + 2, v_2(m+1) + 2\}. \tag{3.11}$$

For any $k \in \mathbb{N}$, we have the implication

$$v_2(k) = a \Rightarrow 2^a \mid k \Rightarrow 2^a \leq k \Rightarrow a \leq \frac{\log k}{\log 2}.$$

So in particular,

$$\max\{v_2(m-1) + 2, v_2(m+1) + 2\} \leq \frac{\log(m+1)}{\log 2} + 2. \tag{3.12}$$

From (3.10), (3.11), (3.12), and (3.8), we obtain

$$v_2(L_m^\ell - 1) \leq \frac{\log(m+1)}{\log 2} + 2 \leq \frac{\log n}{\log 2} + 3. \tag{3.13}$$

On the other hand, $v_2(L_m^\ell - 1) = v_2(F_1 F_2 F_3 \cdots F_n)$. So we obtain by Lemma 2.5 that

$$v_2\left(L_m^\ell - 1\right) = \left\lfloor \frac{n+3}{6} \right\rfloor + 3\left\lfloor \frac{n}{6} \right\rfloor + v_2\left(\left\lfloor \frac{n}{6} \right\rfloor !\right).$$

By Legendre's formula, we know that $v_2(m!) = \sum_{k=1}^{\infty} \left\lfloor \frac{m}{2^k} \right\rfloor$ for any $m \in \mathbb{N}$. It is also well-known that $\left\lfloor \frac{\lfloor x \rfloor}{m} \right\rfloor = \left\lfloor \frac{x}{m} \right\rfloor$ for every $x \in \mathbb{R}$ and $m \in \mathbb{N}$. So in particular,

$$v_2\left(\left\lfloor \frac{n}{6} \right\rfloor !\right) \geq \left\lfloor \frac{\left\lfloor \frac{n}{6} \right\rfloor}{2} \right\rfloor = \left\lfloor \frac{n}{12} \right\rfloor \geq \frac{n}{12} - 1.$$

So the above inequality implies that

$$v_2(L_m^\ell - 1) \geq \frac{n+3}{6} - 1 + 3\left(\frac{n}{6} - 1\right) + \frac{n}{12} - 1 > \frac{3n}{4} - 5. \tag{3.14}$$

By (3.13) and (3.14), we obtain

$$\frac{3n}{4} - 5 < \frac{\log n}{\log 2} + 3. \tag{3.15}$$

However, by considering the function $f : [25, \infty) \to \mathbb{R}$ given by

$$f(x) = \left(\frac{3x}{4} - 5\right) - \left(\frac{\log x}{\log 2} + 3\right),$$

we see that $f'(x) > 0$ for all $x \geq 25$. So f is strictly increasing on $[25, \infty)$. Since $n > 25$, we obtain $f(n) > f(25) > 0$. That is

$$\frac{3n}{4} - 5 > \frac{\log n}{\log 2} + 3,$$

which contradicts (3.15). So there is no solution in this case.

Case 1.3. $a = 1$ and ℓ is odd. Since $n \geq 26$, we obtain by Lemma 2.5 that

$$v_2(F_1F_2F_3 \cdots F_n) \geq v_2(F_1F_2F_3 \cdots F_{26}) = 19.$$

Therefore

$$L_m^\ell \equiv 1 \pmod{2^{19}}. \tag{3.16}$$

For any $x \in \mathbb{Z}$, we have

$$x^\ell - 1 = (x-1)(x^{\ell-1} + x^{\ell-2} + \cdots + x + 1),$$

and since ℓ is odd, the second factor on the right hand side of the above equation is odd. Therefore $2^{19} \mid x^\ell - 1$ if and only if $2^{19} \mid x - 1$. From this, we see that (3.16) is equivalent to

$$L_m \equiv 1 \pmod{2^{19}}. \tag{3.17}$$

By Corollary 2.9(i) and the fact that $m > 1$, we obtain $m > 3 \cdot 2^{18}$. Then by (3.8), we obtain $n \geq \frac{m+1}{2} > 3 \cdot 2^{17}$. Now we can repeat this process as follows. By Lemma 2.5, we have

$$v_2(F_1F_2F_3 \cdots F_n) \geq v_2(F_1F_2F_3 \cdots F_{3\cdot 2^{17}}) \geq 2^{18} =: b.$$

Then (3.16) and (3.17) become $L_m^\ell \equiv 1 \pmod{2^b}$ and $L_m \equiv 1 \pmod{2^b}$, respectively. Applying Corollary 2.9(i) and (3.8), respectively, we obtain

$$m > 3 \cdot 2^{b-1} \text{ and } n > 3 \cdot 2^{b-2} \geq 3 \cdot 2^{2^{18}-2}.$$

Repeating this process, we see that $n \geq M$ for any given positive integer M, a contradiction.

