# COMPOSITION AND WEIGHTED COMPOSITION OPERATORS FROM BLOCH-TYPE TO BESOV-TYPE SPACES

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In this paper, we study the boundedness and compactness of the composition operators and weighted composition operators from Bloch-type to Besov-type spaces by using the hyperbolic analytic *Besov-type class.* Finally, we show the relation between the hyperbolic analytic Besov-type class and the *meromorphic (or spherical) Besov-type class.*

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*Key words:* Bloch-type space, Besov-type space, hyperbolic analytic Besov-type class, meromorphic (or spherical) Besov-type class.

## 1. INTRODUCTION

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$. The space of bounded analytic functions on $\mathbb{D}$ will be denoted by $H^\infty(\mathbb{D})$. Denote by $H(\mathbb{D})$ the class of all complex-valued functions analytic on $\mathbb{D}$. Let $B$ be the family of holomorphic self-maps $\varphi$ of the unit disc $\mathbb{D}$ into itself.

An analytic self-map $\varphi$ of $\mathbb{D}$ induces the *composition operator* $C_\varphi$ on $H(\mathbb{D})$, defined by $C_\varphi(f) = f \circ \varphi$ for $f \in H(\mathbb{D})$. Let $u$ be a fixed analytic function on $\mathbb{D}$. The functions $\varphi$ and $u$ induce a linear operator $uC_\varphi$ on the $H(\mathbb{D})$ as follows:

$$uC_\varphi f = u.(f \circ \varphi), \qquad f \in H(\mathbb{D}),$$

where the dot denotes pointwise multiplication. An operator of the form $uC_\varphi$ is called a *weighted composition operator.* For more details on composition operators we refer to [3] and [13].

For $0 < \alpha < \infty$, the analytic function $f$ on $\mathbb{D}$ is said to belong to the *Bloch-type space* $\mathcal{B}^\alpha$ if

$$\|f\|_{\mathcal{B}^\alpha} = \sup\{(1 - |z|^2)^\alpha |f'(z)| : z \in \mathbb{D}\} < \infty.$$

This defines a semi-norm. We can see that $|f(0)| + \|f\|_{\mathcal{B}^\alpha}$ is a norm on $\mathcal{B}^\alpha$ that makes it a Banach space. For $\alpha = 1$, we obtain the well-known classical Bloch space, simply denoted by $\mathcal{B}$.

We will use the following Lemma which was proved in [14].

Lemma 1.1. *Let $0<\alpha<\infty$. If $f\in\mathcal{B}^\alpha$, then*

$$|f(z)|\le C\begin{cases}\|f\|_{\mathcal{B}^\alpha} & \alpha\in(0,1),\\ \|f\|_{\mathcal{B}^\alpha}\ln\dfrac{e}{1-|z|^2} & \alpha=1,\\ \dfrac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha-1}} & \alpha>1,\end{cases}$$

*for some $C>0$ independent of $f$.*

Lemma 1.2 ([8]). *For $0<\alpha<\infty$, there exist two holomorphic maps $f$ and $g$ in $\mathcal{B}^\alpha$ such that*

$$(1-|z|^2)^\alpha(|f'(z)|+|g'(z)|)\asymp 1,$$

*for all $z\in\mathbb{D}$, where the notation $A\asymp B$ means that there exists a positive constant $C$ such that $C^{-1}B\le A\le CB$.*

For $1<p<+\infty$ and $-1<r<\infty$, an analytic function $f$ on $\mathbb{D}$ is said to belong to the *Besov-type space* $B_{p,r}$ if

$$\|f\|_{B_{p,r}}=\left(\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^r dA(z)\right)^{\frac{1}{p}}<\infty, \tag{1}$$

where $dA(z)$ denote the Lebesgue area measure on $\mathbb{D}$. Also, if we take $1<p<\infty$ and $r=p-2$ in (1), then we get the analytic Besov space, simply denoted by $B_p$. We can see that $|f(0)|+\|f\|_{B_{p,r}}$ is a norm on $B_{p,r}$, that makes it a Banach space.

