ASSOCIATE SEQUENCES OF A BALANCING-LIKE SEQUENCE

GOPAL KRISHNA PANDA and SUSHREE SANGEETA PRADHAN

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There are three associates of the balancing sequence namely, the cobalancing, the Lucas-balancing and the Lucas-cobalancing sequence. Each balancing-like sequence is associated with a Lucas-balancing-like sequence. In the present work, we construct the cobalancing-like and the Lucas-cobalancing-like sequences. Two factor sequences namely the Pell-like and the associated Pell-like sequences are identified. Further, for each balancing-like sequence, triangular-like numbers are defined and certain properties similar to those satisfied by triangular numbers are studied.

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1. INDRODUCTION

A balancing number B is a natural number satisfying the Diophantine equation $1+2+\cdots+(B-1) = (B+1)+\cdots+(B+R)$ for some natural number R, known as the cobalancer of B [2]. If B is a balancing number then $8B^2+1$ is a perfect square and its positive square root is called a Lucas-balancing number [8]. The n^{th} balancing and Lucas-balancing numbers are denoted by B_n and C_n respectively and satisfy the binary recurrences

$$B_{n+1} = 6B_n - B_{n-1},\ C_{n+1} = 6C_n - C_{n-1}$$

with initial values $B_0 = 0,\ B_1 = 1,\ C_0 = 1,\ C_1 = 3.$

A cobalancing number is a natural number b which satisfies $1+2\cdots+b = (b+1)+\cdots+(b+r)$ for some r, known as the cobalancer of b [4]. If b is a cobalancing number then $8b^2+8b+1$ is a perfect square and its positive square root is called a Lucas-cobalancing number [8]. The n^{th} cobalancing and Lucas-cobalancing numbers are denoted by b_n and c_n respectively and satisfy the binary recurrences

$$b_{n+1} = 6b_n - b_{n-1} + 2,\ c_{n+1} = 6c_n - c_{n-1}$$

with initial terms $b_0 = b_1 = 0,\ c_0 = -1,\ c_1 = 1.$

Balancing numbers, cobalancing numbers, balancers and cobalancers are very closely entangled with each other. For example, every balancing number is a cobalancer and every balancer is a cobalancing number and twice the sum of first n balancing numbers is equal to the $(n+1)^{st}$ cobalancing number [4]. These numbers are also connected in many different ways.

The Pell and associated Pell sequences are defined by means of the binary recurrences

$$P_{n+1} = 2P_n + P_{n-1},\ Q_{n+1} = 2Q_n + Q_{n-1}$$

respectively with initial terms $P_0 = 0,\ P_1 = 1,\ Q_0 = 2,\ Q_1 = 1$. These two sequences are connected by $Q_n^2 = 2P_n^2 + (-1)^n,\ n = 1, 2, \cdots$. An important observation about these sequences is that the successive convergents in the continued fraction expansion of $\sqrt{2}$ are $Q_n/P_n,\ n = 1, 2, \cdots$. In addition, $B_n = P_nQ_n,\ n = 1, 2, \cdots$ [6].

As a generalization of the balancing sequence, Panda and Rout [7] studied a family of binary recurrences defined by

$$x_{n+1} = Ax_n - x_{n-1}\ , x_0 = 0,\ x_1 = 1$$

and $A > 2$ is any natural number. Subsequently, these sequences were known as balancing-like sequences [9] since the particular case corresponding to $A = 6$ coincides with the balancing sequence. Khan and Kwong [3] called these sequences as generalized natural number sequences since natural numbers are solutions of the binary recurrence $x_{n+1} = 2x_n - x_{n-1}$ with initial terms $x_0 = 0,\ x_1 = 1$. These names are also supported by some identities satisfied by each balancing-like sequence in which it behaves like the sequence of natural numbers.

If x is a balancing-like number corresponding to a balancing-like sequence with fixed $A > 2$, then $Dx^2 + 1$, where $D = \frac{A^2-4}{4}$, is a perfect square and its positive square root is called a Lucas-balancing-like number [7]. The Lucas-balancing-like sequences are linked with their balancing-like sequences similar to the association of the Lucas-balancing sequence with the balancing sequence. The Lucas-balancing-like sequences corresponding to even values of A have integral terms, while those corresponding to odd values have terms which are odd integral multiples of $\frac{1}{2}$.

As we have discussed, there are three associate sequences of the balancing sequence, namely, the cobalancing sequence, the Lucas-balancing sequence and the Lucas-cobalancing sequence. Each balancing-like sequence has also one associate sequence, its Lucas-balancing-like sequence. In this paper, we construct two more associate sequences, namely, the cobalancing-like and the Lucas-cobalancing-like sequences. We also construct the factor sequences of each balancing-like sequence, and call them Pell-like and associated Pell-like sequences. Lastly, we define triangular-like numbers in such a way that these numbers reduce to triangular numbers when $A = 2$.

