APPROXIMATE BIFLATNESS OF CERTAIN BANACH ALGEBRAS

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In this paper we investigate the approximate biflatness of semigroup algebras. Under some conditions we show that approximate biflatness of semigroup algebra $\ell^1(S)$ implies amenability of semigroup S. Also we study the approximate biflatness of group algebra $L^1(G)$ and its Segal algebra $S(G)$ according to the amenability of G.

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1. INTRODUCTION

The concept of amenability of Banach algebras was introduced by Johnson in [9]. The notions related to amenability in the theory of homological Banach algebras are biflatness and biprojectivity which were introduced by Helemskii [7]. Choi [2], showed, for any semilattice S, that $\ell^1(S)$ is biflat if and only if S is uniformly locally finite. He also proved, for a Clifford semigroup S, that $\ell^1(S)$ is biflat if and only if

(i) $(E(S), \leq)$ is uniformly locally finite and

(ii) each maximal subgroup of S is amenable.

Afterwards Ramsden [13] extended this result for any inverse semigroup S. He showed that

(i) $\ell^1(S)$ is biflat if and only if S is uniformly locally finite and G_P the maximal subgroup of S at $p \in E(S)$ is amenable.

(ii) $\ell^1(S)$ is biprojective if and only if S is uniformly locally finite and G_p the maximal subgroup of S at $p \in E(S)$ is finite.

Samei *et al.* in [17] provided a natural generalization of biflatness, called approximate biflatness. They characterized a sufficient condition for Banach algebra A to be pseudo-amenable. Indeed a Banach algebra A is pseudo-amenable whenever it is approximately biflat and has an approximate identity.

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Also they studied approximate biflatness for various Segal algebra in both group algebra $L^1(G)$ and the Fourier algebra $A(G)$. The question then arises, what will happen when the semigroup algebra $\ell^1(S)$ is approximately biflat?

In this paper, we investigate the concept of approximate biflatness in the category of semigroup algebras. We study the conditions such that approximate biflatness of semigroup algebra $\ell^1(S)$ implies amenability of semigroup S. Also we study the relationship between approximate biflatness of group algebra $L^1(G)$ and Segal algebra $S(G)$ with the amenability group G. Finally we give an example of Banach algebra which is approximately biflat but it is not approximately biprojective. Also we present an example of Banach algebra which is approximately biflat but it is not pseudo-contractible.

2. PRELIMINARIES

The main reference for the semigroup theory is [8]. Suppose that S is a semigroup and $E(S)$ is the set of its idempotents. We have a partial order on $E(S)$, which is defined by

$$i \leq j \Leftrightarrow i = ij = ji \quad (i, j \in E(S)).$$

An idempotent $i \in E(S)$ is called maximal if $i = j$ whenever $i \leq j$.

A semigroup S is called an inverse semigroup, if for every $s \in S$ there exists a unique element $s^* \in S$ such that $s = ss^*s$ and $s^* = s^*ss^*$. Suppose that S is an inverse semigroup. Then there exists a partial order on S defined by $s \leq t \Leftrightarrow s = ss^*t$, where $s, t \in S$. This partial order on S coincides with the partial order on $E(S)$. We denote $(s] = \{t \in S : t \leq s\}$ for any $s \in S$. We recall that S is locally finite (uniformly locally finite), if $|(s]| < \infty$ ($\sup\{|(s]| : s \in S\} < \infty$, respectively) for every $s \in S$.

Suppose that S is an inverse semigroup and $p \in E(S)$. Then $G_p = \{s \in S : ss^* = s^*s = p\}$ is a group with identity p and it is called the maximal subgroup of S at p. A semigroup S is called a Clifford semigroup if it is an inverse semigroup with $ss^* = s^*s \ \ (s \in S)$.

We recall that a semigroup S is a left amenable (right amenable) semigroup if there is $m \in \ell^1(S)^{**}$ such that $s \cdot m = m$ ($m = m \cdot s$) and $\|m\| = m(\phi_S) = 1 \ \ (s \in S)$, where ϕ_S is the augmentation character on $\ell^1(S)$. The semigroup S is amenable, if it is both left and right amenable.

Suppose that A is a Banach algebra. We denote the character space of A by $\Delta(A)$, that is, all non-zero multiplicative linear functionals on A. Suppose that $\phi \in \Delta(A)$. Then $\tilde{\phi} \in \Delta(A^{**})$ is a unique extension of ϕ to A^{**} which is defined by $\tilde{\phi}(F) = F(\phi)$ for every $F \in A^{**}$.