Take $\varphi\in B$. By the Schwarz-Pick lemma, $\sup\limits_{z\in\mathbb{D}}(1-|z|^2)\varphi^*(z)\le 1$, where $\varphi^*(z)$ is the hyperbolic derivative

$$\varphi^*(z)=\frac{|\varphi'(z)|}{1-|\varphi(z)|^2}.$$

*Definition* 1.3. For $1<p<+\infty$ and $-1<r<\infty$, the hyperbolic analytic *Besov-type class* $B^h_{p,r}$ is defined to be the family of all functions $\varphi\in B$ such that

$$\|\varphi\|_{B^h_{p,r}}=\left(\int_{\mathbb{D}}(1-|z|^2)^r(\varphi^*(z))^p dA(z)\right)^{\frac{1}{p}}<\infty. \tag{2}$$

Also, if we take $1<p<\infty$ and $r=p-2$ in (2), then we get the hyperbolic analytic Besov class, simply denoted by $B^h_p$.

Proposition 1.4 ([18]). *$H^\infty(\mathbb{D})\subset\mathcal{B}$. Moreover, $\|f\|_{\mathcal{B}}\le\|f\|_\infty$ for any $f\in H^\infty(\mathbb{D})$.*

Lemma 1.5 ([15]). *Let $X,Y=B_p$ $(1<p<\infty)$ or $\mathcal{B}$. Then $C_\varphi:X\to Y$ is a compact operator if and only if for any bounded sequence $\{f_n\}$ in $X$ with $f_n\to 0$ uniformly on compact sets as $n\to\infty$, $\|C_\varphi f_n\|_Y\to 0$ as $n\to\infty$.*

Throughout this paper C denotes a positive constant which may be different at different occurrences.

In [11], Ohno and Zhao characterized the bounded and compact weighted composition operators on Bloch-type spaces. Composition and weighted composition operators on Bloch-type and some other spaces of holomorphic functions are studied, for example, in [4, 6, 9, 12].

Boundedness and compactness of composition operators on Besov spaces have been studied in [2, 7, 15]. Composition operators from Bloch type spaces to Hardy and Besov spaces were studied by Zhao in [19].

Weighted composition operators on weak vector-valued Bergman spaces and Hardy spaces were studied by M. Hassanlou, H. Vaezi and M. Wang in [5].

In [16], Vaezi studied nearly open weighted composition operators on weighted spaces of continuous functions. Symmetric lifting operators acting on some spaces of analytic functions were studied by Vaezi and Nasresfahani in [17].

In this article, we characterize the boundedness and compactness of the composition and weighted composition operators from Bloch-type to Besov-type spaces, by using the hyperbolic analytic Besov-type class, in section 2. Our results extends some results obtained in [10] about the boundedness and compactness of composition operators from Bloch to Besov spaces. Also we give new criterions for boundedness and compactness of the composition and weighted composition operators from Bloch(type) to Besov(type) spaces in sections 2 and 3. Finally, we show the relation between the hyperbolic analytic Besov-type class and meromorphic (or spherical) Besov-type class.

## 2. COMPOSITION OPERATORS FROM $\mathcal{B}^\alpha$ TO $B_{P,R}$

In this section we characterize the bounded and compact composition operator $C_\varphi : \mathcal{B}^\alpha \to B_{p,r}$.

THEOREM 2.1. *Let $\varphi$ be a holomorphic mapping of $\mathbb{D}$ into itself, $1 < p < \infty$, $-1 < r < \infty$ and $0 < \alpha < \infty$. Then*

(1) *If $\sup_{z\in\mathbb{D}}(1-|\varphi(z)|^2)^{p(1-\alpha)} < \infty$ and $\varphi \in B^h_{p,r}$, then $C_\varphi : \mathcal{B}^\alpha \to B_{p,r}$ is a bounded composition operator.*

(2) *If $\sup_{z\in\mathbb{D}}(1-|\varphi(z)|^2)^{p(\alpha-1)} < \infty$ and $C_\varphi : \mathcal{B}^\alpha \to B_{p,r}$ is a bounded composition operator, then $\varphi \in B^h_{p,r}$.*