2. AUXILIARY RESULTS

To achieve the aforesaid goals, we need some identities involving balancing, cobalancing, Lucas-balancing, Lucas-cobalancing, balancing-like, Lucas-balancing-like, Pell and associated Pell sequences. Many identities of this section will be needed in the subsequent sections for the constructions of new number sequences associated with the balancing-like sequences.

In the following identities the balancing and associated sequences are expressed in terms of the Pell and associated Pell sequences. Proofs are available in [6].

$$B_{-n} = -B_n,\ C_{-n} = C_n$$
$$B_n = P_n Q_n$$
$$P_{2n} = 2B_n$$
$$b_{2n} = P_{2n}Q_{2n-1},\ b_{2n+1} = P_{2n}Q_{2n+1}$$

The following pair of identities can be proved easily using the Binet forms of the balancing, cobalancing, Lucas-balancing, Lucas-cobalancing, Pell and associated Pell sequences.

$$C_n = Q_{2n} = \frac{B_{n+1} - B_{n-1}}{2} = b_n + b_{n+1} + 1$$

$$c_n = Q_{2n-1} = \frac{b_{n+1} - b_{n-1}}{2} = B_n + B_{n-1}$$

The identity

$$B_{n-r}B_{n+r} = B_n^2 - B_r^2$$

[5] confirms that the balancing sequence behaves like natural numbers. In particular for $r = 1$, we have

$$B_{n-1}B_{n+1} = B_n^2 - 1.$$

The balancing-like numbers also satisfy the above two identities. The cobalancing numbers satisfy

$$b_{n-1}b_{n+1} = (b_n - 1)^2 - 1$$

[4] which somehow resemble the last identity. The cobalancing numbers are also related to the balancing numbers as

$$b_n = \frac{-(2B_n + 1) + \sqrt{8B_n^2 + 1}}{2} = \frac{C_n - (2B_n + 1)}{2},\ n = 1, 2, \cdots$$

[2, 4] and are also connected by the sum formula

$$b_n = 2(B_0 + B_1 + B_2 + \cdots + B_{n-1}),\ n = 1, 2, \cdots$$

We have already seen in the last section that the balancing-like sequences are defined recursively as

$$x_{n+1} = Ax_n - x_{n-1},\ x_0 = 0,\ x_1 = 1.$$

Using this recurrence, we can find their Binet form as

$$x_n = \frac{\alpha^n - \beta^n}{\sqrt{A^2-4}},\ n = 1, 2, \cdots$$

where $\alpha = \frac{A+\sqrt{A^2-4}}{2}$ and $\beta = \frac{A-\sqrt{A^2-4}}{2}$. Using the relationship $y_n^2 = Dx_n^2 + 1$ where $D = \frac{A^2-4}{4}$, we can have the Binet form for the Lucas-balancing-like sequences as

$$y_n = \frac{\alpha^n + \beta^n}{2},\ n = 1, 2, \cdots$$

A point of caution in this connection is that all terms of the sequence $\{y_n\}_{n=1}^{\infty}$ corresponding to an odd value of A are not integral; rather, each term of this sequence whose index is a multiple of 3 is integral and rest terms are odd integral multiples of $\frac{1}{2}$. Using these Binet forms, we can obtain the one-step shift formulas of balancing-like and Lucas-balancing-like sequences as

$$x_{n\pm1} = \frac{A}{2}x_n \pm y_n = \frac{A}{2}x_n \pm \sqrt{Dx_n^2+1}$$

and

$$y_{n\pm1} = \frac{A}{2}y_n \pm Dx_n = \frac{A}{2}y_n \pm \sqrt{\frac{y_n^2-1}{D}},\ n = 1, 2, \cdots$$

Taking the help of the identities and ideas of this section, we are now in a position to construct the cobalancing-like and the Lucas-cobalancing-like sequences corresponding to every balancing-like sequence.

3. CONSTRUCTION OF COBALANCING-LIKE SEQUENCES

We have already mentioned in the last section that the cobalancing sequence can be obtained from the balancing sequence in two ways,

$$b_n = \frac{-(2B_n+1) + \sqrt{8B_n^2+1}}{2},\ n = 1, 2, \cdots$$

and

$$b_n = 2(B_0 + B_1 + B_2 + \cdots + B_{n-1}),\ n = 1, 2, \cdots$$

These identities suggest defining the cobalancing-like sequences $\{z_n\}_{n=1}^{\infty}$ corresponding to the balancing-like sequences $\{x_n\}_{n=1}^{\infty}$ by