Suppose that A is a Banach algebra. Then the projective tensor product $A \otimes_p A$ is a Banach A-bimodule with the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A).$$

The product morphism $\pi_A : A \otimes_p A \to A$ is defined by $\pi_A(a \otimes b) = ab$ for every $(a \otimes b) \in A \otimes_p A$. We recall that for every $\phi \in \Delta(A)$ and $\psi \in \Delta(B)\}$, the map $\phi \otimes \psi$ defined by $\phi \otimes \psi(a \otimes b) = \phi(a)\psi(b)$, for every $a \in A$ and $b \in B$, is a character on $A \otimes_p B$.

A Banach algebra A is called approximately biflat if there exists a net $\theta_\alpha : (A \otimes_p A)^* \to A^*$ of bounded A-bimodule morphisms such that

$$\mathrm{W^*OT} - \lim_\alpha \theta_\alpha \circ \pi_A^* = id_{A^*},$$

where W*OT is the weak* operator topology on $B(A^*)$ [17]. We remind that the weak* operator topology on $B(A^*)$ is the locally convex topology determined by the seminorms $\{p_{a,f} : a \in A, f \in A^*\}$, where $p_{a,f}(T) = |\langle a, T(f)\rangle|$.

A Banach algebra A is called approximately biprojective, if there is a net $\eta_\alpha : A \to A \otimes_p A$ of continuous A-bimodule morphisms such that $\pi_A \circ \eta_\alpha(a) \longrightarrow a$ for every $a \in A$. See [19].

We say that a Banach algebra A is left φ-amenable (left φ-contractible), where $\varphi \in \Delta(A)$, if there is $m \in A^{**}$ ($m \in A$) such that $am = \varphi(a)m$ and $\widetilde{\varphi}(m) = 1$ ($\varphi(m) = 1$, respectively), for every $a \in A$. For more details see [10, 12]. Equivalently, a Banach algebra A is called left ϕ-amenable, where $\phi \in \Delta(A)$, if there exists a bounded net $(a_\alpha) \subseteq A$ such that for every $a \in A$

$$\phi(a_\alpha) \longrightarrow 1, \quad \|aa_\alpha - \phi(a)a_\alpha\| \longrightarrow 0$$

and any such net is called a bounded approximate ϕ-mean [10].

We say that a Banach algebra A is pseudo-contractible if there is a (not necessarily bounded) net $(m_\beta)_\beta \subseteq A \otimes_p A$ such that $a \cdot m_\beta = m_\beta \cdot a$ and $\lim_\beta \pi_A(m_\beta)a = a$ for every $a \in A$. See [6].

3. APPROXIMATE BIFLATNESS OF SEMIGROUP ALGEBRAS

Throughout this section, S is a semigroup and $\ell^1(S)$ is its semigroup algebra. We begin with the following lemma.

LEMMA 3.1. *Let A be a Banach algebra and $m \in (A \otimes_p A)^{**}$ such that $a \cdot m = m \cdot a$ for every $a \in A$ and let $\varphi \in \Delta(A)$ such that $\tilde{\varphi} \circ \pi_A^{**}(m) = 1$, where $\tilde{\varphi}$ is extension of φ to A^{**}. Then there exists $\eta \in (A \otimes_p A)^{**}$ such that $a \cdot \eta = \eta \cdot a = \varphi(a)\eta$ for every $a \in A$ and $\tilde{\varphi} \circ \pi_A^{**}(\eta) = 1$.*

Proof. Let φ be a character on A. Then there exists a bounded linear map $T : A \otimes_p A \to A$ defined by $T(a \otimes b) = \varphi(b)a$ [16, proposition 2.2]. Since the adjoint map $T^{**} : (A \otimes_p A)^{**} \to A^{**}$ is a bounded linear map, we show that $T^{**}(m \cdot a) = \varphi(a)T^{**}(m)$ and $T^{**}(a \cdot m) = a \cdot T^{**}(m)$. To see this, let $b^* \in A^*$. Then

$$\begin{aligned} T^{**}(m \cdot a)(b^*) =& (m \cdot a) \circ T^*(b^*) \\ =& (m \cdot a) \circ (b^* \circ T) = m(a \cdot b^* \circ T). \end{aligned} \tag{3.1}$$

