SECOND MAIN THEOREM FOR HOLOMORPHIC CURVES WITH THE MOVING TARGETS ON ANNULI

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In this paper, we obtain the second main theorem for holomorphic curves on annuli crossing a finite set of moving hyperplanes in u-subgeneral position in $\mathbb{P}^n(\mathbb{C})$.

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1. INTRODUCTION AND MAIN RESULTS

There are many results on second main theorem for holomorphic curves from $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ with fixed or moving targets. In 1925, Nevanlinna [8] established the second main theorem for meromorphic functions on $\mathbb{C}$. In 1933, Cartan [3] proposed the second main theorem for holomorphic curves from $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$. In 1983, E. I. Nochka [9] proved the second main theorem in case of hyperplanes in the u-subgeneral position in $\mathbb{P}^n(\mathbb{C})$ which extended the Nevanlinna-Cartan second main theorem and confirmed a conjecture by Cartan. In 1997, M. Ru [12] showed a simple proof of the second main theorem with moving hyperplanes which was originally proved by M. Ru and W. Stoll [14] in 1991. In 2004, M. Ru and T. Y. Wang [15] proved the second main with ramification for a holomorphic curve intersecting a finite set of moving or fixed hyperplanes. For the background of Nevanlinna theory, we refer to [13].

In this paper we mainly consider the case for holomorphic curves from doubly connected domain into $\mathbb{P}^n(\mathbb{C})$. By the Doubly Connected Mapping Theorem [1] each doubly connected domain is conformally equivalent to the annulus $\mathbb{A}(R_1, R_2) = \{z : R_1 < |z| < R_2\}$, $0 \leq R_1 < R_2 \leq +\infty$. We need only consider two cases: $R_1 = 0$, $R_2 = +\infty$ simultaneously and $0 < R_1 < R_2 < +\infty$. In the latter case the homothety $z \mapsto \frac{z}{\sqrt{R_1 R_2}}$ reduces the given domain to the annulus

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$\{z : \frac{1}{R_0} < |z| < R_0\}$, where $R_0 = \sqrt{\frac{R_2}{R_1}}$. Thus, in the two cases every annulus is invariant with respect to the inversion $z \mapsto \frac{1}{z}$.

In recent years, there are some results about holomorphic maps on annuli $\mathbb{A} = \{z \in \mathbb{C} : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. In 2005, A. Y. Khrystiyanyn and A. A. Kondratyuk [5, 6] proved Nevanlinna theory for meromorphic functions on $\mathbb{A}$. Cao, Yi and Xu [2] proved a generalized theorem on the multiple values and uniqueness of meromorphic functions in the annulus $\mathbb{A}$. In 2009, Lund and Ye [7] considered the logarithmic derivatives in annuli. In 2012, Chen and Wu [4] investigated exceptional values of meromorphic functions on annuli. In 2015, H. T. Phuong and N. V. Thin [11] considered the extension of the Nevanlinna-Cartan second main theorem for holomorphic curves from $\mathbb{A}$ into $\mathbb{P}^n(\mathbb{C})$ crossing a finite set of fixed hyperplanes in general position.

THEOREM 1.1 ([11, Theorem 1.2]). *Let $f : \mathbb{A} \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate holomorphic curve, and let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in the general position. Thus, we get*

$$(q - n - 1)T_f(r) \leq \sum_{j=1}^{q} N_f^n(r, H_j) + S(r, f).$$

where

$$S(r, f) = \begin{cases} O(\log r + \log T_f(r)), & \text{for } R_0 = +\infty; \\ O(\log \frac{1}{R_0 - r} + \log T_f(r)), & \text{for } R_0 < +\infty. \end{cases}$$

Throughout this paper, if $R_0 = +\infty$ the inequality (in the second main theorem) holds for $r \in (1, +\infty)$ outside a set Δ'_r satisfying the inequality

$$\int_{\Delta'_r} r^{\lambda - 1} dr < +\infty;$$

if $R_0 < +\infty$ the inequality holds for $r \in (1, R_0)$ outside a set Δ'_r satisfying

$$\int_{\Delta'_r} \frac{1}{(R_0 - r)^{\lambda + 1}} dr < +\infty.$$

Thus, it is natural to ask how about the Nochka's version or Ru's version of second main theorems on annuli, according to the trend of H. T. Phuong and N. V. Thin. [11].

Motivated by this problem, the main purpose of this paper is to adopt the idea of Ru and his coauthors [12, 14, 15], and obtain the second theorem for holomorphic curves on annuli into complex projective spaces intersecting moving hyperplanes targets in subgeneral position.

THEOREM 1.2. *Let $f : \mathbb{A} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map. Let $\mathcal{G}$ be a finite set of moving hyperplanes $H_1, \dots, H_q$ which define respectively holomorphic maps $a_1, \dots, a_q : \mathbb{A} \to \mathbb{P}^n(\mathbb{C})$ with $T_{a_j(r)} = o(T_f(r))$. Assume that $\mathcal{G}$ is in u-subgeneral position. Let $\mathcal{R}_{\mathcal{G}}$ be the small field with contains $\mathbb{C}$ and all $a_{j\mu}/a_{j\nu}$ with $a_{j\nu} \not\equiv 0$. If f is linearly non-degenerate over $\mathcal{R}_{\mathcal{G}}$. Then*

$$\sum_{j=1}^{q} m_f(r, H_j) \leq (2u - n + 1)T_f(r) + S(r, f).$$

where

$$S(r,f) = \begin{cases} O(\log r + \log T_f(r)), & \text{for } R_0 = +\infty; \\ O(\log \frac{1}{R_0 - r} + \log T_f(r)), & \text{for } R_0 < +\infty. \end{cases}$$

Remark that by the first main theorem 2.3 [11, Theorem 1.1], we get from the conclusion of Theorem 1.2 that

$$(q - 2u + n - 1)T_f(r) \leq \sum_{j=1}^{q} N_f(r, H_j) + S(r, f),$$

which is obviously an extension of Theorem 1.1. It will be an interesting problem to considered the truncated second theorem by considering the truncated counting function $N_f^n(r, H_j)$ instead of $N_f(r, H_j)$, similarly like the case of holomorphic curve from $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$. Furthermore, it will be worth to use Theorem 1.2 for the uniqueness problem for holomorphic curves sharing slowly moving hyperplanes from annulus into $\mathbb{P}^n(\mathbb{C})$, for this we refer to [10].