$$z_n = \frac{-(2x_n+1) + \sqrt{Dx_n^2+1}}{2},\ n = 1, 2, \cdots$$

or

$$z_n = 2(x_0 + x_1 + x_2 + \cdots + x_{n-1}),\ n = 1, 2, \cdots .$$

If we use the former definition then its terms may be negative and/or fractional in some cases. For example, if $A = 3$ then $x_1 = 1,\ x_2 = 3,\ x_3 = 8$ and consequently $z_1 = \frac{-3}{4},\ z_2 = \frac{-7}{4},\ z_3 = -4$. Hence, we choose the later definition. The terms of the cobalancing-like sequence corresponding to $A = 3$ are then $z_1 = 0,\ z_2 = 2,\ z_3 = 8,\ z_4 = 24,\ z_5 = 66,\ z_6 = 176$ and so on, one can easily verify that

$$z_{n+1} = 3z_n - z_{n-1} + 2$$

and

$$z_{n-1}z_{n+1} = (z_n - 1)^2 - 1.$$

These two identities resemble the corresponding identities of the cobalancing sequence. A natural question is: "Do these identities hold good for arbitrary values of A?" The following theorem answers this question in affirmative.

THEOREM 3.1. *Let $A > 2$ be an fixed but arbitrary integer and $\{x_n\}_{n=1}^{\infty}$ be a binary recurrence sequence defined by $x_{n+1} = Ax_n - x_{n-1}$ with initial values $x_0 = 0,\ x_1 = 1$. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence derived from $\{x_n\}_{n=1}^{\infty}$ by setting $z_0 = 0$ and $z_n = 2(x_0 + x_1 + x_2 + \cdots + x_{n-1}),\ n = 1, 2, \cdots$ Then $\{z_n\}_{n=1}^{\infty}$ satisfies $z_{n+1} = Az_n - z_{n-1} + 2$ and $z_{n-1}z_{n+1} = (z_n - 1)^2 - 1$.*

Proof. If $z_n = 2(x_0 + x_1 + x_2 + \cdots + x_{n-1})$, then

$$\begin{aligned} Az_n - z_{n-1} + 2 &= 2A(x_0 + x_1 + x_2 + \cdots + x_{n-1}) \\ &\qquad - 2(x_0 + x_1 + x_2 + \cdots + x_{n-2}) + 2 \\ &= 2\{(Ax_1 - x_0) + (Ax_2 - x_1) + \cdots + (Ax_{n-1} - x_{n-2})\} + 2 \\ &= 2(x_2 + x_3 + \cdots + x_n) + 2 = z_{n+1} - 2 + 2 = z_{n+1} \end{aligned}$$

from which the first part of the theorem follows. To prove the second part, we observe that

$$A = \frac{z_{n+1} + z_{n-1} - 2}{z_n} = \frac{z_n + z_{n-2} - 2}{z_{n-1}}$$

from which it follows that

$$z_{n+1}z_{n-1} - z_n^2 + 2z_n = z_n z_{n-2} - z_{n-1}^2 + 2z_{n-1}$$

and hence

$$z_{n+1}z_{n-1} - z_n^2 + 2z_n = z_2 z_0 - z_1^2 + 2z_1 = 0.$$

Consequently

$$z_{n+1}z_{n-1} = z_n^2 - 2z_n$$

from which the second part follows. □

4. CONSTRUCTION OF LUCAS-COBALANCING-LIKE SEQUENCES

The Lucas-cobalancing sequence $\{c_n\}_{n=1}^{\infty}$ is associated with balancing sequence $\{B_n\}_{n=1}^{\infty}$ by the relation

$$c_n = B_{n-1} + B_n,\ n = 1, 2, \cdots .$$

This motivates us to define the Lucas-cobalancing-like sequence $\{w_n\}_{n=1}^{\infty}$ associated with balancing-like sequence $\{x_n\}_{n=1}^{\infty}$ as

$$w_n = x_{n-1} + x_n,\ n = 1, 2, \cdots .$$

It is easy to see that the sequence $\{w_n\}_{n=1}^{\infty}$ satisfies the binary recurrence

$$w_{n+1} = Aw_n - w_{n-1}$$

with initial values $w_0 = -1,\ w_1 = 1$ which is consistent with the fact that the recurrence relation of the Lucas-cobalancing-like sequence is identical with that of the balancing-like sequence [8].

For each $n \in \mathbb{Z}^+$, $8b_n^2 + 8b_n + 1$ is a perfect square and by definition, $c_n^2 = 8b_n^2 + 8b_n + 1$. In the following theorem, we establish similar relationship in the cobalancing-like and and Lucas-cobalancing-like sequences corresponding to any given value of A.

