APPROXIMATELY DUAL FRAMES IN BANACH SPACES VIA SEMI-INNER PRODUCTS

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Communicated by Dan Timotin

In this paper, we develop the concept of dual and approximately dual frames in Banach spaces via semi-inner products and some properties of dual and approximate dual frames are investigated. Also, we introduce g-dual frames in these spaces and some relationships between g-duals and approximate duals are stated. Finally, the ϵ-nearly g-dual frames and their relations with g-duals are studied in Banach spaces using semi-inner products.

AMS 2010 Subject Classification: Primary 42C15; Secondary 46B15, 46C50.

Key words: X_d-frame; semi-inner product; duality mapping; approximately duals; pseudo-duals; g-dual.

1. INTRODUCTION AND PRELIMINARIES

The concept of frame was introduced by Duffin and Schaeffer [12] in 1952. After some decades, Young reintroduced frames in abstract Hilbert spaces [30]. Daubechies, Grossmann and Meyer studied frames deeply in 80's [8]. Feichtinger and Grochenig [16, 22] extended the concept of frames from Hilbert spaces to Banach spaces and defined atomic decomposition and Banach frames. Frames have many nice properties which make them very useful in sampling [13, 14], signal processing [17, 28], filter bank theory [3], and many other fields. Recent applications of the frames in compressed sensing was given in [4] and applications of the frames to operator theory was given in [21]. A sequence $\{f_j\}_{j\in\mathcal{J}}$ in $\mathcal{H}$ is said to be a frame for $\mathcal{H}$ if there exist positive real numbers A, B such that

$$A\|f\|^2 \leq \sum_{j\in\mathcal{J}} |\langle f, f_j\rangle|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}.$$

The elements A and B are called the lower and the upper frame, respectively. Suppose that $\{f_j\}_{j\in\mathcal{J}}$ is a frame of $\mathcal{H}$. The operator $T : \mathcal{H} \to \ell^2(\mathcal{J})$ defined by $T(f) = \{\langle f, f_j\rangle\}_{j\in\mathcal{J}}$ is called the analysis operator. T^* is called the synthesis operator. The operator $S = T^*T$ is called the frame operator of $\{f_j\}_{j\in\mathcal{J}}$.

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A very useful property of a frame $\{f_j\}_{j\in\mathcal{J}}$ for a Hilbert space $\mathcal{H}$ is that $\{f_j\}_{j\in\mathcal{J}}$ has a dual frame $\{g_j\}_{j\in\mathcal{J}}$, i.e. there exists a frame $\{g_j\}_{j\in\mathcal{J}}$ for $\mathcal{H}$ such that for all $f \in \mathcal{H}$,

$$f = \sum_{j\in\mathcal{J}} \langle f, g_j\rangle f_j = \sum_{j\in\mathcal{J}} \langle f, f_j\rangle g_j.$$

It is not easily to find a dual for a frame in general. A more general concept, namely, approximate dual is introduced by O. Christensen and R. S. Laugesen [6], which are more available. In this paper, we intend to introduce these concepts on Banach spaces and so some necessary concepts are introduced as follows.

A sequence space X_d is called a BK-space, if it is a Banach space and the coordinate functionals are continuous on X_d. If the canonical vectors forms a Schauder basis for X_d, then X_d is called a CB-space and its canonical basis is denoted by $\{e_j\}_1^\infty$. If X_d is reflexive and a CB-space, then X_d is called an RCB-space. Also, the dual of X_d is denoted by X_d^*.

The spaces $\ell^\infty, c, c_0, \ell^p(1 \le p < \infty)$ are BK-spaces with their natural norms. Also the space ℓ^∞ has no Schauder basis, since it is not separable and the spaces c_0 and $\ell^p(1 \le p < \infty)$ have $\{e_j\}_1^\infty$ as their Schauder bases.

The concept of semi-inner product, which was introduced in 1961 by G. Lumer [27] and modified by other researchers, is presented in the following definition.

Definition 1.1. [23] Let X be a complex (real) vector space. A semi-inner product (in short s.i.p.) on X is a function from $X \times X \to \mathbb{C}$, denoted by [.,.], such that for all $f, g, h \in X$ and $\lambda \in \mathbb{C}$,

1. $[\lambda f + g, h] = \lambda[f, h] + [g, h]$ and $[f, \lambda g] = \bar{\lambda}[f, g]$,
2. $[f, f] \ge 0$, for all $f \in X$ and $[f, f] = 0$ implies $f = 0$,
3. $|[f, g]|^2 \le [f, f][g, g]$.

However an s.i.p. space need not satisfy the following properties

(*i*) $[f, g] = \overline{[g, f]}$,

(*ii*) $[f, g + h] = [f, g] + [f, h]$.

If $[.,.]$ is a s.i.p. on X then $\|f\| := [f, f]^{\frac{1}{2}}$ is a norm on X. Conversely, if X is a normed vector space then it has a s.i.p. that induces its norm in this manner which is called the compatible semi-inner product [27].

Let X be a Banach space. We define a duality map $\Phi_X : X \to X^*$ as follows. Given $f \in X$, by the Hahn-Banach theorem, there exists an $f^* \in X^*$ such that $\|f\| = \|f^*\|$ and $f^*(f) = \|f\|^2$. Set $\Phi_X(f) = f^*$, and $\Phi_X(\lambda f) = \bar{\lambda} f^*$, and define Φ_X on the rest of X in the same manner. In general, Φ_X is not unique, linear or continuous. The duality map Φ_X induces a semi-inner product

$[.,.]$ if we set $[f,g] = g^*(f)$ [29]. We shall use this definition for g^*, $g \in X$. Note that if X is a Hilbert space, then the duality map is unique [29].