From Cases 1.1, 1.2, and 1.3, we see that (3.1) has no solutions when $a = 1$. Next we consider (3.1) when $a = -1$.

Case 2. $a = -1$ and ℓ is even. Then L_m^ℓ is a perfect square. So $L_m^\ell + 1 \equiv 1, 2 \pmod 4$, contradicting the fact that $L_m^\ell + 1 = F_1F_2F_3 \cdots F_n$ which is divisible by 4.

Case 3. $a = -1$ and ℓ is odd. Then A in (3.3) is an integer. Then we can apply Lemma 2.3(iv) to obtain the same set of equations (3.6), (3.7), (3.4), and (3.5), respectively. So we can use the same argument to obtain (3.8). That is $m \leq 2n - 1$. Then we can follow the argument in Case 1.3 to obtain, respectively,

$$v_2(F_1F_2F_3 \cdots F_n) \geq 19 \text{ and } L_m^\ell \equiv -1 \pmod{2^{19}}.$$

Since ℓ is odd, we see that for any $x \in \mathbb{Z}$,

$$x^\ell + 1 = (x+1)\left(x^{\ell-1} - x^{\ell-2} + x^{\ell-3} - x^{\ell-4} + \cdots - x + 1\right)$$

and the second factor on the right hand side of the above equation is odd. So we can still use the argument in Case 1.3 to obtain

$$L_m \equiv -1 \pmod{2^{19}}.$$

Then by Corollary 2.9(ii), we obtain $m \geq 2^{18}$. So $n \geq \frac{m+1}{2} > 2^{17}$. Then repeating this process as in Case 1.3, we reach a contradiction. This completes the proof. $\square$

THEOREM 3.2. *Let $a \in \{1, -1\}$ and $\ell \geq 3$. Then the Diophantine equation*

$$L_1L_2L_3 \cdots L_n + a = L_m^\ell \tag{3.18}$$

has no solution in positive integers m, n.

Proof. Since some parts of the proof is similar to those in Theorem 3.1, we leave some details to the reader. We first check using MAPLE that (3.18) has no solutions when $1 \leq n \leq 25$ or when $m = 1$. So we suppose $n \geq 26$ and $m \geq 2$. If $m \leq n$, then we would have

$$3 \leq L_m \mid L_m^\ell - L_1L_2L_3 \cdots L_n = a,$$

which is a contradiction. Therefore

$$m > n. \tag{3.19}$$

Let $A = \frac{L_m^\ell - a}{L_m - a}$. We first consider the case $a = 1$. Then $A \in \mathbb{Z}$. Similar to the proof of Theorem 3.1, we apply Lemma 2.3(iii) and the identity $F_{2k} = F_kL_k$ to write (3.18) according to the residue class of m modulo 4 as follows:

$$\text{for } m \equiv 0 \pmod 4,\ \frac{F_2}{F_1}\frac{F_4}{F_2}\frac{F_6}{F_3}\cdots\frac{F_{2n}}{F_n}\frac{F_m}{F_{\frac{m}{2}}}F_{\frac{3m}{2}} = F_{3m}A, \tag{3.20}$$

$$\text{(3.21)} \qquad \text{for } m \equiv 1 \pmod 4,\ L_1L_2L_2\cdots L_n = 5F_{\frac{m+1}{2}}F_{\frac{m-1}{2}}A,$$

$$\text{(3.22)} \qquad \text{for } m \equiv 2 \pmod 4,\ \frac{F_2}{F_1}\frac{F_4}{F_2}\frac{F_6}{F_3}\cdots\frac{F_{2n}}{F_n}F_{\frac{m}{2}} = F_{\frac{3m}{2}}A,$$

$$\text{(3.23)} \qquad \text{for } m \equiv 3 \pmod 4,\ \frac{F_2}{F_1}\frac{F_4}{F_2}\frac{F_6}{F_3}\cdots\frac{F_{2n}}{F_n}F_{\frac{m+1}{2}}F_{\frac{m-1}{2}} = F_{m+1}F_{m-1}A.$$