The remaining of this paper is organized as follows. In the next section, we will introduce the basic notations of Nevanlinna theory and some lemmas. In Section 3, we first prove a result on the second main theorem for holomorphic curves intersecting moving hyperplanes in general position, and then give the proof of Theorem 1.2.

2. PRELIMINARIES AND LEMMAS

Let $R_0 > 1$ be a fixed positive real number or $+\infty$, let

$$\mathbb{A} = \{z \in \mathbb{C} : \frac{1}{R_0} < |z| < R_0\}$$

be an annulus in $\mathbb{C}$. For any real number r such that $1 < r < R_0$, we denote

$$\mathbb{A}_r = \{z \in \mathbb{C} : \frac{1}{r} < |z| < r\},\ \mathbb{A}_{1,r} = \{z \in \mathbb{C} : \frac{1}{r} < |z| \leq 1\},$$
$$\mathbb{A}_{2,r} = \{z \in \mathbb{C} : 1 < |z| < r\}.$$

Let $f = [f_0 : \dots : f_n] : \mathbb{A} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map, where $f_0, \dots, f_n$ are holomorphic functions without common zeros in $\mathbb{A}$. Let $\mathbf{f} = (f_0 : \dots : f_n)$

be a reduced representation of f. For $1 < r < R_0$, the Nevanlinna-Cartan's characteristic function $T_f(r)$ of f is defined by

$$T_f(r) = \int_0^{2\pi} \log \|\mathbf{f}(re^{i\theta})\| \frac{d\theta}{2\pi} + \int_0^{2\pi} \log \|\mathbf{f}(r^{-1}e^{i\theta})\| \frac{d\theta}{2\pi},$$

where

$$\|\mathbf{f}(z)\| = \max\{|f_0(z)|, \ldots, |f_n(z)|\}.$$

Note that the characteristic does not depend on the choice of the reduced representation of f.

Let $H_j, 1 \leq j \leq q$ be (fixed) moving hyperplanes in $\mathbb{P}^n(\mathbb{C})$ given by

$$H_j = \{[x_0, \ldots, x_n] | a_{j0}x_0 + \ldots + a_{jn}x_n = 0\},$$

where $a_{j0}, \ldots, a_{jn}$ are (constant) entire functions without common zeros. Let $a_j = (a_{j0}, \ldots, a_{jn}) : \mathbb{A} \to \mathbb{P}^n(\mathbb{C})$ be the moving vector associated with H_j respectively.

The proximity function of f with respect to a hyperplane H in $\mathbb{P}^n(\mathbb{C})$ is defined by

$$m_f(r, H) = \int_0^{2\pi} \log \frac{\|\mathbf{f}(re^{i\theta})\| \|a\|}{| < a, \mathbf{f} > (re^{i\theta})|} \frac{d\theta}{2\pi} + \int_0^{2\pi} \log \frac{\|\mathbf{f}(r^{-1}e^{i\theta})\| \|a\|}{| < a, \mathbf{f} > (r^{-1}e^{i\theta})|} \frac{d\theta}{2\pi},$$

where $a = (a_0, \ldots, a_n)$ is the nonzero vector associate with H, and $< a, \mathbf{f} >$ is the inner product. Here, we assume that $< a, \mathbf{f} > \not\equiv 0$. Further, we denote by $n_{1,f}(r, H)$ the number of zeros of $< a, \mathbf{f} >$ in $\mathbb{A}_{1,r}$, counting multiplicities, and by $n_{2,f}(r, H)$ the number of zeros of $< a, \mathbf{f} >$ in $\mathbb{A}_{2,r}$, counting multiplicities. We set

$$N_{1,f}(r, H) = \int_{r^{-1}}^{1} \frac{n_{1,f}(t, H)}{t} dt,$$
$$N_{2,f}(r, H) = \int_{1}^{r} \frac{n_{2,f}(t, H)}{t} dt.$$

The counting function of f with respect to H is defined as

$$N_f(r, H) = N_{1,f}(r, H) + N_{2,f}(r, H).$$

Definition 2.1. [12] The fixed hyperplanes $H_1, \ldots, H_q$ are said to be in general position if for any injective map $\mu : \{0, 1, \ldots, n\} \to \{1, \ldots, q\}$, $a_{\mu(0)}, \ldots, a_{\mu(n)}$ are linearly independent. The moving hyperplanes $H_1, \ldots, H_q$ are said to be in general position if $H_1(z), \ldots, H_q(z)$ are in general position for some (hence for almost all) $z \in \mathbb{A}$.

Let $\mathcal{G} = \{H_j | 1 \leq j \leq q\}$ be a finite set of moving hyperplanes in general position. Define, for $z \in \mathbb{A}$,

$$\Gamma(\mathcal{G})(z) = \min\{\frac{\|a_{\mu(0)}(z) \wedge \ldots \wedge a_{\mu(n)}(z)\|}{\|a_{\mu(0)}(z)\| \ldots \|a_{\mu(n)}(z)\|} | \mu : Z[0, n] \to \{1, \ldots, q\}, \text{is injective}\}.$$

By the general position assumption,

$$S = \{z \in \mathbb{A} | \Gamma(\mathcal{G})(z) = 0\}$$

is a closed set of isolated points.

Definition 2.2. [12] The moving hyperplanes $H_1, \ldots, H_q$ are said to be in u-subgeneral position if for every $1 \leq i_0 < \ldots < i_u \leq q$, the linear span of $a_{i_0}(z), \ldots, a_{i_u}(z)$ is $\mathbb{C}^{n+1}$ for some (hence for almost all) z.