Recall that a Banach space X is called strictly convex, if for any pair of vectors $f, g \neq 0$ in X, the equation $\|f+g\|_X = \|f\|_X + \|g\|_X$, implies that there exists a $\lambda > 0$ such that $f = \lambda g$ [11]. In these spaces, the duality mapping from X to X^* is unique and bijective when X is reflexive [11, 15]. In other words, for each $f^* \in X^*$ there exists a unique $g \in X$ such that $f^*(g) = [g, f]$, for all $g \in X$. Moreover, we have $\|f^*\|_{X^*} = \|f\|_X$. Also, $[f^*, g^*]_* := [g, f]$, $f, g \in X$, defines a compatible semi-inner product on X^* [23]. Note that, in this case $g^{**} = g$, indeed for any $f \in X$

$$\hat{g}(f^*) = f^*(g) = [g, f] = [f^*, g^*]_* = g^{**}(f^*),$$

where $\hat{g}$ is the Gelfand transform of g in X^{**}.

A Banach space X will be said to be uniformly convex if to each ε, $0 < \varepsilon \leq 2$, there corresponds a $\delta(\varepsilon) > 0$ such that the conditions $\|f\|_X = \|g\|_X = 1$, $\|f-g\|_X \geq \varepsilon$ imply $\|\frac{f+g}{2}\|_X \leq 1 - \delta(\varepsilon)$ [7]. We recall that Hilbert spaces, L^p and ℓ^p for $1 < p < \infty$ are uniformly convex and $C[0,1]$ is not uniformly convex [5, 7].

We know that a uniformly convex Banach space is reflexive [5], but a reflexive Banach space is not necessarily uniformly convex [9]. Also, every uniformly convex Banach space is strictly convex [5].

In 2011, H. Zhang and J. Zhang [31] introduced frames in Banach space X via s.i.p. that is presented in the following definition.

Definition 1.2. [31] Let X be a separable Banach space and $[.,.]$ be a compatible semi-inner product on X. Also let X be reflexive and strictly convex and X_d be an CB- space. Then a sequence $\{f_j\}_{j\in\mathcal{J}} \subseteq X$ is called an X_d-frame for X if for any $f \in X$

(i) $\{[f, f_j]\}_{j\in\mathcal{J}} \in X_d$,

(ii) there exist positive constants A, B such that

$$A\|f\|_X \leq \|\{[f, f_j]\}_{j\in\mathcal{J}}\|_{X_d} \leq B\|f\|_X\,, \quad f \in X.$$

If the right side of this inequality holds then we say that $\{f_j\}_{j\in\mathcal{J}}$ is an X_d-Bessel sequence for X.

Recall that an indexed set $F = \{f_j\}_{j\in\mathcal{J}} \subseteq X$ is an X_d-Riesz basis for X if $\overline{span}\{f_j\}_{j\in\mathcal{J}} = X$ and $\sum_{j\in\mathcal{J}} c_j f_j$ converges in X for all $c = \{c_j\}_{j\in\mathcal{J}} \in X_d$ and there exists $0 < A \leq B < \infty$ such that

$$A\|\{c_j\}_{j\in\mathcal{J}}\|_{X_d} \leq \|\sum_{j\in\mathcal{J}} c_j f_j\|_X \leq B\|\{c_j\}_{j\in\mathcal{J}}\|_{X_d}\,, \quad c = \{c_j\}_{j\in\mathcal{J}} \in X_d.$$

Let $F = \{f_j\}_{j\in\mathcal{J}}$ be an X_d-Bessel sequence. The analysis operator $U_F : X \to X_d$ is defined by $U_F(f) := \{[f, f_j]\}_{j\in\mathcal{J}}$ and the adjoint $U_F^* : X_d^* \to$

X^* of U_F is called the synthesis operator which is given by $U_F^*(\{c_j\}_{j\in\mathcal{J}}) := \sum_{j\in\mathcal{J}} c_j f_j^*$.

Let X be a strictly convex separable Banach space, X_d be a uniformly convex BK-space and $F = \{f_j\}_{j\in\mathcal{J}}$ and $F^* := \{f_j^*\}_{j\in\mathcal{J}}$ be X_d and X_d^*-Bessel sequences with analysis operators U_F and U_{F^*} for X and X^*, respectively. We define the X_d-frame operator $S_F : X \to X$ for $\{f_j\}_{j\in\mathcal{J}}$ by $S_F f := U_{F^*}^* U_F f = \sum_{j\in\mathcal{J}}[f, f_j]f_j$, for any $f \in X$, that is well-defined bounded linear operator. S_F is not bijective in general (see [31] for more details).

We need the following results of [31] in our study.

Proposition 1.3. *A subset $F = \{f_j\}_{j\in\mathcal{J}} \subseteq X$ forms an X_d-Bessel sequence for X if and only if $\sum_{j\in\mathcal{J}} b_j f_j^*$ converges in X^* for all $b \in X_d^*$ and*

$$\|\sum_{j\in\mathcal{J}} b_j f_j^*\|_{X^*} \leq B\|b\|_{X_d^*}.$$

Proposition 1.4. *A sequence $F = \{f_j\}_{j\in\mathcal{J}} \subseteq X$ is an X_d-frame for X if and only if the operator U_F^* is bounded and surjective from X_d^* to X^*.*

Theorem 1.5. *Suppose that $F = \{f_j\}_{j\in\mathcal{J}} \subseteq X$ and $F^* = \{f_j^*\}_{j\in\mathcal{J}} \subseteq X^*$ are X_d-Bessel sequence and X_d^*-Bessel sequence with analysis operators U_F and U_{F^*} for X and X^*, respectively. Then the operator $S_F : X \to X$ is bijective and bounded if and only if $\{f_j\}_{j\in\mathcal{J}}$ is an X_d-frame and $\{f_j^*\}_{j\in\mathcal{J}}$ is an X_d^*-frame and $\{f_j\}_{j\in\mathcal{J}}$ is an $R(U)$-Riesz basis for X and in this case we have*

$$f = \sum_{j\in\mathcal{J}}[f, f_j]S_F^{-1} f_j\ , \quad f \in X$$

and

$$f^* = \sum_{j\in\mathcal{J}}[f_j, f](S_F^{-1})^* f_j^* = \sum_{j\in\mathcal{J}}[S_F^{-1} f_j, f]f_j^*\ , \quad f \in X.$$