*Proof.* (1) Suppose that $\sup_{z\in\mathbb{D}}(1-|\varphi(z)|^2)^{p(1-\alpha)} < \infty$ and $\varphi \in B^h_{p,r}$. For any $f \in \mathcal{B}^\alpha$ there exists a constant $C$ such that,

$$\begin{aligned}
\|C_\varphi(f)\|^p_{B_{p,r}} &= \int_{\mathbb{D}} |\varphi'(z)|^p|f'(\varphi(z))|^p(1-|z|^2)^r dA(z) \\
&= \int_{\mathbb{D}} (1-|z|^2)^r(\varphi^*(z))^p(1-|\varphi(z)|^2)^p|f'(\varphi(z))|^p dA(z) \\
&\leq \|f\|^p_{\mathcal{B}^\alpha} \int_{\mathbb{D}} (1-|z|^2)^r(\varphi^*(z))^p(1-|\varphi(z)|^2)^{p(1-\alpha)} dA(z) \\
&\leq C\|f\|^p_{\mathcal{B}^\alpha} \int_{\mathbb{D}} (1-|z|^2)^r(\varphi^*(z))^p dA(z) \\
&\leq C\|f\|^p_{\mathcal{B}^\alpha}\|\varphi\|^p_{B^h_{p,r}} < \infty.
\end{aligned}$$

(2) By using Lemma 1.2, we choose the functions $f$ and $g$ in $\mathcal{B}^\alpha$ such that

$$|f'(z)| + |g'(z)| \geqslant \frac{C}{(1-|z|^2)^\alpha}.$$

Then for every $p > 1$,

$$|f'(z)|^p + |g'(z)|^p \geqslant \frac{2^{1-p}C}{(1-|z|^2)^{\alpha p}}.$$

Hence,

$$\begin{aligned}
\|\varphi\|^p_{B^h_{p,r}} &= \int_{\mathbb{D}} (1-|z|^2)^r(\varphi^*(z))^p dA(z) \\
&= \int_{\mathbb{D}} (1-|z|^2)^r \frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^p} dA(z) \\
&\leqslant \frac{C}{2^{1-p}} \int_{\mathbb{D}} (1-|z|^2)^r|\varphi'(z)|^p(1-|\varphi(z)|^2)^{p(\alpha-1)}(|f'(\varphi(z))|^p \\
&\qquad\qquad + |g'(\varphi(z))|^p) dA(z) \\
&\leqslant \frac{C}{2^{1-p}} \int_{\mathbb{D}} (1-|z|^2)^r|\varphi'(z)|^p|f'(\varphi(z))|^p(1-|\varphi(z)|^2)^{p(\alpha-1)} dA(z) \\
&\quad + \frac{C}{2^{1-p}} \int_{\mathbb{D}} (1-|z|^2)^r|\varphi'(z)|^p|g'(\varphi(z))|^p(1-|\varphi(z)|^2)^{p(\alpha-1)} dA(z) \\
&\leqslant \frac{C}{2^{1-p}}(\|f\circ\varphi\|^p_{B_{p,r}} + \|g\circ\varphi\|^p_{B_{p,r}}) \\
&= \frac{C}{2^{1-p}}(\|C_\varphi(f)\|^p_{B_{p,r}} + \|C_\varphi(g)\|^p_{B_{p,r}}) < \infty.
\end{aligned}$$

Thus, $\varphi \in B^h_{p,r}$. $\square$

THEOREM 2.2. *Let $\varphi$ be a holomorphic mapping of $\mathbb{D}$ into itself, $1 < p < \infty$, $-1 < r < \infty$, $0 < \alpha < \infty$ and $\sup(1-|\varphi(z)|^2)^{p(1-\alpha)} < \infty$. If $\varphi \in B^h_{p,r}$, then $C_\varphi : \mathcal{B}^\alpha \to B_{p,r}$ is a compact composition operator.*