On the other hand, for all $(f \otimes g) \in A \otimes_p A$ we have

$$\begin{aligned} a \cdot (b^* \circ T)(f \otimes g) =& b^* \circ T(f \otimes ga) = b^*(T(f \otimes ga)) \\ =& b^*(\varphi(ga)f) = \varphi(a)b^*(\varphi(g)f) = \varphi(a)b^*(T(f \otimes_p g)). \end{aligned}$$

Therefore

$$a \cdot (b^* \circ T) = \varphi(a)b^* \circ T. \tag{3.2}$$

Using (3.2) from (3.1), we obtain

$$\begin{aligned} T^{**}(m \cdot a)(b^*) =& m(\varphi(a)b^* \circ T) \\ =& \varphi(a)(m(b^* \circ T)) \\ =& \varphi(a)(m \circ T^*(b^*)) \\ =& \varphi(a)T^{**}(m)(b^*). \end{aligned}$$

Hence $T^{**}(m \cdot a) = \varphi(a)T^{**}(m)$. By a similar argument, we have $T^{**}(a \cdot m) = a \cdot T^{**}(m)$. Now, we note that

$$\|T^{**}(a \cdot m) - T^{**}(m \cdot a)\| = \|T^{**}(a \cdot m - m \cdot a)\| \leq \|T^{**}\| \|a \cdot m - m \cdot a\| = 0.$$

So we have

$$T^{**}(a \cdot m) = T^{**}(m \cdot a) = a \cdot T^{**}(m) = \varphi(a)T^{**}(m). \tag{3.3}$$

Since $m \in (A \otimes_p A)^{**}$, by Goldstine's theorem, there exists a bounded net $(m_\alpha) \subseteq A \otimes_p A$ such that $m_\alpha \longrightarrow m$ in the weak*-topology on $(A \otimes_p A)^{**}$ and $\varphi \circ \pi_A(m_\alpha) \longrightarrow 1$. Since T^{**} is a bounded linear map on $(A \otimes_p A)^{**}$, we have $T^{**}(m_\alpha) = T(m_\alpha) \longrightarrow T^{**}(m)$ in the weak*-topology on A^{**}. Since $a \cdot m = m \cdot a$ for every $a \in A$, by the equation (3.3), we have $\varphi(a)T^{**}(m) = a \cdot T^{**}(m)$ and hence

$$\varphi(a)T(m_\alpha) - a \cdot T(m_\alpha) \longrightarrow 0 \tag{3.4}$$

in the weak*-topology on A^{**}. Since $(m_\alpha) \subseteq A \otimes_p A$ one can assume that the equation (3.4) holds in the weak-topology. As in the proof of [6, Proposition 2.3], by using Mazur's theorem we can replace the weak-topology on A by a norm-topology on A, so that the equation (3.4) holds in the norm-topology. Take $n_\alpha = T(m_\alpha)$. We have $\varphi(\pi_A(m_\alpha)) = \varphi(T(m_\alpha)) = \varphi(n_\alpha) \longrightarrow 1$ and

$\varphi(a)n_\alpha - a \cdot n_\alpha \longrightarrow 0$. By replacing (n_α) with $(\frac{n_\alpha}{\varphi(n_\alpha)})$ we can assume that $\varphi(n_\alpha) = 1$ for every α. Since (n_α) is a bounded net by Alaoglu's theorem there exists $N_1 \in A^{**}$ such that $n_\alpha \longrightarrow N_1$ in the weak*-topology on A^{**} such that $a \cdot N_1 = \varphi(a)N_1$ for every $a \in A$ and $\tilde{\varphi}(N_1) = 1$. Similarly, we can find $N_2 \in A^{**}$ such that $N_2 \cdot a = \varphi(a)N_2$ for every $a \in A$ and $\tilde{\varphi}(N_2) = 1$. Now, take $M = N_1 \otimes N_2 \in A^{**} \otimes_p A^{**}$. By [5, Lemma 1.7], there is a continuous linear map $\psi : A^{**} \otimes_p A^{**} \to (A \otimes_p A)^{**}$ such that for every $a, b, x \in A$ and $m \in A^{**} \otimes_p A^{**}$, we have

$$\psi(a \otimes b) = a \otimes b, \quad \psi(m) \cdot x = \psi(m \cdot x),$$

$$x \cdot \psi(m) = \psi(x \cdot m), \quad \pi_A^{**}(\psi(m)) = \pi_{A^{**}}(m).$$

By taking $\eta = \psi(M) \in (A \otimes_p A)^{**}$ we have $a \cdot \eta = \eta \cdot a = \varphi(a)\eta$ and $\tilde{\varphi} \circ \pi_A^{**}(\eta) = 1$. $\square$