THEOREM 4.1. *If $A > 2$ is an arbitrary integer and $\{z_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ are the cobalancing-like and Lucas-cobalancing-like sequences corresponding to the balancing-like sequence $\{x_n\}_{n=1}^{\infty}$, then $w_n^2 = Dz_n^2 + (A+2)z_n + 1$ for each $n \in \mathbb{Z}^+$.*

Proof. By definition, for each $n \in \mathbb{Z}^+$, $w_n^2 = (x_{n-1} + x_n)^2$. We set $t_n = Dz_n^2 + (A+2)z_n + 1$. Observe that

$$\begin{aligned} w_{n+1}^2 - w_n^2 &= (x_n + x_{n+1})^2 - (x_{n-1} + x_n)^2 \\ &= x_{n+1}^2 - x_{n-1}^2 + 2x_n(x_{n+1} - x_{n-1}) \\ &= (x_{n+1} + x_{n-1} + 2x_n)(x_{n+1} - x_{n-1}) \\ &= (A+2)x_n(x_{n+1} - x_{n-1}). \end{aligned} \tag{1}$$

In view of the definition of the cobalancing-like sequence, $2x_n = z_{n+1} - z_n$ for each $n \in \mathbb{Z}^+$. Letting $M = A/2$, we get

$$\begin{aligned} x_{n+1} - x_{n-1} &= Ax_n - 2x_{n-1} = 2Mx_n - 2x_{n-1} \\ &= M(z_{n+1} - z_n) - (z_n - z_{n-1}) \\ &= M(z_{n+1} + z_n) - (Az_n - z_{n-1} + 2) - z_n + 2 \\ &= M(z_{n+1} + z_n) - (z_{n+1} + z_n) + 2 \\ &= (M-1)(z_{n+1} + z_n) + 2. \end{aligned} \tag{2}$$

Further,

$$
\begin{aligned}
t_{n+1} - t_n &= D(z_{n+1}^2 - z_n^2) + (A+2)(z_{n+1} - z_n) \\
&= (M^2-1)(z_{n+1}^2 - z_n^2) + 2(M+1)(z_{n+1} - z_n) \qquad (3)\\
&= (A+2)x_n \,[\,(M-1)(z_{n+1} + z_n) + 2\,].
\end{aligned}
$$

From equations (1), (2) and (3), $w_{n+1}^2 - w_n^2 = t_{n+1} - t_n$ follows. Since, $w_1 = t_1 = 1$, it follows that $w_n^2 = t_n$ for each $n \in \mathbb{Z}^+$ and this completes the proof. □

Note that if $x_n = B_n$, $n \in \mathbb{Z}^+$, then $A = 6$ and hence $M = 3$, $D = M^2 - 1 = 8$, $z_n = b_n$, $n \in \mathbb{Z}^+$ and the identity $w_n^2 = Dz_n^2 + (A+2)z_n + 1$ reduces to $c_n^2 = 8b_n^2 + 8b_n + 1$, the well-known relationship between cobalancing and Lucas-cobalancing numbers [4].

The cobalancing numbers enjoy the shift formulas

$$b_{n+1} = 3b_n + \sqrt{8b_n^2 + 8b_n + 1} + 1$$

and

$$b_{n-1} = 3b_n - \sqrt{8b_n^2 + 8b_n + 1} + 1,$$

$n = 1, 2, \cdots$ [4]. A natural question is "Can there be similar shift formulas for the cobalancing-like sequences?" The following theorem answers this question in affirmative.

THEOREM 4.2. *Let $A > 2$ be an fixed but arbitrary integer, $M = A/2$ and $\{z_n\}_{n=1}^{\infty}$ be the cobalancing-like sequence corresponding to the balancing-like sequence $\{x_n\}_{n=1}^{\infty}$. Then for $n \in \mathbb{Z}^+$*

(a) $z_{n+1} = Mz_n + \sqrt{Dz_n^2 + (A+2)z_n + 1} + 1$

(b) $z_{n-1} = Mz_n - \sqrt{Dz_n^2 + (A+2)z_n + 1} + 1.$

Proof. We will prove (a) only. The proof of (b) is similar. Observe that $\sqrt{Dz_n^2 + (A+2)z_n + 1} = w_n$. We set $s_n = Mz_n + w_n + 1$. In view of $z_2 = s_1 = 2$, we need only to show that $z_{n+2} - z_{n+1} = s_{n+1} - s_n$ for $n \in \mathbb{Z}^+$. From the definition of the sequence $\{z_n\}_{n=1}^{\infty}$, it follows that

$$z_{n+2} - z_{n+1} = 2x_{n+1} \qquad (4)$$

and

$$s_{n+1} - s_n = M(z_{n+1} - z_n) + (w_{n+1} - w_n) = 2Mx_n + x_{n+1} - x_{n-1} = 2x_{n+1} \qquad (5)$$

Comparing (4) and (5), one can get the desired result. □

The balancing-like sequences are generalizations of the balancing sequence and these sequences are termed as natural sequences since the case $A = 2$ defines the natural numbers via a binary recurrence. What we are going to establish in the following theorem makes the balancing-like sequence corresponding to $A = 3$ special.