A holomorphic map $f = [f_0, \ldots, f_n] : \mathbb{A} \to \mathbb{P}^n(\mathbb{C})$ is said to be linearly non-degenerate over $\mathcal{R}_{\mathcal{G}}$ if $f_0, \ldots, f_n$ are linearly independent over $\mathcal{R}_{\mathcal{G}}$.

LEMMA 2.3 ([11, Theorem 1.1]). *Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ and let $f = [f_0 : \ldots : f_n] : \mathbb{A} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve whose image is not contain in H. Then, for any $1 < r < R_0$, we have*

$$T_f(r) = m_f(r, H) + N_f(r, H) + O(1).$$

LEMMA 2.4 ([11, Theorem 1.2]). *Let $f = [f_0 : \ldots : f_n] : \mathbb{A} \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate holomorphic curve and let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in the general position. Then,*

$$\begin{aligned} &\int_0^{2\pi} \max_K \sum_{j\in K} \log \frac{\|\mathbf{f}(re^{i\theta})\| \|a_j\|}{|< a_j, \mathbf{f} > |(re^{i\theta})} \frac{d\theta}{2\pi} \\ + &\int_0^{2\pi} \max_K \sum_{j\in K} \log \frac{\|\mathbf{f}(r^{-1}e^{i\theta})\| \|a_j\|}{|< a_j, \mathbf{f} > |(r^{-1}e^{i\theta})} \frac{d\theta}{2\pi} \\ \leq &(n+1)T_f(r) - N_W(r, 0) + S(r, f), \end{aligned}$$

where

$$S(r, f) = \begin{cases} O(\log r + \log T_f(r)), & \text{for } R_0 = +\infty; \\ O(\log \frac{1}{R_0 - r} + \log T_f(r)), & \text{for } R_0 < +\infty. \end{cases}$$

Here, the maximum is taken over all subsets K of $\{1, \ldots, q\}$ such that $a_j, j \in K$ are linearly independent.

LEMMA 2.5 ([16]). *Given $\mathcal{G}$, a set of moving hyperplanes $H_1, \ldots, H_q$ in u-subgeneral position with $q \geq 2u - n + 1$, there exists a function $\omega : \{1, \ldots, q\} \to R(0, 1]$ called the Nochka weight and a real number $\theta \geq 1$ called the Nochka constant satisfying the following properties:*

(i) *If $j \in \{1, \ldots, q\}$, then $0 < \omega(j)\theta \leq 1$.*

(ii) $q - 2u + n - 1 = \theta(\sum_{j=1}^q \omega(j) - n - 1)$.

(iii) *If* $\emptyset \neq B \subset \{1, \ldots, q\}$ *with* $\sharp B \leq u+1$, *then*

$$\sum_{j \in P} \omega(j) \leq \dim L(B).$$

(iv) $1 \leq (u+1)/(n+1) \leq \theta \leq (2u-n+1)/(n+1)$.

(v) *Let* $\{E_1, \ldots, E_q\}$ *be a family of functions* $E_j : \mathbb{C} - S \to R[1, +\infty)$. *Given any* $A \subset \{1, \ldots, q\}$ *with* $0 < \sharp A \leq u+1$. *Take a* $z \in \mathbb{C} - S$. *Then there is a subset* $B(z)$ *of* A *such that* $\sharp B(z) = \dim L(z, A) = d(A)$ *and such that* $\{a_j(z) | j \in B(z)\}$ *is a base of* $L(z, A)$, *and such that*

$$\prod_{j \in A} E_j^{\omega(j)} \leq \prod_{j \in B} E_j,$$

where $L(z, A)$ *is the linear subspaces of* $\mathbb{C}^{n+1}$ *spanned by* $\{a_j(z) | j \in A\}$ *and* $d(A)$ *is the dimension of linear span of* A.

According to Product to The Sum Estimate [14, Theorem 6.2] and using a almost the same proof as the proof of [12, Lemma 3.2], we have the following result.

Lemma 2.6. *Let* $f = [f_0 : \ldots : f_n] : \mathbb{A} \to \mathbb{P}^n(\mathbb{C})$ *be a holomorphic map and let* $\mathcal{G}$ *be a finite set of hyperplanes* $H_1, \ldots, H_q$. *Assume that* $\mathcal{G}$ *is in the general position. Then for every* $z \in \mathbb{A} - S$ *with* $< \mathbf{f}(z), a(z) > \neq 0$ *for all* $a \in \mathcal{G}$, *there exist* $i(z,0), \ldots, i(z,n)$ *among* $1, \cdots, q$ *such that*

$$\prod_{a \in \mathcal{G}} \Big(\frac{\|\mathbf{f}(z)\| \|a(z)\|}{| < a(z), \mathbf{f}(z) > |}\Big) \leq \Big(\frac{2(n+1)}{\Gamma(\mathcal{G})(z)}\Big)^{q-n-1} \prod_{l=0}^{n} \Big(\frac{\|\mathbf{f}(z)\| \|a_{i(z,l)}(z)\|}{| < a_{i(z,l)}(z), \mathbf{f}(z) > |}\Big).$$

3. PROOF OF THEOREM 1.2

To prove Theorem 1.2, we propose the annulus version of second main theorem for moving targets in general position as follows.

Theorem 3.1. *Let* $f = [f_0 : \ldots : f_n] : \mathbb{A} \to \mathbb{P}^n(\mathbb{C})$ *be a holomorphic map. Let* $\mathcal{G}$ *be a finite set of moving hyperplanes* $H_1, \ldots, H_q$ *defining respectively holomorphic map* $a_1, \ldots, a_q : \mathbb{A} \to \mathbb{P}^n(\mathbb{C})$ *with* $T_{a_j(r)} = o(T_f(r))$. *Assume that* $\mathcal{G}$ *is in general position. Let* $\mathcal{R}_{\mathcal{G}}$ *be the small field with contains* $\mathbb{C}$ *and all* $a_{j\mu}/a_{j\nu}$ *with* $a_{j\nu} \not\equiv 0$. *If* f *is linearly non-degenerate over* $\mathcal{R}_{\mathcal{G}}$. *Then*

$$\sum_{j=1}^{q} m_f(r, H_j) \leq (n+1) T_f(r) + S(r, f),$$

where

$$S(r,f)=\begin{cases}O(\log r+\log T_f(r)), & \text{for } R_0=+\infty;\\ O(\log\frac{1}{R_0-r}+\log T_f(r)), & \text{for } R_0<+\infty.\end{cases}$$