THEOREM 3.2. *Let S be a semigroup and $\mathbf{Z}(S) \neq \emptyset$, where $\mathbf{Z}(S)$ is the center of semigroup S. Suppose that $\ell^1(S)$ is approximately biflat. Then S is amenable.*

Proof. Since $\ell^1(S)$ is approximately biflat, there exists a net $\theta_\alpha : (\ell^1(S) \otimes_p \ell^1(S))^* \to \ell^1(S)^*$ of bounded $\ell^1(S)$-bimodule morphisms such that

$$\mathrm{W^*OT} - \lim_\alpha \theta_\alpha \circ \pi^*_{\ell^1(S)} = id_{\ell^1(S)^*}.$$

Clearly $\theta^*_\alpha : \ell^1(S)^{**} \to (\ell^1(S) \otimes_p \ell^1(S))^{**}$ is a net of bounded $\ell^1(S)$-bimodule morphisms. Suppose that ϕ_S is the augmentation character on $\ell^1(S)$ and $\widetilde{\phi_S}$ is its extension to $\ell^1(S)^{**}$. Then $\phi_S(\delta_{s_0}) = 1$ for every $s_0 \in \mathbf{Z}(S)$. Since $\theta^*_\alpha|_{\ell^1(S)} : \ell^1(S) \to (\ell^1(S) \otimes_p \ell^1(S))^{**}$ is a net of bounded $\ell^1(S)$-bimodule morphisms, we have $\widetilde{\phi_S} \circ \pi^{**}_{\ell^1(S)} \circ \theta^*_\alpha(a) = \pi^{**}_{\ell^1(S)} \circ \theta^*_\alpha(a)(\phi_S)$ and

$$\begin{aligned} \lim_\alpha \langle \phi_S, \pi^{**}_{\ell^1(S)} \circ \theta^*_\alpha(a) \rangle &= \lim_\alpha \langle \phi_S, \theta^*_\alpha(a) \circ \pi^*_{\ell^1(S)} \rangle \\ &= \lim_\alpha \langle a, \theta_\alpha \circ \pi^*_{\ell^1(S)}(\phi_S) \rangle = \langle a, \phi_S \rangle, \end{aligned} \tag{3.5}$$

for every $a \in \ell^1(S)$. By taking $\mu_\alpha = \theta^*_\alpha(\delta_{s_0})$, we have $a \cdot \mu_\alpha = \mu_\alpha \cdot a$. To see this, since θ^*_α is a net of bounded $\ell^1(S)$-bimodule morphisms, we have

$$a \cdot \mu_\alpha = a \cdot \theta^*_\alpha(\delta_{s_0}) = \theta^*_\alpha(a \cdot \delta_{s_0}) = \theta^*_\alpha(\delta_{s_0} \cdot a) = \theta^*_\alpha(\delta_{s_0}) \cdot a = \mu_\alpha \cdot a.$$

In contrast, the equation (3.5) follows that $\widetilde{\phi_S} \circ \pi^{**}_{\ell^1(S)}(\mu_\alpha) \longrightarrow 1$. We may suppose that $\widetilde{\phi_S} \circ \pi^{**}_{\ell^1(S)}(\mu_\alpha) = 1$, by replacing μ_α with $\frac{\mu_\alpha}{\widetilde{\phi_S} \circ \pi^{**}_{\ell^1(S)}(\mu_\alpha)}$. Hence we have $a \cdot \mu_\alpha = \mu_\alpha \cdot a$ and $\widetilde{\phi_S} \circ \pi^{**}_{\ell^1(S)}(\mu_\alpha) = 1$. Since $a \cdot \mu_\alpha = \mu_\alpha \cdot a$ and $\widetilde{\phi_S} \circ \pi^{**}_{\ell^1(S)}(\mu_\alpha) = 1$, by Lemma 3.1 there exists $\eta_\alpha \in (\ell^1(S) \otimes \ell^1(S))^{**}$ such that

$a \cdot \eta_\alpha = \eta_\alpha \cdot a = \phi_S(a)\eta_\alpha$ and $\widetilde{\phi_S}(\pi^{**}_{\ell^1(S)}(\eta_\alpha)) = 1$ for every $a \in \ell^1(S)$. Similarly to [4, Corollary 2.10], one can show that S is amenable $\square$