THEOREM 4.3. *The balancing-like sequence* $\{x_n\}_{n=1}^{\infty}$ *corresponding to* $A = 3$ *coincides with the sequence of even indexed Fibonacci numbers. Furthermore, the corresponding Lucas-balancing-like sequence is the sequence of odd indexed Lucas numbers.*

Proof. First of all we will show that the sequence of even indexed Fibonacci numbers satisfy a recurrence relation identical with that of the Balancing-like sequence corresponding to $A = 3$. Setting $G_n = F_{2n},\ n = 1, 2, \cdots$ and in view the relationship $F_{k+1} = F_k + F_{k-1}$, we have $3G_n - G_{n-1} = 3F_{2n} - F_{2n-2} = 2F_{2n} + F_{2n-1} = F_{2n} + F_{2n+1} = F_{2n+2} = G_{n+1}$. Further, $G_0 = F_0 = 0$ and $G_1 = F_2 = 1$. Thus, the sequence $\{G_n\}$ coincides with the balancing-like sequence corresponding to $A = 3$. Hence, $x_n = F_{2n}$.

We next show that the Lucas-balancing-like sequence corresponding to $A = 3$ coincides with the sequence of odd indexed Lucas numbers. Since $x_n = F_{2n}$ and $F_{k-1} + F_{k+1} = L_k$, it follows that

$$w_n = x_n + x_{n-1} = F_{2n} + F_{2n-2} = L_{2n-1},\ n = 1, 2, \cdots$$

This completes the proof. □

5. FACTORIZATION OF THE BALANCING-LIKE SEQUENCES

It is well-known that each balancing number except 1 is composite and is product of a Pell number and an associated Pell number. The same thing happens for cobalancing numbers also. Panda and Ray [6] proved that for each $n \in \mathbb{Z}^+$

$$B_n = P_n Q_n,\ b_{2n} = P_n Q_{2n-1},\ b_{2n-1} = P_{n-1} Q_{2n-1}.$$

The identity $F_{2n} = F_n L_n$ confirms that the balancing-like sequence corresponding to $A = 3$ admits a factorization like the balancing sequence. Further, one can check that the terms of each balancing-like sequence, except the first and second, are composite. A natural question is: Is there a factorization of every balancing-like sequence similar to the balancing sequence? In this section, we answer this question in mostly affirmative. For each balancing-like sequence $\{x_n\}_{n=1}^{\infty}$, we construct two sequences $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ such that $x_n = p_n q_n,\ n = 1, 2, \cdots$. We call the sequence $\{p_n\}_{n=1}^{\infty}$, a Pell-like sequence while we call $\{q_n\}_{n=1}^{\infty}$, the corresponding associated Pell-like sequence. The identities $P_{2n} = 2B_n$, $P_{2n-1} = B_n - B_{n-1}$, $Q_{2n} = (B_{n+1} - B_{n-1})/2$ and $Q_{2n-1} = B_n + B_{n-1}$ prompt us to define

$$p_{2n} = 2x_n,\ p_{2n-1} = x_n - x_{n-1},$$

$$q_{2n} = \frac{(x_{n+1} - x_{n-1})}{2},\ q_{2n-1} = x_n + x_{n-1},\ n \in \mathbb{Z}^+.$$

Then

$$p_{2n}q_{2n} = x_n(x_{n+1} - x_{n-1})$$

and using the identity $x_2 + x_4 + \cdots + x_{2n} = x_n x_{n+1}$ [5], one can see that the right hand side of the above identity is equal to x_{2n}. Further,

$$p_{2n-1}q_{2n-1} = (x_n - x_{n-1})(x_n + x_{n-1}) = x_n^2 - x_{n-1}^2$$

and using the identity $x_1 + x_3 + \cdots + x_{2n-1} = x_n^2$ [5], the right side of the identity becomes x_{2n-1}. Hence, $x_n = p_n q_n$ for all $n \in \mathbb{Z}^+$.

After factorizing the balancing-like sequences in terms of corresponding Pell-like and associated Pell-like sequences, it is interesting to see what recurrence relations these factor sequences are obeying. The following theorem is important in this regard.

Theorem 5.1. *The Pell-like sequences $\{p_n\}_{n=1}^{\infty}$ satisfy the recurrence relations*

$$p_{n+1} = \begin{cases} 2p_n + p_{n-1} & \text{if } n \text{ is odd} \\ (M-1)p_n + p_{n-1} & \text{if } n \text{ is even} \end{cases}$$

with initial terms $p_0 = 0, p_1 = 1, p_2 = 2, p_3 = A - 1$. Further, the associated Pell-like sequences $\{q_n\}_{n=1}^{\infty}$ satisfy

$$q_{n+1} = \begin{cases} (M-1)q_n + q_{n-1} & \text{if } n \text{ is odd} \\ 2q_n + q_{n-1} & \text{if } n \text{ is even} \end{cases}$$

with initial terms $q_0 = q_1 = 1,\ q_2 = M,\ q_3 = A + 1$ where $M = A/2$.