Proof. We only prove the case $R_0=+\infty$. It is similar to prove the case $R_0<+\infty$. Using the idea of Ru, we give a similar proof as the second main theorem with moving targets [12]. First, we show that for every $\epsilon>0$ the inequality

$$\begin{aligned}\int_0^{2\pi}\max_K\sum_{l\in K}\log\frac{\|\mathbf{f}(re^{i\theta})\|\|a_{\mu(l)}(re^{i\theta})\|}{|<a_{\mu(l)},\mathbf{f}>(re^{i\theta})|}\frac{d\theta}{2\pi} &\le (n+1+\epsilon)\int_0^{2\pi}\log\|\mathbf{f}(re^{i\theta})\|\frac{d\theta}{2\pi}\\ &+ O(\log r+\log T_f(r))\end{aligned}$$

holds for any $z\in\mathbb{A}_{2,R_0},|z|=r$. Here, $\max_K$ is taken over all subsets $K\in\{1,\dots,q\}$ such that $a_j(z)$ for $j\in K$ are linearly independent for some $z\in\mathbb{A}$.

Without loss of generality, we can assume that $q\ge n+1$, and $\sharp K=n+1$. Let T be the set of all maps $\mu:\{0,1,\dots,n\}\to\{1,\dots,q\}$ such that $a_{\mu(0)}(z),\dots,a_{\mu(n)}(z)$ are linearly independent for some (thus for almost all) $z\in\mathbb{A}$.

For each $1\le j\le q$, chose an index $\hat{j}$ with $0\le\hat{j}\le n$ and $a_{j,\hat{j}}\not\equiv 0$, and define

$$\zeta_{j,l}(z)=a_{j,l}(z)/a_{j,\hat{j}}(z), j=1,\dots,q, l=0,\dots,n.$$

Let $\mathcal{L}(s)$ be the vector space generate over $\mathbb{C}$ by

$$\{\zeta_{1,0}^{n_{1,0}}\cdots\zeta_{1,0}^{n_{q,0}}\cdots\zeta_{1,n}^{n_{1,n}}\cdots\zeta_{q,n}^{n_{q,n}}|n_{j,l}\in N,\sum_{j=1}^{q}\sum_{l=0}^{n}n_{j,l}=s\},$$

we have $\mathcal{L}(s)\subset\mathcal{L}(s+1)$. Let $\{b_1,\dots,b_{\ell(s+1)}\}$ be a basis of $\mathcal{L}(s+1)$ such that $\{b_1,\dots,b_{\ell(s)}\}$ is a basis of $\mathcal{L}(s)$, where $\ell(s)=\dim\mathcal{L}(s)$.

Let $F:\mathbb{A}\to\mathbb{P}^{(n+1)\ell(s+1)}(\mathbb{C})$ be the holomorphic map defined by

$$\begin{aligned}F(z)=[&g(z)b_1(z)f_0(z):\dots:g(z)b_{\ell(s+1)}(z)f_0(z):\\ &g(z)b_1(z)f_1(z):\dots:g(z)b_{\ell(s+1)}(z)f_n(z)],\end{aligned}\tag{1}$$

where $g(z)$ is a holomorphic function on $\mathbb{A}$ such that $g(z)b_1(z),\dots,g(z)b_{\ell(s+1)}(z)$ are all holomorphic on $\mathbb{A}$ and $T_g(r)=o(T_f(r))$.

Since f is linearly non-degenerate over $\mathcal{R}_{\mathcal{G}}$, F is linearly non-degenerate. For each $\mu\in T$, let $h_j, 1\le j\le q$ be the meromorphic function on $\mathbb{A}$ defined by

$$h_j(z)=\zeta_{j,0}+\sum_{l=0}^{n}\zeta_{j,l}(z)\frac{f_l(z)}{f_0(z)}(j=1,\dots,q).\tag{2}$$

Noticing that $b_j\zeta_{k,l} \in \mathcal{L}(s+1)$ for $1 \le j \le \ell(s), 1 \le k \le q$ and $0 \le l \le n$, so it can be written as a linear combination of $b_r, 1 \le r \le \ell(s+1)$.

Thus the functions $b_j h_{\mu(l)}, 1 \le j \le \ell(s), 0 \le l \le n$, can be written as a linear combination of $b_r, 1 \le r \le \ell(s+1)$, and $b_\alpha(f_\beta/f_0), 1 \le \alpha \le \ell(s+1), 0 \le \beta \le n$. In other words, there is an $(n+1)\ell(s) \times ((n+1)\ell(s+1))$ matrix $C(\mu)$ such that

$$\begin{pmatrix} b_l h_{\mu(0)} \\ \vdots \\ b_{\ell(s)} h_{\mu(0)} \\ \vdots \\ b_l h_{\mu(n)} \\ \vdots \\ b_{\ell(s)} h_{\mu(n)} \end{pmatrix} = C(\mu) \begin{pmatrix} b_1 \\ \vdots \\ b_{\ell(s+1)} \\ b_1(f_1/f_0) \\ \vdots \\ b_{\ell(s+1)}(f_1/f_0) \\ \vdots \\ b_1(f_n/f_0) \\ \vdots \\ b_{\ell(s+1)}(f_n/f_0) \end{pmatrix}. \tag{3}$$