*Proof.* Let $b(\mathcal{B}^\alpha)$ be the unit ball in $\mathcal{B}^\alpha$. Suppose that $\{g_k\} \in b(\mathcal{B}^\alpha)$ and $\{g_k\}$ converge to 0 uniformly on compact subsets of $\mathbb{D}$. We show that $\|C_\varphi(g_k)\|_{B_{p,r}} \to 0$. Since $\varphi \in B^h_{p,r}$, for every $\epsilon > 0$ there exists a compact set $K \subset \mathbb{D}$ such that

$$\int_{\mathbb{D}/K} (1-|z|^2)^r (\varphi^*(z))^p dA(z) < \epsilon$$

and there exists a number $N$ such that

$$\sup_{w\in\varphi(K)} (1-|w|^2)^\alpha |g_k'(w)| < \epsilon^{\frac{1}{p}}, \tag{3}$$

for any $k \geqslant N$. Then

$$\begin{aligned}
\|C_\varphi(g_k)\|^p_{B_{p,r}} &= \int_{\mathbb{D}} |\varphi'(z)|^p |g_k'(\varphi(z))|^p (1-|z|^2)^r dA(z)\\
&= \int_K (1-|z|^2)^r (\varphi^*(z))^p (1-|\varphi(z)|^2)^p |g_k'(\varphi(z))|^p dA(z)\\
&\quad + \int_{\mathbb{D}/K} (1-|z|^2)^r (\varphi^*(z))^p (1-|\varphi(z)|^2)^p |g_k'(\varphi(z))|^p dA(z)\\
&\leq \epsilon \int_K (1-|z|^2)^r (\varphi^*(z))^p (1-|\varphi(z)|^2)^{p(1-\alpha)} dA(z)\\
&\quad +1 \int_{\mathbb{D}/K} (1-|z|^2)^r (\varphi^*(z))^p (1-|\varphi(z)|^2)^{p(1-\alpha)} dA(z)\\
&\leq \epsilon C \|\varphi\|^p_{B^h_{p,r}} + 1C\epsilon < \epsilon C.
\end{aligned}$$

So, $\|C_\varphi(g_k)\|^p_{B_p} \to 0$ and $C_\varphi$ is compact by Lemma 1.5. □

If we take $\alpha = 1$, from Theorems 2.1 and 2.2, we obtain the boundedness and compactness criterions for composition operators from Bloch to Besov-type spaces.

**Note 1**. If $C_\varphi : \mathcal{B} \to B_{p,r}$ is compact, then $C_\varphi$ is bounded. So, $\varphi \in B^h_{p,r}$. Hence, from Theorems 2.1 and 2.2, we obtain the following corollary.

Corollary 2.3. *Let $\varphi$ be a holomorphic mapping of $\mathbb{D}$ into itself, $1 < p < \infty$ and $-1 < r < \infty$. The operator $C_\varphi : \mathcal{B} \to B_{p,r}$ is compact if and only if $\varphi \in B^h_{p,r}$.*

If we take $\alpha = 1$ and $r = p - 2$, from Theorems 2.1 and 2.2, we obtain the results about the boundedness and compactness of weighted composition operators from Bloch to Besov spaces.

**Note 2**. If $C_\varphi : \mathcal{B} \to B_p$ is compact, then $C_\varphi$ is bounded. So, $\varphi \in B^h_p$. Hence we obtain the following corollary.

Corollary 2.4. *Let $\varphi$ be a holomorphic mapping of $\mathbb{D}$ into itself, $1 < p < \infty$. The operator $C_\varphi : \mathcal{B} \to B_p$ is compact if and only if $\varphi \in B^h_p$.*

## 3. WEIGHTED COMPOSITION OPERATORS FROM $\mathcal{B}^\alpha$ TO $B_{P,R}$

In this section, we characterize the bounded and compact weighted composition operator $uC_\varphi : \mathcal{B}^\alpha \to B_{p,r}$.