Proof. The identities

$$2p_{2k-1} + p_{2k-2} = 2(x_k - x_{k-1}) + 2x_{k-1} = 2x_k = p_{2k}$$

and

$$(M-1)p_{2k} + p_{2k-1} = 2(M-1)x_k + x_k - x_{k-1} = x_{k+1} - x_k = p_{2k+1}$$

establish the recurrence relation of the Pell-like sequences as stated in the theorem. Further, recurrence relations of the associated Pell-like sequences follow from

$$\begin{aligned}(M-1)q_{2k-1}+q_{2k-2} &= (M-1)(x_k+x_{k-1})+\frac{x_k-x_{k-2}}{2}\\ &= \frac{(Ax_k-x_{k-1})+(Ax_{k-1}-x_{k-2})-(x_k+x_{k-1})}{2}\\ &= \frac{(x_{k+1}+x_k)-(x_k+x_{k-1})}{2}=\frac{x_{k+1}-x_{k-1}}{2}=q_{2k}\end{aligned}$$

and

$$2q_{2k}+q_{2k-1}=(x_{k+1}-x_{k-1})+(x_k+x_{k-1})=(x_{k+1}+x_k)=q_{2k+1}.$$

□

With the help of the factor sequences $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ corresponding to a given $A>2$, we can factorize the corresponding cobalancing-like sequence. In view of the identities $x_1+x_3+\cdots+x_{2n-1}=x_n^2$ and $x_2+x_4+\cdots+x_{2n}=x_nx_{n+1}$, it follows that

$$\begin{aligned}p_{2k}q_{2k-1} &= 2x_k(x_{k-1}+x_k)\\ &= 2[(x_2+x_4+\cdots+x_{2k-2})+(x_1+x_3+\cdots+x_{2k-1})]\\ &= 2(x_1+x_2+\cdots+x_{2k-1})=z_{2k}\end{aligned}$$

and

$$\begin{aligned}p_{2k}q_{2k+1} &= 2x_k(x_k+x_{k+1})\\ &= 2[(x_1+x_3+\cdots+x_{2k-1})+(x_2+x_4+\cdots+x_{2k})]\\ &= 2(x_1+x_2+\cdots+x_{2k})=z_{2k+1}.\end{aligned}$$

Thus, the cobalancing-like numbers have the factorizations

$$z_n=\begin{cases}p_nq_{n-1} & \text{if } n \text{ is even}\\ p_{n-1}q_n & \text{if } n \text{ is odd.}\end{cases}$$

It is important to note that the factorization $x_n=F_{2n}=F_nL_n$ of the balancing-like sequence corresponding to $A=3$ is not same as that stated prior to Theorem 5.1. The numeric values of the Pell-like and associated Pell-like sequences corresponding to $A=3$ are $1,2,2,6,\ldots$ and $1,\frac{3}{2},4,\frac{7}{2},\ldots$ respectively. The factor sequences of a balancing-like sequence corresponding to an even A are integral while if A is odd then the terms of the corresponding associated Pell-like sequence are odd positive integer multiple of $\frac{1}{2}$.

For each value of A, each factor sequence is defined by means a composite recurrence relations. However, the recurrence relations of these factor sequences are identical (except the initial terms) whenever $M-1=2$, that is $A=6$, which, in turn, corresponds to the balancing sequence. This shows the importance of the balancing sequence among the balancing-like sequences excepting the case $A = 3$, which corresponds to the even indexed Fibonacci numbers.

6. TRIANGULAR-LIKE NUMBERS ASSOCIATED WITH BALANCING-LIKE SEQUENCES

As we have already seen, the balancing-like sequences, in some way, are generalizations of the natural number sequence. In this connection, Panda [5] proved some properties of balancing numbers. The natural numbers contain a subclass known as triangular numbers. In combinatorics, these numbers appear in choosing two objects out of n objects. If the Pascals triangle is arranged in an infinite square, these numbers appear in the third row. The n^{th} triangular number is defined as $T_n = \frac{n(n+1)}{2}$. There are three important properties of triangular numbers.

1. The sum of two consecutive triangular numbers is a perfect square.
2. If T is a triangular number then $8T + 1$ is a perfect square.
3. If T is a triangular number then $9T + 1$ is also a triangular number.

We define the n^{th} triangular-like number τ_n corresponding the balancing like sequence $\{x_n\}_{n=1}^{\infty}$ defined by the recurrence $x_{n+1} = Ax_n - x_{n-1}$ with initial values $x_0 = 0,\ x_1 = 1$ as

$$\tau_n = \frac{x_n x_{n+1}}{x_2} = \frac{x_n x_{n+1}}{A}.$$

With this definition, for $n > 1$ we have

$$\tau_{n-1} + \tau_n = \frac{x_n(x_{n-1} + x_{n+1})}{A} = \frac{x_n A x_n}{A} = x_n^2$$

which is consistent with the first property of triangular numbers mentioned above. A property similar to the second one is given in the following theorem.