For $l = 0, \ldots, n$ and $j = 1, \ldots, \ell(s)$, let $\hat{H}_{l,j}(\mu)$ be the (fixed) hyperplane in $\mathbb{P}^{(n+1)\ell(s+1)}(\mathbb{C})$ defined by he corresponding row in $C(\mu)$, i.e. if we denote $c_{ij}(\mu)$ the elements of $C(\mu)$, then

$$\begin{aligned} \hat{H}_{l,j}(\mu) \quad = \quad & \{[y_{1,0} : \ldots : y_{\ell(s+1),0} : y_{1,1} : \ldots : y_{\ell(s+1),1} : \ldots : \\ & y_{1,n} : \ldots : y_{\ell(s+1),n}] \in P(C^{(n+1)\ell(s+1)})| \\ & c_{l\ell(s)+j,1}(\mu) y_{1,0} + \ldots + c_{l\ell(s)+j,(n+1)\ell(s+1)}(\mu) y_{\ell(s+1),n} = 0\}. \end{aligned} \tag{4}$$

Let $\hat{a}_{l,j}(\mu)$ be the vector with $\hat{H}_{l,j}(\mu)$. Since $a_{\mu(0)}(z), \ldots, a_{\mu(n)}(z)$ are linearly independent for some z and $f_0(z), \ldots, f_n(z)$ are linearly independent over $\mathcal{R}_\mathcal{G}$, $h_{\mu(0)}, \ldots, h_{\mu(n)}$ are linearly independent over $\mathcal{R}_\mathcal{G}$. Thus, by the choice of $b_1, \ldots, b_{\ell(s)}$, the set $\{b_j h_{\mu(l)}, j = 1, \ldots, \ell(s); l = 0, \ldots, n\}$ is linearly independent over $\mathbb{C}$. Hence, $\hat{H}_{l,j}(\mu), l = 0, \ldots, n, j = 1, \ldots, \ell(s)$ are linearly independent for each $\mu \in T$.

Applying Lemma 2.4 for F, with the hyperplanes $\{\hat{H}_{l,j}(\mu) | l = 0, \ldots, n, j = 1, \ldots, \ell(s)\}$, we obtain

$$\begin{aligned} & \int_0^{2\pi} \max_{\mu \in T} \sum_{l=0}^{n} \sum_{j=1}^{\ell(s)} \log \frac{\|F(re^{i\theta})\| \|\hat{a}_{l,j}(\mu)\|}{|<\hat{a}_{l,j}(\mu), F>(re^{i\theta})|} \frac{d\theta}{2\pi} \\ & \le ((n+1)\ell(s+1)) \int_0^{2\pi} \log \|F(re^{i\theta})\| \frac{d\theta}{2\pi} + O(\log r + \log T_F(r)). \end{aligned} \tag{5}$$

Now, we compare $T_f(r)$ and $T_F(r)$. In fact, for each $z \in \mathbb{A}$, not in the set of the poles of $b_1, \ldots, b_{\ell(s+1)}$, by (1), we have,

$$
\begin{aligned}
\|F(z)\| &= \max(|g(z)b_1(z)f_0(z)|, \ldots, \\
&\qquad |g(z)b_{\ell(s+1)}(z)f_0(z)|, \ldots, |g(z)b_{\ell(s+1)}(z)f_n(z)|) \\
&= |g(z)| \max(|f_0(z)|, \ldots, |f_n(z)|) \cdot \max_{1\le\alpha\le\ell(s+1)}(|b_\alpha(z)|) + O(1) \qquad (6)\\
&= \|\mathbf{f}(z)\| |g(z)| \cdot \max_{1\le\alpha\le\ell(s+1)}(|b_\alpha(z)|) + O(1).
\end{aligned}
$$

This gives that for any $z \in \mathbb{A}_{2,R_0}, |z| = r$,

$$
\begin{aligned}
\int_0^{2\pi} \log \|F(re^{i\theta})\| \frac{d\theta}{2\pi} &= \int_0^{2\pi} \log[\|\mathbf{f}(re^{i\theta})\| \cdot \max_{1\le\alpha\le\ell(s+1)}(|b_\alpha(re^{i\theta})|) \cdot |g(re^{i\theta})|] \frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} \log \|\mathbf{f}(re^{i\theta})\| \frac{d\theta}{2\pi} + o(T_f(r)).
\end{aligned}
$$

Similar discussion as above, we conclude that for any $z \in \mathbb{A}_{1,R_0}, |z| = \frac{1}{r}$,

$$
\int_0^{2\pi} \log \|F(r^{-1}e^{i\theta})\| \frac{d\theta}{2\pi} = \int_0^{2\pi} \log \|\mathbf{f}(r^{-1}e^{i\theta})\| \frac{d\theta}{2\pi} + o(T_f(r)).
$$

Therefore, we obtain

$$
T_F(r) = T_f(r) + o(T_f(r)).
$$

Next, we compare

$$
\int_0^{2\pi} \log \frac{\|F(re^{i\theta})\| \|\hat{a}_{l,j}(\mu)\|}{|<\hat{a}_{l,j}(\mu), F>(re^{i\theta})|}
$$

and

$$
\int_0^{2\pi} \log \frac{\|\mathbf{f}(re^{i\theta})\| \|a_{\mu(l)}(re^{i\theta})\|}{|<a_{\mu(l)}, \mathbf{f}>(re^{i\theta})|} \frac{d\theta}{2\pi},
$$

for each $\mu \in T$. By (2), (3), (4) and (6), for $0 \le l \le n$,

$$
\begin{aligned}
&\int_0^{2\pi} \log \frac{\|F(re^{i\theta})\| \|\hat{a}_{l,j}(\mu)\|}{|<\hat{a}_{l,j}(\mu), F>(re^{i\theta})|} \frac{d\theta}{2\pi} \\
= \;& \int_0^{2\pi} \log \|\mathbf{f}(re^{i\theta})\| \frac{d\theta}{2\pi} + \int_0^{2\pi} \log(\max_{1\le\alpha\le\ell(s+1)} |b_\alpha(re^{i\theta})|) \frac{d\theta}{2\pi} \\
&- \int_0^{2\pi} \log |b_j(re^{i\theta}) h_{\mu(l)(re^{i\theta})} f_0(re^{i\theta})| \frac{d\theta}{2\pi} + O(1) \\
= \;& \int_0^{2\pi} \log \frac{\|\mathbf{f}(re^{i\theta})\| \|a_{\mu(l)}(re^{i\theta})\|}{|<a_{\mu(l)}, \mathbf{f}>(re^{i\theta})|} \frac{d\theta}{2\pi} - \int_0^{2\pi} \log \frac{|b_j(re^{i\theta})|}{\max_{1\le\alpha\le\ell(s+1)} |b_\alpha(re^{i\theta})|} \frac{d\theta}{2\pi} \\
&- \int_0^{2\pi} \log \frac{\max_{0\le t\le n} |a_{\mu(l),t}(re^{i\theta})|}{|a_{\mu(l),\hat{\mu}(l)}(re^{i\theta})|} \frac{d\theta}{2\pi} + O(1)
\end{aligned}
$$