THEOREM 3.1. *Let $\varphi$ be a holomorphic mapping of $\mathbb{D}$ into itself, $u \in H^\infty(\mathbb{D})$, $1 < p < \infty$, $-1 < r < \infty$ and $0 < \alpha < \infty$.*

(1) *If $\sup(1-|\varphi(z)|^2)^{p(1-\alpha)} < \infty$, $\varphi \in B^h_{p,r}$ and*

i) *If $\alpha \in (0,1)$, $\int_{\mathbb{D}} (1-|z|^2)^{(r-\alpha p)} dA(z)$ is finite,*

ii) *If $\alpha = 1$, $\int_{\mathbb{D}} (1-|z|^2)^{(r-\alpha p)} (\ln \frac{e}{1-|\varphi(z)|^2})^p dA(z)$ is finite,*

iii) *If $\alpha > 1$, $\int_{\mathbb{D}} (1-|z|^2)^{(r-\alpha p)} (\frac{1}{(1-|\varphi(z)|^2)^{\alpha-1}})^p dA(z)$ is finite,*

*then $uC_\varphi : \mathcal{B}^\alpha \to B_{p,r}$ is a bounded weighted composition operator.*

(2) *If $\sup(1-|\varphi(z)|^2)^{p(\alpha-1)} < \infty$ and $uC_\varphi : \mathcal{B}^\alpha \to B_{p,r}$ is a bounded weighted composition operator, then $\varphi \in B^h_{p,r}$.*

*Proof.* (1) Let $\sup(1-|\varphi(z)|^2)^{p(1-\alpha)} < \infty$ and $\varphi \in B^h_{p,r}$. For any $f \in \mathcal{B}^\alpha$,

$$\begin{aligned}
\|uC_\varphi(f)\|^p_{B_{p,r}} &= \int_{\mathbb{D}} |[u(z)f(\varphi(z))]'|^p (1-|z|^2)^r dA(z) \\
&= \int_{\mathbb{D}} |u'(z)|^p |f(\varphi(z))|^p (1-|z|^2)^r dA(z) \\
&\quad + \int_{\mathbb{D}} |u(z)|^p |\varphi'(z)|^p |f'(\varphi(z))|^p (1-|z|^2)^r dA(z) \\
&= \int_{\mathbb{D}} (1-|z|^2)^r |u'(z)|^p |f(\varphi(z))|^p dA(z) \\
&\quad + \int_{\mathbb{D}} |u(z)|^p (1-|z|^2)^r (\varphi^*(z))^p (1-|\varphi(z)|^2)^p |f'(\varphi(z))|^p dA(z) \\
&\leq \|u\|^p_{\mathcal{B}^\alpha} \int_{\mathbb{D}} (1-|z|^2)^{(r-\alpha p)} |f(\varphi(z))|^p dA(z) \\
&\quad + \|u\|^p_\infty \|f\|^p_{\mathcal{B}^\alpha} \int_{\mathbb{D}} (1-|z|^2)^r (\varphi^*(z))^p (1-|\varphi(z)|^2)^{p(1-\alpha)} dA(z) \\
&\leqslant \|u\|^p_\infty \int_{\mathbb{D}} (1-|z|^2)^{(r-\alpha p)} |f(\varphi(z))|^p dA(z) \\
&\quad + \|u\|^p_\infty \|f\|^p_{\mathcal{B}^\alpha} C \|\varphi\|^p_{B^h_{p,r}}.
\end{aligned}$$

By using Lemma 1.1, we consider the following cases.

Case $\alpha \in (0,1)$: Since $|f(\varphi(z))| \leqslant C\|f\|_{\mathcal{B}^\alpha}$, so

$$\|uC_\varphi(f)\|^p_{B_{p,r}} \le C\|u\|^p_\infty \|f\|^p_{\mathcal{B}^\alpha} \int_{\mathbb{D}} (1-|z|^2)^{(r-\alpha p)} dA(z) + \|u\|^p_\infty \|f\|^p_{\mathcal{B}^\alpha} C\|\varphi\|^p_{B^h_{p,r}} < \infty.$$