THEOREM 6.1. *If τ_n is the n^{th} triangular-like number corresponding to the balancing-like sequence $\{x_n\}_{n=1}^{\infty}$ defined by the recurrence $x_{n+1} = Ax_n - x_{n-1}$ with initial values $x_0 = 0,\ x_1 = 1$, then $(A^2 + 2A)\tau_n + 1$ and $(A^2 - 2A)\tau_n + 1$ are perfect squares.*

Proof. Using the identity $x_{n-1}x_{n+1} = x_n^2 - 1$ [5], we get

$$\begin{aligned}(A^2 + 2A)\tau_n + 1 &= (A + 2)x_n x_{n+1} + 1\\ &= (x_{n+1} + x_{n-1})x_{n+1} + 2x_n x_{n+1} + 1\end{aligned}$$

$$= x_{n+1}^2 + x_n^2 - 1 + 2x_n x_{n+1} + 1 = (x_{n+1} + x_n)^2.$$

In a similar manner

$$(A^2 - 2A)\tau_n + 1 = (x_{n+1} - x_n)^2$$

follows. □

It is also well-known that if $8T + 1$ is a perfect square for some natural number T then T is a triangular number. However, if $(A^2 + 2A)\tau_n + 1$ is a perfect square for some natural number τ, then τ need not be a triangular-like number. Similarly, if $(A^2 - 2A)s + 1$ is a perfect square for some natural number s, then s need not be a triangular-like number. The following examples clarify the last two statements.

If τ is a triangular-like number corresponding to $A = 3$, then $15\tau + 1$ and $3\tau + 1$ are perfect squares. However, $15.13 + 1 = 14^2$ and $15.17 + 1 = 14^2$, but neither 13 nor 17 are triangular-like numbers. Further, $3.5 + 1 = 4^2$, but 5 is not a triangular-like number. Indeed, if τ is any triangular-like number corresponding to $A = 3$ and if $\tau' = 16\tau + 1$, then it is easy to check that $15\tau' + 1$ is a perfect square, but τ' is not a triangular-like number.

The following theorem, which is the converse of Theorem 6.1, provides conditions, the fulfillment of which makes a natural number a triangular-like number.

THEOREM 6.2. *If A and τ are natural numbers such that $(A^2 + 2A)\tau_n + 1$ and $(A^2 - 2A)\tau_n + 1$ are perfect squares then τ is a triangular-like number corresponding to the balancing-like sequence $\{x_n\}_{n=1}^{\infty}$ defined by the recurrence $x_{n+1} = Ax_n - x_{n-1}$ with initial values $x_0 = 0,\ x_1 = 1$.*

Proof. Let us assume that

(6) $$(A^2 - 2A)\tau + 1 = g^2$$

and

(7) $$(A^2 + 2A)\tau + 1 = h^2$$

We distinguish two cases:
Case I: A is even. Let $A = 2K$. In this case, equations (6) and (7) reduce to

(8) $$8T_{K-1}\tau + 1 = g^2$$

and

(9) $$8T_K\tau + 1 = h^2.$$

It follows from equations (8) and (9) that

$$\frac{g^2-1}{h^2-1} = \frac{T_{K-1}}{T_K} = \frac{K-1}{K+1}$$

which reduces to

$$(K+1)g^2 - (K-1)h^2 = 2$$

and the substitution $G = (K+1)g$ converts the last equation to

$$G^2 - (K^2-1)h^2 = 2(K+1) \tag{10}$$

which is a generalized Pells equation. The fundamental solution of

$$G^2 - (K^2-1)h^2 = 1$$

is $u + v\sqrt{K^2-1} = K + \sqrt{K^2-1}$ and in view of [1], the fundamental solutions of (10) corresponds to

$$0 \le h \le 1,\ 0 \le |g| \le K+1.$$

Since $y = 0$ is not a feasible solution, in view of equation (7), there is just one fundamental solution of (10) namely, $G + h\sqrt{K^2-1} = K+1+\sqrt{K^2-1}$. Hence, the general solution of (10) is given by

$$G_n + h_n\sqrt{K^2-1} = \left(K+1+\sqrt{K^2-1}\right)\left(K+\sqrt{K^2-1}\right)^{n-1},\ n = 1, 2, \cdots$$

Thus,

$$\begin{aligned} h_n &= \frac{\left(K+1+\sqrt{K^2-1}\right)\left(K+\sqrt{K^2-1}\right)^{n-1} - \left(K+1-\sqrt{K^2-1}\right)\left(K-\sqrt{K^2-1}\right)^{n-1}}{2\sqrt{K^2-1}} \\ &= \frac{\left(K+\sqrt{K^2-1}\right)^{n} - \left(K-\sqrt{K^2-1}\right)^{n}}{2\sqrt{K^2-1}} + \frac{\left(K+\sqrt{K^2-1}\right)^{n-1} - \left(K-\sqrt{K^2-1}\right)^{n-1}}{2\sqrt{K^2-1}} \end{aligned}$$