THEOREM 3.3. *Suppose that $\ell^1(S)$ has a central approximate identity. Let $\ell^1(S)$ be approximately biflat. Then S is amenable.*

Proof. Since $\ell^1(S)$ is approximately biflat, there exists a net $\vartheta_\gamma : (\ell^1(S)\otimes_p \ell^1(S))^* \to \ell^1(S)^*$ of bounded $\ell^1(S)$-bimodule morphisms such that

$$\mathrm{W^*OT} - \lim_\gamma \vartheta_\gamma \circ \pi^*_{\ell^1(S)} = id_{\ell^1(S)^*}.$$

Clearly $\vartheta^*_\gamma : \ell^1(S)^{**} \to (\ell^1(S) \otimes_p \ell^1(S))^{**}$ is a net of bounded $\ell^1(S)$-bimodule morphisms. Suppose that ϕ_S is the augmentation character on $\ell^1(S)$ and $\widetilde{\phi_S}$ is its extension to $\ell^1(S)^{**}$. Suppose that (e_λ) is a central approximate identity for $\ell^1(S)$. Then $\phi_S(e_\lambda) \longrightarrow 1$ and for every $a \in A$ and $\psi \in \ell^1(S)^*$ we have

$$\begin{aligned}
\lim_\gamma \lim_\lambda \langle \psi, \pi^{**}_{\ell^1(S)}(\vartheta^*_\gamma(e_\lambda)) \cdot a\rangle &= \lim_\gamma \lim_\lambda \langle \psi, \pi^{**}_{\ell^1(S)}(\vartheta^*_\gamma(e_\lambda \cdot a))\rangle \\
&= \lim_\gamma \lim_\lambda \langle \psi, \vartheta^*_\gamma(e_\lambda \cdot a) \circ \pi^*_{\ell^1(S)}\rangle \\
&= \lim_\gamma \langle e_\lambda \cdot a, \vartheta_\gamma(\pi^*_{\ell^1(S)}(\psi))\rangle \\
&= \lim_\gamma \langle a, \vartheta_\gamma(\pi^*_{\ell^1(S)}(\psi))\rangle = \langle a, \psi\rangle.
\end{aligned} \tag{3.6}$$

Since ϑ^*_γ is a net of bounded $\ell^1(S)$-bimodule morphisms, we have

$$\lim_\gamma \lim_\lambda a \cdot \vartheta^*_\gamma(e_\lambda) - \vartheta^*_\gamma(e_\lambda) \cdot a = \lim_\gamma \lim_\lambda \vartheta^*_\gamma(a \cdot e_\lambda - e_\lambda \cdot a) = 0. \tag{3.7}$$

Suppose that $Z = \Lambda \times \Gamma^\Lambda$ is directed by the product ordering and, for every $\alpha = (\lambda, (\gamma_{\lambda'})_{\lambda' \in \Lambda}) \in Z$, define $m_\alpha = \vartheta^*_{\gamma_\lambda}(e_\lambda)$. By iterated limit theorem [11, P. 69], the equation (3.7) gives the following

$$a \cdot m_\alpha - m_\alpha \cdot a \longrightarrow 0 \tag{3.8}$$

and the equation (3.6) implies that $\pi^{**}_{\ell^1(S)}(m_\alpha)a \xrightarrow{w^*} a$ in the weak*-topology on $\ell^1(S)^{**}$ and so we have $\widetilde{\phi_S}(\pi^{**}_{\ell^1(S)}(m_\alpha)a) \longrightarrow \widetilde{\phi_S}(a) = \phi_S(a)$ for all $a \in \ell^1(S)$. One can easily see that $\widetilde{\phi_S}(\pi^{**}_{\ell^1(S)}(m_\alpha)) \longrightarrow 1$. Now, we show that $a \cdot m_\alpha = m_\alpha \cdot a$ for every $a \in \ell^1(S)$ and $\alpha \in Z$. To see this, since ϑ^*_γ is a net of bounded $\ell^1(S)$-bimodule morphisms, we have

$$a \cdot m_\alpha = a \cdot \vartheta^*_{\gamma_\lambda}(e_\lambda) = \vartheta^*_{\gamma_\lambda}(ae_\lambda) = \vartheta^*_{\gamma_\lambda}(e_\lambda a) = \vartheta^*_{\gamma_\lambda}(e_\lambda) \cdot a = m_\alpha \cdot a.$$