Theorem 1.6. *If $F = \{f_j\}_{j\in\mathcal{J}} \subseteq X$ is an X_d-frame for X and $R(U_F)$ has an algebraic complement in X_d then there exists an X_d^*-frame $G^* = \{g_j^*\}_{j\in\mathcal{J}} \subseteq X^*$ for X^* such that*

$$f = \sum_{j\in\mathcal{J}}[f, f_j]g_j\ , \quad f \in X \tag{1.1}$$

and

$$f^* = \sum_{j\in\mathcal{J}}[g_j, f]f_j^*\ , \quad f \in X. \tag{1.2}$$

The content of the present paper is as follows. In section 2, we introduce the dual, pseudo-dual and approximate dual of an X_d-Bessel sequence in

Banach space via s.i.p. and some properties and relations between of these concepts are given. In section 3, we describe the notions of g-dual frame and some necessary and sufficient condition for their existence are discussed. Finally, we study the concept of ϵ-nearly g-dual frame for an X_d-Bessel sequence and some results on them are obtained.

Throughout this paper, we assume that X is an uniformly convex separable Banach space, X_d is an uniformly convex BK-space, $\mathcal{J}$ a countable index set and I_X is the identity operator on X. For two Banach spaces X and Y, we denote by $B(X,Y)$ the collection of all bounded linear operators between X and Y. Also, we write $B(X)$ instead of $B(X,X)$.

2. DUAL AND APPROXIMATELY DUAL VIA S.I.P

Theorem 2 leads us to introduce the dual of an X_d-Bessel sequence as follows.

Definition 2.1. Let $F=\{f_j\}_{j\in\mathcal{J}}\subseteq X$ be an X_d-Bessel sequence for X. An X_d^*-Bessel sequence $G^*=\{g_j^*\}_{j\in\mathcal{J}}\subseteq X^*$ is called a dual of F if

$$f=\sum_{j\in\mathcal{J}}[f,f_j]g_j\ ,\quad f\in X.$$

If G^* is a dual of F then we can write $f^*=\sum_{j\in\mathcal{J}}[g_j,f]f_j^*$, for all $f\in X$. Note that, the relation $f=\sum_{j\in\mathcal{J}}[f,g_j]f_j$, is not true, in general. This is established in the following example.

Example. Consider the space $X:=\ell^3(\mathbb{N}_2)$ with the semi-inner product

$$[a,b]:=\|b\|_X^{-1}\sum_{j\in\mathcal{J}}a_j\bar{b_j}|b_j|.$$

Let X_d be an arbitrary BK-space. For any $f\in X$, we have $f^*=\frac{\bar{f}|f|}{\|f\|_X}$. Let $F=\{f_1,f_2\}=\{(1,1),(4,1)\}\subseteq X$. The facts that $\overline{span}F^*=\overline{span}\{f_1^*,f_2^*\}=\overline{span}\{\frac{(1,1)}{(2)^{\frac{1}{3}}},\frac{(16,1)}{(65)^{\frac{1}{3}}}\}=\ell^{\frac{3}{2}}(\mathbb{N}_2)$ and $\overline{span}F=\ell^3(\mathbb{N}_2)$, imply that F and F^* are X_d and X_d^*-frame for X and X^*, respectively. Now consider

$$G=\{(-\frac{(2)^{\frac{1}{3}}}{15},\frac{16(2)^{\frac{1}{3}}}{15}),(\frac{(65)^{\frac{1}{3}}}{15},-\frac{(65)^{\frac{1}{3}}}{15})\}.$$

Then G^* is a dual of F and we can write $f=\sum_{j\in\mathcal{J}}[f,f_j]g_j$, for all $f\in X$. But $f=\sum_{j\in\mathcal{J}}[f,g_j]f_j$ does not hold for all $f\in X$, for example, if we take $f=(0,2)\in X$ then

$$\sum_{j\in\mathcal{J}}[f,g_j]f_j=(-1+(16)^3)^{\frac{-1}{3}}(\frac{32(2)^{\frac{1}{3}}}{15})(1,1)\neq(0,2).$$

Remark 2.2. If $U_F : X \to X_d$, $U_F(f) = \{[f, f_j]\}_{j\in\mathcal{J}}$ is the analysis operator of X_d-Bessel sequence $\{f_j\}_{j\in\mathcal{J}} \subseteq X$ for X with the adjoint operator $U_F^* : X_d^* \to X^*$, $U_F^*(\{c_j\}_{j\in\mathcal{J}}) = \sum_{j\in\mathcal{J}} c_j f_j^*$ and $U_{G^*} : X^* \to X_d^*$, $U_{G^*}(g^*) := \{[g_j, g]\}_{j\in\mathcal{J}}$ is the analysis operator of X_d^*-Bessel sequence $\{g_j^*\}_{j\in\mathcal{J}} \subseteq X^*$ for X^* with the adjoint operator $U_{G^*}^* : X_d \to X$, $U_{G^*}^*(\{d_j\}_{j\in\mathcal{J}}) = \sum_{j\in\mathcal{J}} d_j g_j$, then the relation (1.1), (1.2) can be written as follows

$$f = U_{G^*}^* U_F(f)\,, \quad f \in X,\ i.e.\ I_X = U_{G^*}^* U_F$$

and hence

$$f^* = U_F^* U_{G^*}(f^*)\,, \quad f \in X,\ i.e.\ I_{X^*} = U_F^* U_{G^*}.$$

Recall that $A \in B(X)$ is called an adjoint abelian operator if there exists a duality map $\Phi_X : X \to X^*$, such that $A^*\Phi_X = \Phi_X A$ (equivalently, $(Ax)^* = A^*x^*$, for all $x \in X$ or $[Ax, y] = [x, Ay]$, for all $x, y \in X$). It is well-known that if A is bijective and adjoint abelian then A^{-1} is also adjoint abelian (see [29]).