$$= \int_0^{2\pi} \log \frac{\|\mathbf{f}(re^{i\theta})\| \|a_{\mu(l)}(re^{i\theta})\|}{|< a_{\mu(l)}, \mathbf{f} > (re^{i\theta})|} \frac{d\theta}{2\pi} + o(T_f(r)).$$

Thus, combining this with (5), we have

$$\begin{aligned}
&\ell(s) \int_0^{2\pi} \max_{\mu \in T} \sum_{l=0}^{n} \log \frac{\|\mathbf{f}(re^{i\theta})\| \|a_{\mu(l)}(re^{i\theta})\|}{|< a_{\mu(l)}, \mathbf{f} > (re^{i\theta})|} \frac{d\theta}{2\pi} \\
= &\int_0^{2\pi} \max_{\mu \in T} \sum_{l=0}^{n} \sum_{j=1}^{\ell(s)} \log \frac{\|F(re^{i\theta})\| \|\hat{a}_{l,j}(\mu)\|}{|< \hat{a}_{l,j}(\mu), F > (re^{i\theta})|} \frac{d\theta}{2\pi} + O(T_f(r)) \\
\leq &((n+1)\ell(s+1)) \int_0^{2\pi} \log \|F(re^{i\theta})\| \frac{d\theta}{2\pi} + O(\log r + \log T_F(r)) \\
\leq &((n+1)\ell(s+1)) \int_0^{2\pi} \log \|\mathbf{f}(re^{i\theta})\| \frac{d\theta}{2\pi} + O(\log r + \log T_f(r)).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_0^{2\pi} \max_{\mu \in T} \sum_{l \in K} \log \frac{\|\mathbf{f}(re^{i\theta})\| \|a_{\mu(l)}(re^{i\theta})\|}{|< a_{\mu(l)}, \mathbf{f} > (re^{i\theta})|} \frac{d\theta}{2\pi} \\
\leq &((n+1)\frac{\ell(s+1)}{\ell(s)}) \int_0^{2\pi} \log ||\mathbf{f}(re^{i\theta})|| \frac{d\theta}{2\pi} + O(\log r + \log T_f(r)).
\end{aligned}$$

We have $0 \leq \ell(s) \leq \binom{q(n+1)+s-1}{s}$ for each s and therefore

$$\liminf_{s \to \infty} \frac{\ell(s+1)}{\ell(s)} = 1.$$

So, for every $\epsilon > 0$ and for any $z \in \mathbb{A}_{2,R_0}, |z| = r$,

$$\begin{aligned}
&\int_0^{2\pi} \max_{K} \sum_{l \in K} \log \frac{\|\mathbf{f}(re^{i\theta})\| \|a_{\mu(l)}(re^{i\theta})\|}{|< a_{\mu(l)}, \mathbf{f} > (re^{i\theta})|} \frac{d\theta}{2\pi} \\
&\leq (n+1+\epsilon) \int_0^{2\pi} \log \|\mathbf{f}(re^{i\theta})\| \frac{d\theta}{2\pi} + O(\log r + \log T_f(r)).
\end{aligned} \tag{7}$$

Similarly, we conclude that for for any $z \in \mathbb{A}_{1,R_0}, |z| = \frac{1}{r}$,

$$\begin{aligned}
&\int_0^{2\pi} \max_{K} \sum_{l \in K} \log \frac{\|\mathbf{f}(r^{-1}e^{i\theta})\| \|a_{\mu(l)}(r^{-1}e^{i\theta})\|}{|< a_{\mu(l)}, \mathbf{f} > (r^{-1}e^{i\theta})|} \frac{d\theta}{2\pi} \\
&\leq (n+1+\epsilon) \int_0^{2\pi} \log \|\mathbf{f}(r^{-1}e^{i\theta})\| \frac{d\theta}{2\pi} + O(\log r + \log T_f(r)).
\end{aligned} \tag{8}$$

Combining (7) and (8), we have

$$\begin{aligned}
&\int_0^{2\pi} \max_K \sum_{l\in K} \log \frac{\|\mathbf{f}(re^{i\theta})\|\|a_{\mu(l)}(re^{i\theta})\|}{|< a_{\mu(l)}, \mathbf{f} > (re^{i\theta})|}\frac{d\theta}{2\pi} \\
&\quad + \int_0^{2\pi} \max_K \sum_{l\in K} \log \frac{\|\mathbf{f}(r^{-1}e^{i\theta})\|\|a_{\mu(l)}(r^{-1}e^{i\theta})\|}{|< a_{\mu(l)}, \mathbf{f} > (r^{-1}e^{i\theta})|}\frac{d\theta}{2\pi} \\
&\leq (n+1)T_f(r) + O(\log r + \log T_f(r)).
\end{aligned} \tag{9}$$