Since 5 does not divide any Lucas number, (3.21) is impossible. Next we consider (3.20). If $3m > 2n$, then by Theorem 2.1, there exists a prime p such that $p \mid F_{3m}$ but p does not divide the left hand side of (3.20), which is a contradiction. So $3m \le 2n$. Similarly, $\frac{3m}{2} \le 2n$ in (3.22) and $m+1 \le 2n$ in (3.23). In any case, $m \le 2n-1$.

Next we consider the case $a = -1$ and ℓ is odd. Then $A \in \mathbb{Z}$ and we can apply Lemma 2.3(iv) and the identity $F_{2k} = F_kL_k$ to obtain the same set of equations (3.22), (3.23), (3.20), and (3.21), respectively. Then by applying Theorem 2.1, we obtain the same inequality $m \le 2n-1$. Therefore we conclude that

$$\text{(3.24)} \qquad m \le 2n-1 \quad \text{when } a = 1 \text{ or } a = -1 \text{ and } \ell \text{ is odd.}$$

Since $n \ge 26$, we obtain by Lemma 2.5(ii) that

$$\text{(3.25)} \qquad v_2(L_m^\ell - a) = v_2(L_1L_2L_3\cdots L_n) \ge v_2(L_1L_2L_3\cdots L_{26}) = 12.$$

Case 1. $a = 1$ and ℓ is odd. Let $x = L_m$. Then

$$x^\ell - 1 = (x-1)(x^{\ell-1} + x^{\ell-2} + \cdots + x + 1)$$

and the second factor on the right hand side of the above equation is odd. So for any $k \in \mathbb{N}$, $2^k \mid x^\ell - 1$ if and only if $2^k \mid x-1$. By (3.25), we have $2^{12} \mid x^\ell - 1$. So $2^{12} \mid x-1$. That is

$$\text{(3.26)} \qquad L_m \equiv 1 \pmod{2^{12}}.$$

From (3.19), we have $m > n \ge 26$. By Corollary 2.9(i) and (3.26), we obtain $m \ge 3\cdot 2^{11}$ and therefore we obtain by (3.24) that $n \ge \frac{m+1}{2} > 3\cdot 2^{10}$. Then we repeat this process as follows. Let $b = 2^{10}$. Since $n \ge 3b$, we obtain by Lemma 2.5 that

$$v_2(L_m^\ell - 1) = v_2(L_1L_2L_3\cdots L_n) \ge v_2(L_1L_2L_3\cdots L_{3b}) \ge b.$$

So we obtain $L_m^\ell \equiv 1 \pmod{2^b}$, which implies $L_m \equiv 1 \pmod{2^b}$. Then we obtain by Corollary 2.9(i) and (3.24) that

$$m \ge 3\cdot 2^{b-1} \text{ and } n \ge \frac{m+1}{2} > 3\cdot 2^{b-2} = 3\cdot 2^{2^{10}-2}.$$

By repeating this process, we see that $n \ge M$ for any given positive integer M, a contradiction. So there is no solution in this case.

Case 2. $a=1$, ℓ is even, and m is even. Then by Lemma 2.3(i), we obtain

$$\frac{F_{3m}}{F_m} = L_m^2 - 1 \mid L_m^\ell - 1 = L_1L_2L_3\cdots L_n.$$

So

$$F_{3m} \mid L_1L_2L_3\cdots L_nF_m. \tag{3.27}$$

By (3.19), we have $3m > 3n > 2n$, so there exists a prime p such that $p \mid F_{3m}$ but p does not divide F_m and F_{2k} for any $1 \le k \le n$. Therefore

$$p \nmid \frac{F_2}{F_1}\frac{F_4}{F_2}\cdots\frac{F_{2n}}{F_n}F_m = L_1L_2L_3\cdots L_nF_m,$$

which contradicts (3.27).
Case 3. $a=1$, ℓ is even, and m is odd. Then by Lemma 2.3(i), we obtain