For example, if X is a Hilbert space, then the adjoint abelian operators are precisely the self-adjoint ones [29], and every adjoint abelian operator on $C(K)$, (K compact) or $L^p(1 < p < \infty, p \neq 2)$ is a multiple of an isometry whose square is the identity [18]. As another example of adjoint abelian operator, if X is the ℓ^p sum of a one dimensional and a two dimensional space, then

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{bmatrix}$$

is adjoint abelian on that space [19].

The following lemma shows that adjoint abelian operators preserve X_d-Bessel sequences. For the ordinary frames, it has been shown that if $\{f_j\}_{j\in\mathcal{J}}$ is a frame for Hilbert space $\mathcal{H}$ and $T \in B(\mathcal{H})$ then $\{Tf_j\}_{j\in\mathcal{J}}$ is a frame for $\mathcal{H}$ if and only if T is surjective. In the Banach setting we may have the following lemma.

Lemma 2.3. *Suppose that $F = \{f_j\}_{j\in\mathcal{J}} \subseteq X$ is an X_d-Bessel sequence for X with the bound B and $T \in B(X)$ is an adjoint abelian operator then*

(i) *$TF = \{Tf_j\}_{j\in\mathcal{J}} \subseteq X$ is an X_d-Bessel sequence for X with the bound $\|T\|B$.*

(ii) *Let F be an X_d-frame for X and $T \in B(X)$ is an adjoint abelian operator then $T^* \in B(X^*)$ is surjective if and only if TF is an X_d-frame for X.*

Proof. (i) For any $\{c_j\} \in X_d^*$, by Proposition 1, one can see that

$$\|\sum_{j\in\mathcal{J}} c_j (Tf_j)^*\|_{X^*} = \|T^* \sum_{j\in\mathcal{J}} c_j f_j^*\|_{X^*} \leq \|T^*\| \|\sum_{j\in\mathcal{J}} c_j f_j^*\|_{X^*}$$

$$\leq \quad B\|T^*\|\|\{c_j\}\|_{X_d^*} = B\|T\|\|\{c_j\}\|_{X_d^*}.$$

(ii) First, suppose that $T \in B(X)$ is an adjoint abelian operator such that $T^* \in B(X^*)$ is surjective. Let F be an X_d-frame for X, then U_F^* is bounded and surjective. On the other hand, the synthesis operator of TF is $U_{TF}^* : X_d^* \to X^*$ which is of the following form

$$U_{TF}^*(\{c_j\}_{j\in\mathcal{J}}) = \sum_{j\in\mathcal{J}} c_j(Tf_j)^* = \sum_{j\in\mathcal{J}} c_jT^*f_j^* = T^*\sum_{j\in\mathcal{J}} c_jf_j^* = T^*U_F^*(\{c_j\}_{j\in\mathcal{J}}).$$

Now, since $T^*U_F^*$ is bounded and surjective hence, by Proposition 2, TF is an X_d-frame for X. For the inverse, let TF be an X_d-frame for X then $U_{TF}^* = T^*U_F^*$ is bounded and surjective and thus T^* is surjective. □

Now, the notions pseudo-dual and approximate dual of an X_d-Bessel sequence for X are introduced and their relations and properties are established.

Definition 2.4. Let $F = \{f_j\}_{j\in\mathcal{J}} \subseteq X$ be an X_d-Bessel sequence for X and $G^* = \{g_j^*\}_{j\in\mathcal{J}} \subseteq X^*$ be an X_d^*-Bessel sequence for X^* then F and G^* are said to be

(i) approximately dual X_d-frames if

$$\|I_X - U_{G^*}^*U_F\|_X < 1 \text{ or } \|I_{X^*} - U_F^*U_{G^*}\|_{X^*} < 1.$$

(ii) pseudo-dual X_d-frames if $U_{G^*}^*U_F$ or $U_F^*U_{G^*}$ is bijection on X and X^*, respectively.

Note that, if $F = \{f_j\}_{j\in\mathcal{J}}$ and $G^* = \{g_j^*\}_{j\in\mathcal{J}}$ are approximately dual X_d-frames then trivially F and G^* are pseudo-dual X_d-frames.

Now, let F and G^* be approximately dual X_d-frames. In this case, under some condition on $G = \{g_j\}_{j\in\mathcal{J}}$ we may construct a dual of F. In fact, since $\|I_X - U_{G^*}^*U_F\|_X < 1$ then $U_{G^*}^*U_F$ is invertible and for any $f \in X$ we have $f = (U_{G^*}^*U_F)^{-1}(U_{G^*}^*U_F)f = (U_{G^*}^*U_F)^{-1}\sum_{j\in\mathcal{J}}[f,f_j]g_j = \sum_{j\in\mathcal{J}}[f,f_j](U_{G^*}^*U_F)^{-1}g_j$.

Put $H = \{h_j\}_{j\in\mathcal{J}} = \{(U_{G^*}^*U_F)^{-1}g_j\}_{j\in\mathcal{J}}$ and let $H^* = \{h_j^*\}_{j\in\mathcal{J}} \subseteq X^*$ be an X_d^*-Bessel sequence for X^* and U_H^* and $U_{H^*}^*$ are synthesis operators of H and H^*, respectively, then we can write $f = U_{H^*}^*U_Ff$, $f \in X$, and this means that H^* is a dual of F.