We may suppose that $\widetilde{\phi_S}(\pi^{**}_{\ell^1(S)}(m_\alpha)) = 1$ by replacing m_α with $\frac{m_\alpha}{\widetilde{\phi_S}(\pi^{**}_{\ell^1(S)}(m_\alpha))}$. Hence there exists $m_\alpha \in (\ell^1(S) \otimes_p \ell^1(S))^{**}$ such that $a \cdot m_\alpha = m_\alpha \cdot a$ for every

$a \in \ell^1(S)$ and $\widetilde{\phi_S}(\pi^{**}_{\ell^1(S)}(m_\alpha)) = 1$ and so S is amenable by a similar argument as in the previous Theorem. $\square$

Suppose that A is a Banach algebra and Λ is a non-empty set. The Banach algebra $\mathbb{M}_\Lambda(A)$ is the set of $\Lambda \times \Lambda$-matrices over A with finite ℓ^1-norm and matrix multiplication.

In the following example, we show that there exists an approximately biflat Banach algebra which is not pseudo-contractible.

Example 3.4. Suppose that $A = \mathbb{M}_\Lambda(\mathbb{C})$, where Λ is not a finite set. Then A is biflat [13, Proposition 2.7] and so A is approximately biflat. Now let A be pseudo-contractible. Then A has a central approximate identity. So Λ must be finite [4, Theorem 2.2] which is a contradiction.

In the following example, we show that there exists an approximately biflat Banach algebra which is not approximately biprojective.

Example 3.5. Let S be a uniformly locally finite inverse semigroup such that all of its maximal subgroups are amenable and at least one of its maximal subgroups is infinite. Then $\ell^1(S)$ is biflat [13, Theorem 3.7] and so it is approximately biflat but $\ell^1(S)$ is not biprojective [13, Theorem 3.7]. In contrast, $\ell^1(S)$ is biprojective if and only if it is approximately biprojective [15, Theorem 3.6]. Therefore $\ell^1(S)$ is not approximately biprojective.

4. APPROXIMATE BIFLATNESS OF GROUP ALGEBRA AND SEGAL ALGEBRA

Let G be a locally compact group. A linear subspace $S(G)$ of $L^1(G)$ is said to be a Segal algebra on G if it satisfies the following conditions

(i) $S(G)$ is dense in $L^1(G)$,

(ii) $S(G)$ with the norm $||\cdot||_{S(G)}$ is a Banach space and $||f||_{L^1(G)} \leq ||f||_{S(G)}$ for every $f \in S(G)$,

(iii) for every $f \in S(G)$ and $y \in G$ we have $L_y f \in S(G)$ and the map $y \mapsto L_y f$ of G into $S(G)$ is continuous, where $L_y f(x) = f(y^{-1}x)$,

(iv) $||L_y f||_{S(G)} = ||f||_{S(G)}$ for every $f \in S(G)$ and $y \in G$.

It is well-known that $S(G)$ has a left approximate identity and by [1, Lemma 2.2] its character space is $\Delta(S(G)) = \{\phi|_{S(G)} : \phi \in \Delta(L^1(G))\}$.

We start with the following basic theorem.

THEOREM 4.1. *Suppose that A is an approximately biflat Banach algebra and A has a left approximate identity (right approximate identity) and $\psi \in \Delta(A)$. Then A is left ψ-amenable (right ψ-amenable, respectively).*

Proof. Since A is approximately biflat, there is a net $\theta_\alpha : (A \otimes_p A)^* \to A^*$ of bounded A-bimodule morphisms such that $\mathrm{W^*OT} - \lim \theta_\alpha \circ \pi_A^* = id_{A^*}$. Suppose that $\mathbb{K} = \ker \psi$. Define $\zeta_\alpha : (id_A \otimes q)^{**} \circ \theta_\alpha^*|_A : A \to (A \otimes_p \frac{A}{\mathbb{K}})^{**}$, where $q : A \to \frac{A}{\mathbb{K}}$ is a quotient map. We have $\overline{A\mathbb{K}} = \mathbb{K}$, because A has a left approximate identity. Hence for every $k_1 \in \mathbb{K}$, there exist $a \in A$ and $k_2 \in \mathbb{K}$ such that $k_1 = ak_2$ and so for every $k_1 \in \mathbb{K}$, we have

$$\zeta_\alpha(k_1) = (id_A \otimes q)^{**} \circ \theta_\alpha^*(k_1) = (id_A \otimes q)^{**} \circ \theta_\alpha^*(ak_2). \tag{4.1}$$

Since ζ_α is a net of A-bimodule morphisms, the equation (4.1) follows that $\zeta_\alpha(k_1) = 0$ for every $k_1 \in \mathbb{K}$. Hence we can drop ζ_α on $\frac{A}{\mathbb{K}}$ for every α and so $\zeta_\alpha : \frac{A}{\mathbb{K}} \to (A \otimes_p \frac{A}{\mathbb{K}})^{**}$ is a left A-module morphism. Define a character $\tilde{\psi}$ on $\frac{A}{\mathbb{K}}$ by $\tilde{\psi}(a + \mathbb{K}) = \psi(a)$.