$$5 \mid 5F_{m-1}F_{m+1} = L_m^2 - 1 \mid L_m^\ell - 1 = L_1L_2L_3\cdots L_n,$$

which contradicts the fact that 5 does not divide any Lucas number.
Case 4. $a=-1$ and ℓ is even. Then L_m^ℓ is a perfect square. So $L_m^\ell - a = L_m^\ell + 1 \equiv 1, 2 \pmod 4$, which contradicts (3.25).
Case 5. $a=-1$ and ℓ is odd. Then we can apply (3.24) and follow the argument in Case 1. Let $x = L_m$. Then

$$x^\ell + 1 = (x+1)(x^{\ell-1} - x^{\ell-2} + \cdots - x + 1)$$

and the second factor is odd. So $2^{12} \mid x^\ell + 1$ if and only if $2^{12} \mid x+1$. This leads to the congruence

$$L_m \equiv -1 \pmod{2^{12}}.$$

By Corollary 2.9(ii) and (3.24), we obtain, respectively $m \ge 2^{11}$ and $n \ge \frac{m+1}{2} > 2^{10}$. Then repeating the process just like in Case 1, we reach a contradiction. This completes the proof. □

THEOREM 3.3. *Let $a \in \{1,-1\}$ and $\ell \ge 3$. Then the Diophantine equation*

$$L_1L_2L_3\cdots L_n + a = F_m^\ell \tag{3.28}$$

has no solution in positive integers m, n.

Proof. Some parts of the proof are similar to those of Theorems 3.1 and 3.2, so we leave some details to the reader. First, by using MAPLE we can suppose that $n \ge 26$. Then the left hand side of (3.28) is not divisible by 2 and 3, so $m \ge 5$. Let $A = \frac{F_m^\ell - a}{F_m - a}$. Similar to the proof of Theorems 3.1 and 3.2, we rewrite (3.28) according to the residue class of m modulo 4 by applying

Lemma 2.2(iii) if $a = 1$ and Lemma 2.2(iv) if $a = -1$ and ℓ is odd, together with the identity $F_{2k} = F_k L_k$. Then we obtain the equations similar to (3.20) to (3.23). By applying Theorem 2.1 to each case, we can conclude that

$$m \leq 2n + 2 \quad \text{when } a = 1 \text{ or when } a = -1 \text{ and } \ell \text{ is odd.} \tag{3.29}$$

Case 1. $a = 1$ and ℓ is odd. This case is similar to Case 1.3 of Theorem 3.1 and to Case 1 of Theorem 3.2. We first apply Lemma 2.5 to obtain

$$v_2(F_m^\ell - 1) = v_2(L_1 L_2 \cdots L_n) \geq v_2(L_1 L_2 \cdots L_{26}) = 12.$$

Therefore $F_m^\ell \equiv 1 \pmod{2^{12}}$ which leads to $F_m \equiv 1 \pmod{2^{12}}$. Then by Lemma 2.7(i) and the fact that $m \geq 5$, we conclude that $m \geq 3 \cdot 2^{11} - 1$. Then by (3.29), we obtain

$$n \geq \frac{m-2}{2} > 2^{10}.$$

By repeating this process just like in Case 1.3 of Theorem 3.1 or Case 1 of Theorem 3.2, we see that $n \geq M$ for any given positive integer M, which is a contradiction.

Case 2. $a = 1$, ℓ is even and m is even. If $m \leq 2n$, then we have

$$3 \leq L_{\frac{m}{2}} \mid F_m^\ell - L_1 L_2 L_3 \cdots L_n = a,$$

which is a contradiction. So $m > 2n$. Then by Lemma 2.2(i), we have

$$F_{m-2} F_{m+2} = F_m^2 - 1 \mid F_m^\ell - 1 = L_1 L_2 L_3 \cdots L_n.$$

But $m + 2 > 2n + 2 > 2n$, so by Theorem 2.1, there exists a prime p dividing F_{m+2} but p does not divide $\frac{F_2}{F_1} \frac{F_4}{F_2} \frac{F_6}{F_3} \cdots \frac{F_{2n}}{F_n} = L_1 L_2 L_3 \cdots L_n$, a contradiction.