Proposition 2.5. *Let $F = \{f_j\}_{j\in\mathcal{J}}$ and $G^* = \{g_j^*\}_{j\in\mathcal{J}}$ be X_d and X_d^*-Bessel sequences for X and X^*, respectively, then*

(i) *If G^* is a dual frame for F, then F and G^* are approximately X_d-dual frames.*

(ii) *If F and G^* are approximately dual X_d-frames, then F and G^* are pseudo-dual X_d-frames.*

(iii) *If F and G^* are pseudo-dual X_d-frames and $T \in B(X)$ is bijection such that T^* is an adjoint abelian operator, then F and $T^*G^* = \{T^*g_j^*\}_{j\in\mathcal{J}}$ are pseudo-dual X_d-frames.*

(iv) *If F and G^* are pseudo-dual X_d-frames and $(U_{G^*}^*U_F)^{-1}$ is an adjoint abelian operator, then $H^* = \{((U_{G^*}^*U_F)^{-1}g_j)^*\}_{j\in\mathcal{J}}$ is a dual of F.*

Proof. The proofs of (i) and (ii) are trivial by definitions.

For the proof of (iii), by Lemma 1, T^*G^* is an X_d^*-Bessel sequence, thus the synthesis operator for T^*G^* is $U_{T^*G^*}^* = TU_{G^*}^*$, since

$$U_{T^*G^*}^*(\{c_j\}_{j\in\mathcal{J}}) = \sum_{j\in\mathcal{J}} c_j(T^*g_j^*)^* = \sum_{j\in\mathcal{J}} c_jTg_j = T\sum_{j\in\mathcal{J}} c_jg_j = TU_{G^*}^*(\{c_j\}_{j\in\mathcal{J}}).$$

The assumptions that F and G^* are pseudo-dual frames and T is bijection imply that $U_{T^*G^*}^*U_F = TU_{G^*}^*U_F$ is bijection and then F and T^*G^* are pseudo-dual X_d-frames.

For (iv), note that, if F and G^* are pseudo-dual X_d-frames then $(U_{G^*}^*U_F)^{-1}$ exists and is bounded and hence H^* is an X_d^*-Bessel sequence. In this case we have:

$$\sum_{j\in\mathcal{J}}[f,f_j](U_{G^*}^*U_F)^{-1}g_j = ((U_{G^*}^*U_F)^{-1}\sum_{j\in\mathcal{J}}[f,f_j]g_j = ((U_{G^*}^*U_F)^{-1}(U_{G^*}^*U_F)f = f.$$

Therefore, F and H^* are dual frames. □

Under some conditions on the X_d-frame operator S_F, we may construct a dual of the X_d-frame $F = \{f_j\}_{j\in\mathcal{J}}$ as follows.

PROPOSITION 2.6. *Let $S_F := U_{F^*}^*U_F : X \to X$ be a bijective, bounded and adjoint abelian operator, then $\{(S_F^{-1}f_j)^*\}_{j\in\mathcal{J}}$ is a dual of $\{f_j\}_{j\in\mathcal{J}}$.*

Proof. Suppose that S_F is bijective, bounded and adjoint abelian operator, then by Theorem 1, $\{f_j\}_{j\in\mathcal{J}} \subseteq X$ and $\{f_j^*\}_{j\in\mathcal{J}} \subseteq X^*$ are X_d-frame and X_d^*-frame for X and X^*, respectively, and thus $\{(S_F^{-1}f_j)^*\}_{j\in\mathcal{J}}$ is an X_d^*-Bessel sequence since S_F^{-1} is adjoint abelian operator and hence

$$f = \sum_{j\in\mathcal{J}}[f,f_j]S_F^{-1}f_j, \quad f \in X,$$

i.e. $\{(S_F^{-1}f_j)^*\}_{j\in\mathcal{J}}$ is a dual frame of $\{f_j\}_{j\in\mathcal{J}}$. □

THEOREM 2.7. *Suppose that $F = \{f_j\}_{j\in\mathcal{J}} \subseteq X$ and $G^* = \{g_j^*\}_{j\in\mathcal{J}} \subseteq X^*$ are approximately dual X_d-frames then the following holds,*

(i) *If $U_{G^*}^*U_F$ is an adjoint abelian operator then the dual frame $H^* = \{((U_{G^*}^*U_F)^{-1}g_j)^*\}_{j\in\mathcal{J}}$ of F can be written as follows*

$$((U_{G^*}^* U_F)^{-1} g_j)^* = g_j^* + (\sum_{j\in\mathcal{J}} (I_{X^*} - (U_{G^*}^* U_F)^*))^n g_j^*.$$

(ii) *Let $N \in \mathbb{N}$ be given, consider the corresponding partial sum,*

$$\begin{aligned} \gamma_j^{(N)} &= g_j + \sum_{n=1}^{N} (I_X - U_{G^*}^* U_F)^n g_j \\ &= \sum_{n=0}^{N} (I_X - U_{G^*}^* U_F)^n g_j \end{aligned}$$

and let $\Gamma^* = \{(\gamma_j^{(N)})^*\}_{j\in\mathcal{J}}$ be a X_d^*-Bessel sequence then Γ^* is an approximate dual of F. Denoting its associated synthesis operator by $U_{\Gamma^*}^*$ we have

$$\|I_X - U_{\Gamma^*}^* U_F\|_X \leq \|I_X - U_{G^*}^* U_F\|_X^{N+1} \to 0\,, \quad when\, N \to \infty.$$

Proof. (i) Since $f = (U_{G^*}^* U_F)^{-1}(U_{G^*}^* U_F) f = (U_{G^*}^* U_F)^{-1} \sum_{j\in\mathcal{J}} [f, f_j] g_j = \sum_{j\in\mathcal{J}} [f, f_j](U_{G^*}^* U_F)^{-1} g_j\,, f \in X$ then H^* is a dual of F and we have

$$\begin{aligned} ((U_{G^*}^* U_F)^{-1})^* &= (I_{X^*} - (I_{X^*} - (U_{G^*}^* U_F)^*))^{-1} \\ &= \sum_{n=0}^{\infty} (I_{X^*} - (U_{G^*}^* U_F)^*)^n. \end{aligned}$$