By Lemma 2.6, we have

$$\begin{aligned}
\sum_{j=1}^q m_f(r, H_j) &= \sum_{j=1}^q \int_0^{2\pi} \log \frac{\|\mathbf{f}(re^{i\theta})\|\|a_j(re^{i\theta})\|}{|< a_j, \mathbf{f} > (re^{i\theta})|}\frac{d\theta}{2\pi} + \\
&\qquad \sum_{j=1}^q \int_0^{2\pi} \log \frac{\|\mathbf{f}(r^{-1}e^{i\theta})\|\|a_j(r^{-1}e^{i\theta})\|}{|< a_j, \mathbf{f} > (r^{-1}e^{i\theta})|}\frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} \log \prod_{j=1}^q \frac{\|\mathbf{f}(re^{i\theta})\|\|a_j(re^{i\theta})\|}{|< a_j, \mathbf{f} > (re^{i\theta})|}\frac{d\theta}{2\pi} \\
&\qquad + \int_0^{2\pi} \log \prod_{j=1}^q \frac{\|\mathbf{f}(r^{-1}e^{i\theta})\|\|a_j(r^{-1}e^{i\theta})\|}{|< a_j, \mathbf{f} > (r^{-1}e^{i\theta})|}\frac{d\theta}{2\pi},
\end{aligned}$$

and thus,

$$\begin{aligned}
\sum_{j=1}^q m_f(r, H_j) &\leq \int_0^{2\pi} \log \max_{\mu\in T} \prod_{l=0}^n \frac{\|\mathbf{f}(re^{i\theta})\|\|a_{\mu(l)}(re^{i\theta})\|}{|< a_{\mu(l)}, \mathbf{f} > (re^{i\theta})|}\frac{d\theta}{2\pi} \\
&\quad + \int_0^{2\pi} \log \max_{\mu\in T} \prod_{l=0}^n \frac{\|\mathbf{f}(r^{-1}e^{i\theta})\|\|a_{\mu(l)}(r^{-1}e^{i\theta})\|}{|< a_{\mu(l)}, \mathbf{f} > (r^{-1}e^{i\theta})|}\frac{d\theta}{2\pi} + o(T_f(r)) \\
&= \int_0^{2\pi} \max_{\mu\in T} \log \prod_{l=0}^n \frac{\|\mathbf{f}(re^{i\theta})\|\|a_{\mu(l)}(re^{i\theta})\|}{|< a_{\mu(l)}, \mathbf{f} > (re^{i\theta})|}\frac{d\theta}{2\pi} \\
&\quad + \int_0^{2\pi} \max_{\mu\in T} \log \prod_{l=0}^n \frac{\|\mathbf{f}(r^{-1}e^{i\theta})\|\|a_{\mu(l)}(r^{-1}e^{i\theta})\|}{|< a_{\mu(l)}, \mathbf{f} > (r^{-1}e^{i\theta})|}\frac{d\theta}{2\pi} + o(T_f(r)).
\end{aligned}$$

This yields

$$\begin{aligned}
\sum_{j=1}^q m_f(r, H_j) &\leq \int_0^{2\pi} \max_K \sum_{l\in K} \log \frac{\|\mathbf{f}(re^{i\theta})\|\|a_{\mu(l)}(re^{i\theta})\|}{|< a_{\mu(l)}, \mathbf{f} > (re^{i\theta})|}\frac{d\theta}{2\pi} \\
&\quad + \int_0^{2\pi} \max_K \sum_{l\in K} \log \frac{\|\mathbf{f}(r^{-1}e^{i\theta})\|\|a_{\mu(l)}(r^{-1}e^{i\theta})\|}{|< a_{\mu(l)}, \mathbf{f} > (r^{-1}e^{i\theta})|}\frac{d\theta}{2\pi} + O(1).
\end{aligned}$$

So, combining this with (9), we get

$$\sum_{j=1}^{q} m_f(r, H_j) \le (n+1)T_f(r) + S(r, f).$$

Hence, Theorem 3.1 is proved. $\square$

Proof of Theorem 1.2. We mainly modify the method from [12]. We assume that $q > 2u - n + 1$. Let $\mathcal{G} = \{H_j | 1 \le j \le q\}$ be a finite set of moving hyperplanes in u-subgeneral position. Define, for $z \in \mathbb{A}$,

$$\Gamma(\mathcal{G})(z) = \min\{\frac{\|a_{\mu(0)}(z) \wedge \ldots \wedge a_{\mu(n)}(z)\|}{\|a_{\mu(0)}(z)\| \ldots \|a_{\mu(n)}(z)\|} | \mu : Z[0, n] \to \{1, \ldots, q\} \text{ is injective}\}.$$

By the u-subgeneral position assumption,

$$S = \{z \in \mathbb{A} | \Gamma(\mathcal{G})(z) = 0\}$$

is a closed set of isolated points. Fix a $z_0 \in \mathbb{A} - S$ and let $\omega(j), 1 \le j \le q$ be the Nochka weights associated with the hyperplanes $H_j(z_0)$.

Since $H_1, \ldots, H_q$ are in u-subgeneral position, there is an embedding $\mathbb{P}^n(\mathbb{C}) \hookrightarrow \mathbb{P}^u(\mathbb{C})$ and hyperplanes $H'_1, \ldots, H'_q$ in $\mathbb{P}^u(\mathbb{C})$ such that $H'_j \cap \mathbb{P}^n(\mathbb{C}) = H_j$ for all j and such that $H'_1, \ldots, H'_q$ are in general position.

So by Lemma 2.6, for every $z \in \mathbb{A} - S$, there exists $i_{(z,0)}, \ldots, i_{(z,u)}$ among $1, \ldots, q$ such that

(10)

$$\prod_{a \in \mathcal{G}} (\frac{\|\mathbf{f}(z)\|\|a(z)\|}{| < a(z), \mathbf{f}(z) > |})^{\omega(j)} \le (\frac{2(u+1)}{\Gamma(\mathcal{G})(z)})^{q-u-1} \prod_{l=0}^{u} (\frac{\|\mathbf{f}(z)\|\|a_{i(z,l)}(z)\|}{| < a_{i(z,l)}(z), \mathbf{f}(z) > |})^{\omega(i(z,l))}.$$

Let $A = \{i_{(z,0)}, \ldots, i_{(z,u)}\}$, then $d(A) = n + 1$. Define,

$$\lambda_H(f(z)) = \log \frac{\|\mathbf{f}(z)\|\|a(z)\|}{| < a(z), \mathbf{f}(z) > |}.$$