Case $\alpha = 1$: Since $|f(\varphi(z))| \leqslant C\|f\|_{\mathcal{B}^\alpha} \ln \dfrac{e}{1-|\varphi(z)|^2}$, so

$$\begin{aligned}\|uC_\varphi(f)\|^p_{B_{p,r}} \le{}& C\|u\|^p_\infty \|f\|^p_{\mathcal{B}^\alpha} \int_{\mathbb{D}} (1-|z|^2)^{(r-\alpha p)} (\ln \frac{e}{1-|\varphi(z)|^2})^p dA(z)\\ &+\|u\|^p_\infty \|f\|^p_{\mathcal{B}^\alpha} C\|\varphi\|^p_{B^h_{p,r}} < \infty.\end{aligned}$$

Case $\alpha > 1$: Since $|f(\varphi(z))| \leqslant C \dfrac{\|f\|_{\mathcal{B}^\alpha}}{(1-|\varphi(z)|^2)^{\alpha-1}}$, so

$$\begin{aligned}\|uC_\varphi(f)\|^p_{B_{p,r}} \le{}& C\|u\|^p_\infty \|f\|^p_{\mathcal{B}^\alpha} \int_{\mathbb{D}} (1-|z|^2)^{(r-\alpha p)} (\frac{1}{(1-|\varphi(z)|^2)^{\alpha-1}})^p dA(z)\\ &+\|u\|^p_\infty \|f\|^p_{\mathcal{B}^\alpha} C\|\varphi\|^p_{B^h_{p,r}} < \infty.\end{aligned}$$

(2) By using Lemma 1.2, we choose the functions $f$ and $g$ in $\mathcal{B}^\alpha$ such that

$$|f'(z)| + |g'(z)| \geqslant \frac{C}{(1-|z|^2)^\alpha}.$$

Then for every $p > 1$,

$$|f'(z)|^p + |g'(z)|^p \geqslant \frac{2^{1-p}C}{(1-|z|^2)^{\alpha p}}.$$

Hence,

$$\begin{aligned}
\|\varphi\|^p_{B^h_{p,r}} &= \int_{\mathbb{D}} (1-|z|^2)^r (\varphi^*(z))^p dA(z)\\
&= \int_{\mathbb{D}} (1-|z|^2)^r \frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^p} dA(z)\\
&\leqslant \frac{C}{2^{1-p}} \int_{\mathbb{D}} (1-|z|^2)^r |\varphi'(z)|^p (1-|\varphi(z)|^2)^{p(\alpha-1)} (|f'(\varphi(z))|^p\\
&\quad +|g'(\varphi(z))|^p) dA(z)\\
&\leqslant \frac{C}{2^{1-p}} \int_{\mathbb{D}} (1-|z|^2)^r |\varphi'(z)|^p |f'(\varphi(z))|^p (1-|\varphi(z)|^2)^{p(\alpha-1)} dA(z)\\
&\quad + \frac{C}{2^{1-p}} \int_{\mathbb{D}} (1-|z|^2)^r |\varphi'(z)|^p |g'(\varphi(z))|^p (1-|\varphi(z)|^2)^{p(\alpha-1)} dA(z)\\
&\leqslant \frac{C}{2^{1-p}} (\|f\circ\varphi\|^p_{B_{p,r}} + \|g\circ\varphi\|^p_{B_{p,r}})\\
&= \frac{C}{2^{1-p}} (\|1C_\varphi(f)\|^p_{B_{p,r}} + \|1C_\varphi(g)\|^p_{B_{p,r}}) < \infty.
\end{aligned}$$

Thus, $\varphi \in B^h_{p,r}$. $\square$

THEOREM 3.2. *Let $\varphi$ be a holomorphic mapping of $\mathbb{D}$ into itself, $u \in H^\infty(\mathbb{D})$, $1 < p < \infty$, $-1 < r < \infty$, $0 < \alpha < \infty$, $\sup(1-|\varphi(z)|^2)^{p(1-\alpha)} < \infty$ and $\varphi \in B^h_{p,r}$. Also suppose that*