Case 3. $a = 1$, ℓ is even, and m is odd. We first suppose that $4 \mid \ell$. Then we obtain by Lemmas 2.2(i) and 2.2(ii) that

$$F_{m-2} F_{m-1} F_{m+1} F_{m+2} = F_m^4 - 1 \mid F_m^\ell - 1 = L_1 L_2 L_3 \cdots L_n. \tag{3.30}$$

Since m is odd, $m + 2$ is odd and is larger than 6. By Theorem 2.1, there exists an odd prime p such that $z(p) = m + 2$. Since p and $z(p)$ are odd, we obtain by Lemma 2.4 that $v_p(L_k) = 0$ for every k. So in particular, $p \mid F_{m+2}$ but $p \nmid L_1 L_2 L_3 \cdots L_n$, which contradicts (3.30). Hence $\ell = 2\ell_0$, ℓ_0 is odd, and $\ell_0 \geq 3$. Then by an argument similar to that in Case 1.2 of Theorem 3.1, we obtain

$$\begin{aligned} v_2(F_m^\ell - 1) &= v_2(F_m^2 - 1) = v_2(F_{m-1} F_{m+1}) \\ &\leq \max\{v_2(m-1) + 2, v_2(m+1) + 2\} \\ &\leq \frac{\log(m+1)}{\log 2} + 2 \leq \frac{\log(2n+3)}{\log 2} + 2, \end{aligned} \tag{3.31}$$

where the last inequality is obtained by (3.29). On the other hand, by Lemma 2.5, we obtain

$$v_2(F_m^\ell - 1) = v_2(L_1 L_2 L_3 \cdots L_n) > 2\left(\frac{n+3}{6} - 1\right) + \left(\frac{n}{6} - 1\right) = \frac{n}{2} - 2. \tag{3.32}$$

From (3.31) and (3.32), we obtain

$$\frac{n}{2} - 2 < \frac{\log(2n+3)}{\log 2} + 2. \tag{3.33}$$

However, the function f given by

$$f(x) = \left(\frac{x}{2} - 2\right) - \left(\frac{\log(2x+3)}{\log 2} + 2\right)$$

is increasing on $[25, \infty)$. So $f(n) \geq f(26) > 0$, which contradicts (3.33).

Case 4. $a = -1$ and ℓ is even. Since ℓ is even, we have $F_m^\ell + 1 \equiv 1, 2 \pmod 4$, which contradicts the fact that $F_m^\ell + 1 = L_1 L_2 L_3 \cdots L_n$, which is divisible by 4.

Case 5. $a = -1$ and ℓ is odd. This case is similar to Case 3 of Theorem 3.1 and Case 5 of Theorem 3.2. First, by Lemma 2.5, we obtain $v_2(F_m^\ell + 1) \geq 12$, so $F_m^\ell \equiv -1 \pmod{2^{12}}$, which leads to $F_m \equiv -1 \pmod{2^{12}}$. Then by Lemma 2.7(ii) and (3.29), we obtain, respectively

$$m \geq 3 \cdot 2^{11} - 2 \quad \text{and} \quad n \geq \frac{m-2}{2} > 2^{10}.$$

Then repeating this process just like before, we reach a contradiction. This completes the proof. $\square$

Combining Theorems 3.1, 3.2, and 3.3 with Pongsriiam's results [17], [18], we obtain the proof of Theorem 1.1 as follows.

Proof of Theorem 1.1. For (i), we need to show that all solutions to (1.3) when $a = 1$ are $(n, m, \ell) = (3, 2, 1)$ and $(4, 4, 1)$. It is easy to see that $m = 1$ does not lead to a solution. So assume that $m \geq 2$. If $\ell \geq 3$, then Theorem 3.1 shows that (1.3) has no solution and if $\ell = 2$, then Theorem 3.1 of [17] implies that there is no solution to (1.3) either. For $\ell = 1$, all solutions to (1.3) can be obtained easily from Theorem 3.6 of [18].

For (ii), it is easy to check that $m = 1$ leads to the solution given by $(n, m, \ell) = (3, 1, \ell)$ where ℓ is any positive integer. So assume that $m \geq 2$. Then Theorem 3.2 of [17] implies that (1.3) has no solution when $\ell = 2$ and Theorem 3.1 shows that (1.3) has no solution when $\ell \geq 3$. If $\ell = 1$, we apply Theorem 3.7 of [18] to obtain the solution to (1.3), namely, $(n, m, \ell) = (5, 7, 1)$.