Applying Lemma 2.5 with $E_l = e^{\lambda_{H_{i(z,l)}}(f(z))}, 0 \le l \le u$, there is a subset $B(z)$ of A such that $\sharp B(z) = \dim L(z, A) = d(A) = n + 1$ and such that $\{a_j(z) | j \in B(z)\}$ is a base of $\mathbb{C}^{n+1}$. Moreover,

$$\begin{aligned} \prod_{l=0}^{u} (\frac{\|\mathbf{f}(z)\|\|a_{i(z,l)}(z)\|}{| < a_{i(z,l)}(z), \mathbf{f}(z) > |})^{\omega(i(z,l))} &\le \prod_{j \in B(z)} (\frac{\|\mathbf{f}(z)\|\|a_j(z)\|}{| < a_j(z), \mathbf{f}(z) > |}) \\ &\le \max_{\gamma \in \Gamma} \prod_{t=0}^{n} (\frac{\|\mathbf{f}(z)\|\|a_{\gamma(t)}(z)\|}{| < a_{\gamma(t)}(z), \mathbf{f}(z) > |}) \end{aligned}$$

where Γ is the set of all maps $\gamma : \{0, \ldots, n\} \to \{1, \ldots, q\}$ such that $a_{\gamma(0)}(z), \ldots, a_{\gamma(n)}(z)$ are linearly independent for some (hence for almost all) $z \in \mathbb{A}$.

For any $z \in \mathbb{A}_{2,R_0}, |z| = r$, combining this with (10) gives us

$$\begin{aligned} &\int_0^{2\pi} \sum_{j=1}^{q} \omega(j) \log \frac{\|\mathbf{f}(re^{i\theta})\| \|a_j(re^{i\theta})\|}{|< a_j, \mathbf{f} > (re^{i\theta})|} \frac{d\theta}{2\pi} \\ \leq \; &\int_0^{2\pi} \max_{\gamma \in \Gamma} \sum_{l=0}^{n} \log \frac{\|\mathbf{f}(re^{i\theta})\| \|a_{\gamma(l)}(re^{i\theta})\|}{|< a_{\gamma(l)}, \mathbf{f} > (re^{i\theta})|} \frac{d\theta}{2\pi} + o(T_f(r)). \end{aligned}$$

Applying Theorem 3.1 yields that for every $\epsilon > 0$

$$\begin{aligned} &\int_0^{2\pi} \max_{\gamma \in \Gamma} \sum_{l=0}^{n} \log \frac{\|\mathbf{f}(re^{i\theta})\| \|a_{\gamma(l)}(re^{i\theta})\|}{|< a_{\gamma(l)}, \mathbf{f} > (re^{i\theta})|} \frac{d\theta}{2\pi} \\ &\leq (n+1+\epsilon) \int_0^{2\pi} \log \|\mathbf{f}(re^{i\theta})\| \frac{d\theta}{2\pi} + S(r, f). \end{aligned}$$

Then

$$\begin{aligned} &\int_0^{2\pi} \sum_{j=1}^{q} \omega(j) \log \frac{\|\mathbf{f}(re^{i\theta})\| \|a_j(re^{i\theta})\|}{|< a_j, \mathbf{f} > (re^{i\theta})|} \frac{d\theta}{2\pi} \\ &\leq (n+1+\epsilon) \int_0^{2\pi} \log \|\mathbf{f}(re^{i\theta})\| \frac{d\theta}{2\pi} + S(r, f). \end{aligned}$$

Combining this with Lemma 2.5, we have

$$\begin{aligned} &\sum_{j=1}^{q} \int_0^{2\pi} \log \frac{\|\mathbf{f}(re^{i\theta})\| \|a_j(re^{i\theta})\|}{|< a_j, \mathbf{f} > (re^{i\theta})|} \frac{d\theta}{2\pi} \\ = \; &\sum_{j=1}^{q} (1 - \theta\omega(j)) \int_0^{2\pi} \log \frac{\|\mathbf{f}(re^{i\theta})\| \|a_j(re^{i\theta})\|}{|< a_j, \mathbf{f} > (re^{i\theta})|} \frac{d\theta}{2\pi} \\ &+ \sum_{j=1}^{q} \theta\omega(j) \int_0^{2\pi} \log \frac{\|\mathbf{f}(re^{i\theta})\| \|a_j(re^{i\theta})\|}{|< a_j, \mathbf{f} > (re^{i\theta})|} \frac{d\theta}{2\pi} \\ \leq \; &\sum_{j=1}^{q} (1 - \theta\omega(j)) \int_0^{2\pi} \log \|\mathbf{f}(re^{i\theta})\| \frac{d\theta}{2\pi} \\ &+ \theta(n+1+\epsilon) \int_0^{2\pi} \log \|\mathbf{f}(re^{i\theta})\| \frac{d\theta}{2\pi} + S(r, f) \\ = \; &\{q - \theta(\sum_{j=1}^{q} \omega(j) - n - 1) + \epsilon\} \int_0^{2\pi} \log \|\mathbf{f}(re^{i\theta})\| \frac{d\theta}{2\pi} + S(r, f) \\ = \; &(2u - n + 1 + \epsilon) \int_0^{2\pi} \log \|\mathbf{f}(re^{i\theta})\| \frac{d\theta}{2\pi} + S(r, f). \end{aligned}$$

Similarly, we conclude that for any $z \in \mathbb{A}_{1,R_0}, |z| = \frac{1}{r}$,

$$\sum_{j=1}^{q} \int_0^{2\pi} \log \frac{\|\mathbf{f}(r^{-1}e^{i\theta})\|\|a_j(r^{-1}e^{i\theta})\|}{|< a_j, \mathbf{f} > (r^{-1}e^{i\theta})|} \frac{d\theta}{2\pi}$$

$$\leq \quad (2u - n + 1 + \epsilon) \int_0^{2\pi} \log \|\mathbf{f}(r^{-1}e^{i\theta})\| \frac{d\theta}{2\pi} + S(r, f).$$

Hence,

$$\sum_{j=1}^{q} m_f(r, H_j) \leq (2u - n + 1)T_f(r) + S(r, f).$$

Therefore, Theorem 1.2 is proved. $\square$

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