and in view of the Binet form of the balancing-like sequences, it follows that $h_n = x_n + x_{n-1}$. Substituting this value of h_n for h in (7), we obtain

$$\begin{aligned} \tau &= \frac{(x_n + x_{n-1})^2 - 1}{8T_K} = \frac{(x_n + x_{n-1})^2 - 1}{A(A+2)} \\ &= \frac{x_{n-1}x_{n+1} + x_{n-1}^2 + 2x_n x_{n-1}}{A(A+2)} \\ &= \frac{x_{n-1}(x_{n+1} + x_{n-1} + 2x_n)}{A(A+2)} \\ &= \frac{x_{n-1}x_n}{A} = \tau_{n-1},\ n = 1, 2, \cdots \end{aligned}$$

which shows that the possible values of τ are nothing but triangular-like numbers of the balancing-like sequence defined by means of the binary recurrence $x_{n+1} = Ax_n - x_{n-1}$ with initial values $x_0 = 0,\ x_1 = 1$.

If A is odd then from (6) and (7) we can obtain

$$\frac{x^2-1}{y^2-1}=\frac{A-2}{A+2}$$

and proceeding as in the previous case leads to the generalized Pells equation

$$G^2-(A^2-4)h^2=4(A+2) \tag{11}$$

where $G=(A+2)g$. One fundamental solution of (11) is $A+2+\sqrt{A^2-4}$ and one can easily verify that its conjugate $A+2-\sqrt{A^2-4}$ is in a different class associated with the fundamental solution $A^3+A^2-3A-2+(A^2+A-1)\sqrt{A^2-4}$. Further, the fundamental solution of

$$G^2-(A^2-4)h^2=(A+2)$$

is $\frac{A^2+A-2}{2}+\frac{A+1}{2}\sqrt{A^2-4}$ and hence $A^2+A-2+(A+1)\sqrt{A^2-4}$ is another fundamental solution of (11). Once again, it is easy to see that this solution does not belong to the classes of solutions generated by the earlier two solutions. For some values of A, we may have another fundamental solution corresponding to $G=2\sqrt{A+2},\ h=0$ which is infeasible for the present case since $h^2\geq 1$. Using bounds for fundamental solutions [1], one can verify that there is no more fundamental solutions associated with (11). It is important to note that

$$A^2+A-2+(A+1)\sqrt{A^2-4}=\left(A+2+\sqrt{A^2-4}\right)\alpha,$$

$$A^3+A^2-3A-2+(A^2+A-1)\sqrt{A^2-4}=\left(A+2+\sqrt{A^2-4}\right)\alpha^2$$

where

$$\alpha^3=\frac{A^3-3A}{2}+\frac{A^2-1}{2}\sqrt{A^2-4}$$

is the fundamental solution of

$$G^2-(A^2-4)h^2=1.$$

Hence, the general solution of (11) is given by

$$G_n+h_n\sqrt{A^2-4}=\left(A+2+\sqrt{A^2-4}\right)\alpha^{n-1},\ n=1,2,\cdots$$

Thus,

$$h_n=\frac{\left(A+2+\sqrt{A^2-4}\right)\alpha^{n-1}-\left(A+2-\sqrt{A^2-4}\right)\beta^{n-1}}{2\sqrt{A^2-4}}$$

$$=\frac{(\alpha+1)\alpha^{n-1}-(\beta+1)\beta^{n-1}}{\sqrt{A^2-4}}$$

$$=\frac{\alpha^n-\beta^n}{\sqrt{A^2-4}}+\frac{\alpha^{n-1}-\beta^{n-1}}{\sqrt{A^2-4}}=x_n+x_{n-1}.$$

The remaining part of the proof is similar to that of the case when A is even. This completes the proof. □

7. AN OPEN PROBLEM ASSOCIATED WITH TRIANGULAR-LIKE NUMBERS

It is well-known that if T is a triangular number then $8T+1$ is a perfect square and $9T+1$ is also a triangular number since $8(9T+1)+1$ is a perfect square. If τ is a triangular-like number corresponding to the balancing-like sequence $\{x_n\}_{n=1}^{\infty}$ defined by $x_{n+1} = Ax_n - x_{n-1}$ with initial values $x_0 = 0,\ x_1 = 1$, and $\tau' = (A-1)^2\tau + 1$ then it is easy to check that $(A^2-2A)\tau' + 1$ is a perfect square, while if $\tau' = (A+1)^2\tau + 1$, then $(A^2+2A)\tau' + 1$ is a perfect square. However, for each $A > 2$, an important problem is to find a fixed natural number L such that $\tau' = L\tau + 1$ is a triangular-like number, that is, both $(A^2-2A)\tau' + 1$ and $(A^2+2A)\tau' + 1$ are perfect squares. We leave it as an open problem for the readers.

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National Institute of Technology
Department of Mathematics
Rourkela-769008, Odisha, India
gkpanda_nit@rediffmail.com
515ma3004@nitrkl.ac.in