For (iii), we obtain by Theorem 3.3 that there is no solution to (1.4) when $\ell \geq 3$. So we consider only $\ell \leq 2$. If $\ell = 1$, we apply Theorem 3.1 of [18] to obtain the solutions to (1.4), namely, $(n, m, \ell) = (1, 3, 1), (3, 7, 1)$. If $\ell = 2$, then we apply Theorem 3.3 of [17] to obtain $(n, m, \ell) = (2, 3, 2)$.

For (iv), we obtain by Theorem 3.3 that there is no solution to (1.4) when $\ell \geq 3$ and by Theorem 3.4 of [17], there is no solution to (1.4) when $\ell = 2$ either. If $\ell = 1$, then we apply Theorem 3.3 of [18] to obtain $(n, m, \ell) = (2, 3, 1)$.

For (v), we obtain by Theorem 3.2 that there is no solution to (1.5) when $\ell \geq 3$, and by Theorem 3.6 of [17], there is no solution when $\ell = 2$ either. For $\ell = 1$, we apply Theorem 3.4 of [18] to obtain $(n, m, \ell) = (2, 3, 1)$.

For (vi), we obtain by Theorem 3.2 that there is no solution to (1.5) when $\ell \geq 3$, and by Theorem 3.5 of [17], there is no solution to (1.5) when $\ell = 2$ either. If $\ell = 1$, we apply Theorem 3.5 of [18] to obtain $(n, m, \ell) = (3, 5, 1)$. □

We give the explicit solutions to (1.2) in the next theorem. Since it is similar to the other theorems, we only give overall ideas of the proof.

THEOREM 3.4. *The Diophantine equation*

$$F_1F_2F_3 \cdots F_n + 1 = F_m^\ell$$

has a solution in positive integers if and only if $\ell = 1$. *In this case, there are exactly three solutions, namely,* $(m, n, \ell) = (3, 1, 1)$, $(3, 2, 1)$, *and* $(4, 3, 1)$.

Proof. We can limit the range of n by applying the result of Bravo, Komatsu, and Luca [3, Corollary 1], which is obtained by referring to lower bounds for linear forms in p-adic logarithms. Nevertheless, we can also solve this equation in an elementary way. For $\ell = 1$ or 2, the result follows from [18, Corollary 4.1] and [17, Theorem 3.8]. So we assume $\ell \geq 3$. By using MAPLE and divisibility relation, we see that $n \geq 26$ and $m \geq n + 1$. Then we divide the proof into 2 cases.

Case 1 ℓ is odd. This case is similar to Case 1.3 of Theorem 3.1 and to Case 1 of Theorem 3.2. We first apply Lemma 2.5 to obtain $v_2(F_m^\ell - 1) \geq 19$ which leads to $F_m \equiv 1 \pmod{2^{19}}$. Then we apply Lemma 2.7(i) and repeat the process just like in Case 1.3 of Theorem 3.1 or Case 1 of Theorem 3.2. We see that $n \geq M$ for any given positive integer M, which is a contradiction.

Case 2 ℓ is even. Then we obtain by Lemma 2.2 that

$$F_{m-b}F_{m+b} = F_m^2 - 1 \mid F_m^\ell - 1 = F_1F_2 \cdots F_n,$$

where $b = 1$ or 2. Since $m + b \geq m + 1 > n$, we obtained by the primitive divisor theorem that there exists a prime p dividing F_{m+b} but $p \nmid F_1F_2 \cdots F_n$, which is a contradiction.

This completes the proof. □

Similarly, if 1 is replaced by -1, the solutions are as follows.

THEOREM 3.5. *The solutions to the Diophantine equation*

$$F_1F_2F_3 \cdots F_n - 1 = F_m^\ell$$

are $(m, n, \ell) = (1, 3, a)$, $(2, 3, a)$, *and* $(5, 4, 1)$, *where* a *is any positive integer.*

Proof. The proof is similar to that of Theorem 3.4. The case when m or n are small or $\ell = 1, 2$ can be checked by using MAPLE and the result in [18, Corollary 4.1] and [17, Theorem 3.7]. So assume that $\ell \geq 3$, $m \geq n+1 \geq 27$. If ℓ is odd, we apply Lemmas 2.5 and 2.7(ii) repeatedly to reach a contradiction. If ℓ is even, then $F_m^\ell \equiv 0, 1 \pmod 4$ which contradicts the fact that $F_1F_2\cdots F_n \equiv 0 \pmod 4$ for $n \geq 6$. This completes the proof. □

Acknowledgments. I receive financial support jointly from The Thailand Research Fund and Faculty of Science Silpakorn University grant number RSA5980040. I also thanks Professor Dąbrowski [6], [7] and Professor Szalay [20] for sending me their articles which give me some ideas to do this project.