Since ζ_α is a left A-module morphism, $(id_A \otimes \tilde{\psi})^{**} \circ \zeta_\alpha : \frac{A}{\mathbb{K}} \to A^{**}$ is a left A-module morphism for every α. To see this, for every $a, b \in A$ we have

$$\begin{aligned}(id_A \otimes \tilde{\psi})^{**} \circ \zeta_\alpha(a \cdot (b + \mathbb{K})) =&(id_A \otimes \tilde{\psi})^{**} \circ \zeta_\alpha(ab + \mathbb{K})\\ =&(id_A \otimes \tilde{\psi})^{**} \circ \zeta_\alpha(ab)\\ =&a \cdot (id_A \otimes \tilde{\psi})^{**} \circ \zeta_\alpha(b).\end{aligned} \tag{4.2}$$

Since ψ is a non-zero character on A, there is $c \in A$ such that $\psi(c) = 1$. Define $\vartheta_\alpha = (id_A \otimes \tilde{\psi})^{**} \circ \zeta_\alpha(c + \mathbb{K})$. So we have

$$\begin{aligned}\langle \vartheta_\alpha, \psi \rangle =&\langle (id_A \otimes \tilde{\psi})^{**} \circ \zeta_\alpha(c + \mathbb{K}), \psi \rangle = \langle (id_A \otimes \tilde{\psi})^{**} \circ \zeta_\alpha(c), \psi \rangle\\ =&\langle \zeta_\alpha(c) \circ (id_A \otimes \tilde{\psi})^*, \psi \rangle = \langle \zeta_\alpha(c), (id_A \otimes \tilde{\psi})^*(\psi) \rangle\\ =&\langle \zeta_\alpha(c), \psi \circ (id_A \otimes \tilde{\psi}) \rangle = \langle \zeta_\alpha(c), \psi \otimes \tilde{\psi} \rangle\\ =&\langle (id_A \otimes q)^{**} \circ \theta_\alpha^*(c), \psi \otimes \tilde{\psi} \rangle = \langle \theta_\alpha^*(c) \circ (id_A \otimes q)^*, \psi \otimes \tilde{\psi} \rangle\\ =&\langle \theta_\alpha^*(c), (id_A \otimes q)^* \circ (\psi \otimes \tilde{\psi}) \rangle = \langle \theta_\alpha^*(c), (\psi \otimes \tilde{\psi}) \circ (id_A \otimes q) \rangle\\ =&\langle \theta_\alpha^*(c), \psi \circ \pi_A \rangle = \langle c \circ \theta_\alpha, \pi_A^*(\psi) \rangle = \langle c, \theta_\alpha \circ \pi_A^*(\psi) \rangle.\end{aligned} \tag{4.3}$$

Since $\mathrm{W^*OT} - \lim_\alpha \theta_\alpha \circ \pi_A^* = id_{A^*}$, the equation (4.3) implies that

$$\lim_\alpha \psi(\vartheta_\alpha) = \lim_\alpha \theta_\alpha \circ \pi_A^*(\psi)(c) = id_{A^*}(\psi)(c) = \psi(c) = 1.$$

In contrast, since $c - c^2 \in \mathbb{K}$,

$$\begin{aligned}ac + \mathbb{K} =&(a - \psi(a)c + \psi(a)c)c + \mathbb{K}\\ =&ac - \psi(a)c^2 + \psi(a)c^2 + \mathbb{K}\\ =&\psi(a)c^2 + \mathbb{K} = \psi(a)c + \mathbb{K}.\end{aligned}$$