Now, by the fact that $U_{G^*}^* U_F$ is an adjoint abelian operator we get

$$\begin{aligned} ((U_{G^*}^* U_F)^{-1} g_j)^* &= ((U_{G^*}^* U_F)^*)^{-1} g_j^* \\ &= g_j^* + (\sum_{j\in\mathcal{J}} (I_{X^*} - (U_{G^*}^* U_F)^*)^n g_j^*. \end{aligned}$$

(ii) Note that

$$\begin{aligned} U_{\Gamma^*}^* U_F f &= \sum_{j\in\mathcal{J}} [f, f_j] \gamma_j^{(N)} \\ &= \sum_{j\in\mathcal{J}} [f, f_j] (\sum_{n=0}^{N} (I_X - U_{G^*}^* U_F)^n g_j) \\ &= \sum_{n=0}^{N} (I_X - U_{G^*}^* U_F)^n \sum_{j\in\mathcal{J}} [f, f_j] g_j \\ &= \sum_{n=0}^{N} (I_X - U_{G^*}^* U_F)^n U_{G^*}^* U_F f \\ &= \sum_{n=0}^{N} (I_X - U_{G^*}^* U_F)^n (I_X - (I_X - U_{G^*}^* U_F)) f \end{aligned}$$

$$\begin{aligned} &= \sum_{n=0}^{N}(I_X - U_{G^*}^*U_F)^n f - (I_X - U_{G^*}^*U_F)^{n+1} f \\ &= f - (I_X - U_{G^*}^*U_F)^{N+1} f \end{aligned}$$

and hence

$$\|I_X - U_{\Gamma^*}^*U_F\|_X = \|(I_X - U_{G^*}^*U_F)^{N+1}\|_X \leq \ \|I_X - U_{G^*}^*U_F\|_X^{N+1} < 1.$$

□

In the next proposition, we prove a stability result for having an approximate dual.

PROPOSITION 2.8. *Suppose that* $F = \{f_j\}_{j\in\mathcal{J}}$ *is an* X_d*-Bessel sequence in* X *and* $H = \{h_j\}_{j\in\mathcal{J}}$ *is an* X_d*-Bessel sequence for which*

$$\|\{[f, h_j]\}_{j\in\mathcal{J}} - \{[f, f_j]\}_{j\in\mathcal{J}}\|_{X_d} \leq \ R\|f\|_X \ , \quad f \in \ X$$

for some $R > 0$. *Consider a dual* X_d*-frame* $G^* = \{g_j^*\}_{j\in\mathcal{J}}$ *of* H *with the synthesis operator* U_{G^*} *and assume that* G^* *has upper frame bound* C. *If* $CR < 1$, *then* F *and* G^* *are approximately dual* X_d*-frames, with*

$$\|I_X - U_{G^*}^*U_F\|_X < 1.$$

Proof. From the fact that $G^* = \{g_j^*\}_{j\in\mathcal{J}}$ is a dual for $H = \{h_j\}_{j\in\mathcal{J}}$ hence $U_{G^*}^*U_H = I_X$ and therefore
$\|I_X - U_{G^*}^*U_F\|_X = \|U_{G^*}^*U_H - U_{G^*}^*U_F\| = \|U_{G^*}^*(U_H - U_F)\| \leq \|U_{G^*}\|\|U_H - U_F\| \leq \ CR < 1.$ □

3. G-DUAL AND APPROXIMATELY G-DUAL

The concept of g-dual frames introduced for ordinary frame in [10]. In this section, we are going to express this notion for an X_d- Bessel sequence in Banach space via s.i.p. Also, we present some relations between g-dual and approximate dual. Finally, we define the concept of ϵ-nearly g-dual frame for an X_d- Bessel sequence.

Definition 3.1. Let $F = \{f_j\}_{j\in\mathcal{J}} \subset X_d$ be an X_d- Bessel sequence for X. An X_d^*- Bessel sequence $G^* = \{g_j^*\}_{j\in\mathcal{J}} \subset X_d^*$ is called a generalized dual X_d-frame (or g-dual X_d-frame) for F for X^* if there exists an invertible operator $A \in B(X)$ such that

$$f = \sum_{j\in\mathcal{J}}[Af, f_j]g_j \ , \quad f \in \ X.$$

Example. Let $X := \ell^3(\mathbb{N}_2)$ be endowed with its standard s.i.p and $F = \{f_1, f_2\} = \{(1,1),(1,4)\} \subseteq X$. Also, assume that A is defined by $A(a,b) = (2b, a)$. Then for

$$G = \{(-\frac{(2)^{\frac{1}{3}}}{15}, \frac{8(2)^{\frac{1}{3}}}{15}), (\frac{(65)^{\frac{1}{3}}}{15}, -\frac{(65)^{\frac{1}{3}}}{30})\}$$

one can see that G^* is a g-dual of F with respect to the operator A.