REFERENCES

[1] B. C. Berndt and W. F. Galway, *On the Brocard–Ramanujan Diophantine equation* $n! + 1 = m^2$. Ramanujan J. **4** (2006), *1*, 41–42 .

[2] Y. Bilu, G. Hanrot, and P. M. Voutier, *Existence of primitive divisors of Lucas and Lehmer numbers*, J. Reine Angew. Math. **539** (2001), 75–122.

[3] J. J. Bravo, T. Komatsu, and F. Luca, *On the distance between products of consecutive Fibonacci numbers and powers of Fibonacci numbers.* Indag. Math. **24** (2013), 181–198.

[4] P. Bundschuh and R. Bundschuh, *The sequence of Lucas numbers is not stable modulo* 2 *and* 5. Unif. Distrib. Theory, **5** (2010), *2*, 113–130.

[5] R. D. Carmichael, *On the numerical factors of the arithmetic forms* $\alpha^n \pm \beta^n$. Ann. of Math.(2) **15** (1913), 30–48.

[6] A. Dąbrowski, *On the Brocard–Ramanujan problem and generalizations.* Colloq. Math. **126** (2012), *1*, 105–110.

[7] A. Dąbrowski and M. Ulas, *Variations on the Brocard–Ramanujan equation.* J. Number Theory **133** (2013), *4*, 1168–1185.

[8] E. T. Jacobson, *Distribution of the Fibonacci numbers mod* 2^k. Fibonacci Quart. **30** (1992), *3*, 211–215.

[9] Y. Kunrui, *p-adic logarithmic forms and a problem of Erdös.* Acta Math. **211** (2013), *2*, 315–382.

[10] T. Lengyel, *The order of Fibonacci and Lucas Numbers.* Fibonacci Quart. **33** (1995), *3*, 234–239.

[11] F. Luca, *The Diophantine equation* $P(x) = n!$ *and a result of M. Overholt.* Glas. Mat. Ser. III **37** (2002), *2*, 269–273.

[12] F. Luca and T. N. Shorey, *Diophantine equations with products of consecutive terms in Lucas sequences II.* Acta Arith. **133** (2008), *1*, 53–71.

[13] F. Luca and T. N. Shorey, *Diophantine equations with products of consecutive terms in Lucas sequences.* J. Number Theory **114** (2005), *2*, 298–311.

[14] D. Marques, *The Fibonacci version of the Brocard–Ramanujan Diophantine equation.* Port. Math. **68** (2011), *2*, 185–189.

[15] D. Marques, *The Fibonacci version of a variant of the Brocard–Ramanujan Diophantine equation*. Far East J. Math. Sci. **56** (2011), *2*, 219–224.

[16] P. Pongsriiam, *Factorization of Fibonacci numbers into products of Lucas numbers and related results*. JP J. Algebra Number Theory Appl., **38** (2016), *4*, 363–372.

[17] P. Pongsriiam, *Fibonacci and Lucas Numbers Associated with Brocard-Ramanujan Equation*. Commun. Korean Math. Soc., to appear

[18] P. Pongsriiam, *Fibonacci and Lucas numbers which are one away from their products*. Fibonacci Quart. **55** (2017), *1*, 29–40.

[19] C. L. Stewart, *On divisors of Lucas and Lehmer numbers*. Acta Math. **211** (2013), *2*, 291–314.

[20] L. Szalay, *Diophantine equations with binary recurrences associated to the Brocard–Ramanujan problem*. Port. Math. **69** (2012), *3*, 213–220.

[21] D. D. Wall, *Fibonacci series modulo m*. Amer. Math. Monthly **67** (1960), 525–532.

[22] M. Ward, *The prime divisors of Fibonacci numbers*. Pacific J. Math. **11** (1961), 379–386.

Received June 8, 2017

Silpakorn University,
Faculty of Science,
Department of Mathematics,
Nakhon Pathom 73000, Thailand
prapanpong@gmail.com