Hence by using (4.2) we have

$$a \cdot \vartheta_\alpha = \psi(a) \cdot (id_A \otimes \tilde{\psi})^{**} \circ \zeta_\alpha(c + \mathbb{K}) = \psi(a)\vartheta_\alpha,$$

for every $a \in A$. By replacing (ϑ_α) with $(\frac{\vartheta_\alpha}{\psi(\vartheta_\alpha)})$, we have $\psi(\vartheta_\alpha) = 1$ and $a \cdot \vartheta_\alpha = \psi(a)\vartheta_\alpha$. Hence there is $\vartheta_\alpha \in A^{**}$ such that $\psi(\vartheta_\alpha) = 1$ and $a \cdot \vartheta_\alpha = \psi(a)\vartheta_\alpha$ for every $a \in A$. Thus A is left ψ-amenable. $\square$

Proposition 4.2. *Suppose that G is a locally compact group and A is a unital Banach algebra with $\Delta(A) \neq \emptyset$. Suppose that $A \otimes_p L^1(G)$ is approximately biflat. Then G is amenable.*

Proof. Note that $L^1(G)$ has a bounded approximate identity, say $(e_\alpha)_{\alpha \in I}$. A is a unital Banach algebra with a unit element e_A. So $(e_A \otimes e_\alpha)_{\alpha\in I}$ is a bounded approximate identity for $A \otimes_p L^1(G)$. Since $A \otimes_p L^1(G)$ is approximately biflat, by Theorem 4.1, $A \otimes_p L^1(G)$ is $\psi \otimes \phi$-amenable, where $\psi \in \Delta(A)$ and $\phi \in \Delta(L^1(G))$. [18, Theorem 3.3.2] implies that $L^1(G)$ is ϕ-amenable and so $L^1(G)$ has a bounded approximate ϕ-mean. By an argument similar to the one in [1, Corollary 3.4], we can show that $L^1(G)$ has a bounded approximate ϕ_1-mean, where ϕ_1 is an augmentation character on $L^1(G)$. Therefore G is amenable. $\square$

Proposition 4.3. *Suppose that G is a locally compact group. Then the following are equivalent*

(i) *$M(G) \otimes_p L^1(G)$ is biflat.*

(ii) *$M(G) \otimes_p L^1(G)$ is approximately biflat.*

(iii) *G is discrete and amenable.*

Proof. (i)$\Longrightarrow$(ii) It is obvious.

(ii)$\Longrightarrow$(iii). Let $M(G) \otimes_p L^1(G)$ be approximately biflat. Since $M(G)$ is a unital Banach algebra, by an argument similar to the one in [13, Proposition 2.6], we can show that $M(G)$ is approximately biflat. Hence by [17, Theorem 4.2], $M(G)$ is pseudo-amenable. It is well-known that $M(G)$ is pseudo-amenable if and only if G is discrete and amenable [6, Proposition 4.2].

(iii)$\Longrightarrow$(i). Suppose that G is discrete and amenable. Then by [3] $M(G)$ and $L^1(G)$ is amenable. Therefore $M(G) \otimes_p L^1(G)$ is amenable [9, Proposition 5.4] and so $M(G) \otimes_p L^1(G)$ is biflat [14]. $\square$

Samei *et al.* in [17, Corollary 3.2] prove that if G is a *SIN* group, then the followings are equivalent

(i) G is amenable.

(ii) $S(G)$ is approximately biflat.

(iii) $S(G)$ is pseudo-amenable.

In the following proposition, we study approximate biflatness for Segal algebra $S(G)$ on locally compact group G.

PROPOSITION 4.4. *Let G be a locally compact group and let $S(G)$ be approximately biflat. Then G is amenable.*

Proof. Note that $S(G)$ has a left approximate identity. Since $S(G)$ is approximately biflat, by Theorem 4.1, $S(G)$ is left ϕ-amenable, where $\phi \in \Delta(S(G))$. So [1, Corollary 3.4] implies that G is amenable . $\square$

PROPOSITION 4.5. *Let G be a locally compact group and let $S(G)\otimes_p S(G)$ be approximately biflat. Then G is amenable.*

Proof. It is well-known that $S(G)$ has a left approximate identity say $(e_\alpha)_{\alpha\in I}$. Consider $m_\alpha = e_\alpha \otimes e_\alpha$. It is easy to see that $(m_\alpha)_{\alpha\in I}$ is a left approximate identity for $S(G) \otimes_p S(G)$. Since $S(G) \otimes_p S(G)$ is approximately biflat, $S(G) \otimes_p S(G)$ is left $\phi \otimes \psi$-amenable by Theorem 4.1, where $\phi, \psi \in \Delta(S(G))$. This implies that $S(G)$ is left ϕ-amenable [18, Theorem 3.3.2]. Hence G must be amenable [1, Corollary 3.4]. $\square$

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