Clearly, if $A = I_X$, then G^* is a dual X_d-frame for F. Also, by Theorem 1, when S_F is bijective and bounded, we have

$$f = \sum_{j\in\mathcal{J}} [f, f_j] S_F^{-1} f_j = \sum_{j\in\mathcal{J}} [S_F^{-1} f, f_j] f_j \,, \quad f \in X.$$

Therefore, any frame is a g-dual X_d-frame for itself. Also, if U_F and U_{G^*} are synthesis operators of F and G^*, respectively, then the equality $f = \sum_{j\in\mathcal{J}} [Af, f_j] g_j$ means that $f = U_{G^*}^* U_F A f$ and we can write $I_X = U_{G^*}^* U_F A$ i.e. $A^{-1} = U_{G^*}^* U_F$ and thus

$$A^{-1} f = \sum_{j\in\mathcal{J}} [f, f_j] g_j, \; f \in X.$$

Remark 3.2. Note that, if G^* is a g-dual of F with respect to A, then F^* is not necessarily the dual of G with respect to A. For example, assume that $X := \ell^3(\mathbb{N}_2)$ with the semi-inner product as in Example 1, and X_d-frame $F = \{f_1, f_2\} = \{(1,1),(1,4)\} \subseteq X$. Consider $G^* \subseteq \ell^{\frac{3}{2}}(\mathbb{N}_2)$ by $G = \{g_1, g_2\} = \{(\frac{16(2)^{\frac{1}{3}}}{5}, -\frac{(2)^{\frac{1}{3}}}{5}), (-\frac{(65)^{\frac{1}{3}}}{5}, \frac{(65)^{\frac{1}{3}}}{5})\}$. One can see that G^* is a g-dual of F with corresponding operator $Af = \frac{1}{3} f$, but we can not write $f = \sum_{j\in\mathcal{J}} [Af, g_j] f_j$ for all f. Indeed, for $f = (0,1) \in \ell^3(\mathbb{N}_2)$ we have

$$\sum_{j\in\mathcal{J}} [Af, g_j] f_j = -\frac{1}{15}((2)^{12} - 1)(2)^{\frac{1}{3}}(1,1) \neq (0,1).$$

The relation between g-dual and approximate dual is stated in the following proposition.

PROPOSITION 3.3. *Suppose that F and G^* are X_d and X_d^*-Bessel sequences for X and X^*, respectively. Then F and G^* are approximately dual X_d-frames if and only if G^* is a g-dual of F with respect to some $A \in B(X)$ with $\|I_X - A^{-1}\|_X < 1$.*

Proof. Firstly, assume that F and G^* are approximately dual X_d-frames. Then $U_{G^*}^* U_F$ is invertible and putting $A^{-1} = U_{G^*}^* U_F$. One can write

$$f = (U_{G^*}^* U_F)(U_{G^*}^* U_F)^{-1} f = \sum_{j\in\mathcal{J}} [(U_{G^*}^* U_F)^{-1} f, f_j] g_j \,, \quad f \in X$$

i.e. G^* is a g-dual of F with respect to the operator A. The proof of the inverse is trivial. □

Remark 3.4. (i) Recall that, if X, Y and Z are Banach spaces then we say $T \in B(X,Y)$ majorizes $Q \in B(X,Z)$ if there exists $\lambda > 0$ such that $\|Qf\| \leq \lambda\|Tf\|$ for all $f \in X$ (for more details see [2]).

(ii) As we know, if $A \in B(X)$ and $sp(A) \cap (-\infty, 0) = \emptyset$ then A has a unique square root which is denoted by $A^{\frac{1}{2}}$ (see [20, 26]).

Now, we state a sufficient and necessary condition for two X_d and X_d^*-Bessel sequences for X and X^*, respectively, such that they are g-dual frames. In Hilbert spaces, there is a similar expression for classical frame that is stated in [13].

THEOREM 3.5. *Let F and G^* be X_d and X_d^*-Bessel sequences for X and X^*, respectively, $sp(S_F) \cap (-\infty, 0) = \emptyset$, $S_F^{\frac{1}{2}}$ is an adjoint abelian operator and $\{[f, f_j]\}_{j\in\mathcal{J}} = \{[f_j, f]\}^*_{j\in\mathcal{J}}$. Then G^* is a g-dual of F with respect to an invertible operator $A \in B(X)$ if and only if $U^*_{G^*}U_F$ is invertible and there exists an operator $Q \in B(X)$ such that $U^*_{G^*}U_F = QS_F^{\frac{1}{2}}$.*

Proof. First, if G^* is a g-dual of F with respect to an invertible operator $A \in B(X)$, then $f = \sum_{j\in\mathcal{J}}[Af, f_j]g_j = U^*_{G^*}U_F Af\,, \quad f \in X$.

Also, note that

$$[S_F Af, Af] = \|\{[Af, f_j]\}_{j\in\mathcal{J}}\|^2_{X_d}.$$

indeed

$$\begin{aligned}
[S_F Af, Af] &= [U^*_{F^*}U_F Af, Af] \\
&= (Af)^*(U^*_{F^*}U_F Af) \\
&= U_{F^*}(Af)^*(U_F Af) \\
&= [U_F Af, (U_{F^*}(Af)^*)^*] \\
&= [\{[Af, f_j]\}_{j\in\mathcal{J}}, \{[f_j, Af]\}^*_{j\in\mathcal{J}}] \\
&= [\{[Af, f_j]\}_{j\in\mathcal{J}}, \{[Af, f_j]\}_{j\in\mathcal{J}}] \\
&= \|\{[Af, f_j]\}_{j\in\mathcal{J}}\|^2_{X_d}.
\end{aligned}$$

Now G^* is an X_d^*-Bessel sequence so for some $D \geq 0$

$$\begin{aligned}
\|U^*_{G^*}U_F Af\|_X &= \sup_{\|g^*\|=1, g\in X} g^*(U^*_{G^*}U_F Af) \\
&= \sup_{\|g\|=1} [U^*_{G^*}U_F Af, g] \\
&= \sup_{\|g\|=1} [\sum_{j\in\mathcal{J}} [Af, f_j]g_j, g]
\end{aligned}$$

$$\begin{aligned}
&= \sup_{\|g\|=1} \sum_{j\in\mathcal{J}} [Af, f_j][g_j, g] \\
&\leq \sup_{\|g\|=1} \|\{[Af, f_j]\}_{j\in\mathcal{J}}\|_{X_d} \|\{[g_j, g]\}_{j\in\mathcal{J}}\|_{X_d^*} \\
&\leq \sup_{\|g\|=1} \|\{[Af, f_j]\}_{j\in\mathcal{J}}\|_{X_d} D\|g\|_X \\
&= D\|\{[Af, f_j]\}_{j\in\mathcal{J}}\|_{X_d} \\
&= D[S_F Af, Af]^{\frac{1}{2}} \\
&= D[S_F^{\frac{1}{2}} S_F^{\frac{1}{2}} Af, Af]^{\frac{1}{2}} \\
&= D[S_F^{\frac{1}{2}} Af, S_F^{\frac{1}{2}} Af]^{\frac{1}{2}} \\
&= D\|S_F^{\frac{1}{2}} Af\|_X
\end{aligned}$$

i.e.

$$\|U_{G^*}^* U_F Af\|_X \leq D\|S_F^{\frac{1}{2}} Af\|_X$$

and thus $S_F^{\frac{1}{2}} A$ majorizes $U_{G^*}^* U_F A$. By Proposition 3 [2], there exists the operator $Q \in B(X)$ such that $U_{G^*}^* U_F A = Q S_F^{\frac{1}{2}} A$ and by invertibility of A, we have $U_{G^*}^* U_F = Q S_F^{\frac{1}{2}}$. The opposite implication holds by definition. □

By adding a condition to assumptions of Theorem 4, we obtained the following result.

Corollary 3.6. *Let F and G^* be X_d and X_d^*-Bessel sequences for X and X^*, respectively, $sp(S_F) \cap (-\infty, 0) = \emptyset$, $S_F^{\frac{1}{2}}$ is an adjoint abelian operators and $\{[f, f_j]\}_{j\in\mathcal{J}}^* = \{[f_j, f]\}_{j\in\mathcal{J}}$. Then F and G^* are approximately dual X_d-frames if and only if there exists an operator $Q \in B(X)$ such that $U_{G^*}^* U_F = Q S_F^{\frac{1}{2}}$ and $\|I_X - Q S_F^{\frac{1}{2}}\|_X < 1$.*

Finally, we state the concept of an ϵ-nearly g-dual frame in Banach space (see [24]).

Definition 3.7. Suppose that X is a Banach space and let $F = \{f_j\}_{j\in\mathcal{J}} \subset X$ be an X_d-Bessel sequence for X, also let $0 < \epsilon < 1$. An X_d^*-Bessel sequence $G^* = \{g_j^*\}_{j\in\mathcal{J}} \subset X^*$ is called an ϵ-nearly g-dual frame for F if there exists an invertible operator $A \in B(X)$ such that

$$\|f - \sum_{j\in\mathcal{J}} [Af, f_j] g_j\|_X < \epsilon\|f\|_X, \quad f \in X.$$

Clearly by definition, all g-dual frames and ordinary dual frames of any X_d-frame are ϵ-nearly g-dual frame. Also, if $G^* = \{g_j^*\}_{j\in\mathcal{J}}$ is an ϵ-nearly g-dual frame of $F = \{f_j\}_{j\in\mathcal{J}}$ then it is not necessary that F^* is an ϵ-nearly g-dual frame of G.

PROPOSITION 3.8. *Let $F = \{f_j\}_{j\in\mathcal{J}} \subset X$ and $G^* = \{g_j^*\}_{j\in\mathcal{J}} \subset X^*$ be X_d and X_d^*-Bessel sequence for X and X^*, respectively. Then G^* is an ϵ-nearly g-dual frame of F if and only if G^* is a g-dual frame of F.*

Proof. The necessary part is obvious. For the converse, let G^* be an ϵ-nearly g-dual frame of F. Thus we have $\|I_X - U_{G^*}^* U_F A\|_X < \epsilon < 1$, hence $U_{G^*}^* U_F A$ is an invertible operator and we can write

$$f = (U_{G^*}^* U_F A)(U_{G^*}^* U_F A)^{-1} f = \sum_{j\in\mathcal{J}} [A(U_{G^*}^* U_F A)^{-1} f, f_j] g_j\ , \quad f \in X.$$

So, G^* is a g-dual frame of F. □

THEOREM 3.9. *Let $F = \{f_j\}_{j\in\mathcal{J}} \subset X$ be a X_d-Bessel sequence for X, $G^* = \{g_j^*\}_{j\in\mathcal{J}} \subset X^*$ and $H^* = \{h_j^*\}_{j\in\mathcal{J}} \subset X^*$ be X_d^*-Bessel sequences for X^*. If G^* is a g-dual frame of F with respect to an invertible operator A, then H^* is a g-dual frame of F with respect to A if and only if*

$$Rang(U_{G^*} - U_{H^*}) \subset Ker(U_F^*).$$

Proof. First, if H^* is a g-dual of F, then there exists an invertible operator $A \in B(X)$ such that

$$f = \sum_{j\in\mathcal{J}} [Af, f_j] h_j = U_{H^*}^* U_F A f\ , \quad f \in\ X.$$

Hence $A^{-1} = U_{H^*}^* U_F$, since by assumption, we have $A^{-1} = U_{G^*}^* U_F$, thus $U_{H^*}^* U_F = U_{G^*}^* U_F$, and therefore $Rang(U_{G^*} - U_{H^*}) \subset Ker(U_F^*)$.

For the inverse, assume that $Rang(U_{G^*} - U_{H^*}) \subset Ker(U_F^*)$ holds, then $U_{H^*}^* U_F = U_{G^*}^* U_F$ and thus we have $A^{-1} = U_{H^*}^* U_F = U_{G^*}^* U_F$, now we can write

$$f = A^{-1} A f = U_{H^*}^* U_F A f = \sum_{j\in\mathcal{J}} [Af, h_j] f_j\ , \quad f \in X.$$

□

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Received June 25, 2017

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