1. *If $\alpha \in (0,1)$, $\int_{\mathbb{D}} (1-|z|^2)^{(r-\alpha p)} dA(z) = 0$,*
2. *If $\alpha = 1$, $\int_{\mathbb{D}} (1-|z|^2)^{(r-\alpha p)} (\ln \frac{e}{1-|\varphi(z)|^2})^p dA(z) = 0$,*
3. *If $\alpha > 1$, $\int_{\mathbb{D}} (1-|z|^2)^{(r-\alpha p)} (\frac{1}{(1-|\varphi(z)|^2)^{\alpha-1}})^p dA(z) = 0$.*

*Then $uC_\varphi : \mathcal{B}^\alpha \to B_{p,r}$ is a compact weighted composition operator.*

*Proof.* Let $b(\mathcal{B}^\alpha)$ be the unit ball in $\mathcal{B}^\alpha$. Suppose that $\{g_k\} \in b(\mathcal{B}^\alpha)$ and $\{g_k\}$ converge to 0 uniformly on every compact subset of $\mathbb{D}$. Since $\varphi \in B^h_{p,r}$, for every $\epsilon > 0$ there exists a compact $K \subset \mathbb{D}$ such that

$$\int_{\mathbb{D}/K} (1-|z|^2)^r (\varphi^*(z))^p dA(z) < \epsilon$$

and there exists a number $N$ such that

$$\sup_{w \in \varphi(K)} (1-|w|^2)^\alpha |g'_k(w)| < \epsilon^{\frac{1}{p}}, \tag{4}$$

for any $k \geqslant N$. Then

$$\begin{aligned}
\|uC_\varphi(g_k)\|^p_{B_{p,r}} &= \int_{\mathbb{D}} |[u(z)g_k(\varphi(z))]'|^p (1-|z|^2)^r dA(z) \\
&= \int_{\mathbb{D}} |u'(z)|^p |g_k(\varphi(z))|^p (1-|z|^2)^r dA(z) \\
&\quad + \int_{\mathbb{D}} |u(z)|^p |\varphi'(z)|^p |g'_k(\varphi(z))|^p (1-|z|^2)^r dA(z) \\
&\leq \ \|u\|^p_{\mathcal{B}^\alpha} \int_{\mathbb{D}} (1-|z|^2)^{r-\alpha p} |g_k(\varphi(z))|^p dA(z) \\
&\quad + \int_K (1-|z|^2)^r (\varphi^*(z))^p (1-|\varphi(z)|^2)^p |g'_k(\varphi(z))|^p |u(z)|^p dA(z) \\
&\quad + \int_{\mathbb{D}/K} (1-|z|^2)^r (\varphi^*(z))^p (1-|\varphi(z)|^2)^p |g'_k(\varphi(z))|^p |u(z)|^p dA(z) \\
&\leq \|u\|^p_\infty \int_{\mathbb{D}} (1-|z|^2)^{r-\alpha p} |g_k(\varphi(z))|^p dA(z)
\end{aligned}$$

$$
\begin{aligned}
&+\|u\|_\infty^p \epsilon \int_K (1-|z|^2)^r(\varphi^*(z))^p(1-|\varphi(z)|^2)^{p(1-\alpha)}dA(z)\\
&+\|u\|_\infty^p 1 \int_{\mathbb{D}/K} (1-|z|^2)^r(\varphi^*(z))^p(1-|\varphi(z)|^2)^{p(1-\alpha)}dA(z)\\
\leq{}& \|u\|_\infty^p \int_{\mathbb{D}} (1-|z|^2)^{(r-\alpha p)}|g_k(\varphi(z))|^p dA(z)\\
&+\|u\|_\infty^p \epsilon C\|\varphi\|^p_{B^h_{p,r}} + \|u\|_\infty^p 1C\epsilon.
\end{aligned}
$$

By using Lemma 1.1, we consider the following cases.
Case $\alpha \in (0,1)$: Since $|g_k(\varphi(z))| \leqslant C\|g_k\|_{\mathcal{B}^\alpha}$, so

$$
\begin{aligned}
\|uC_\varphi(g_k)\|^p_{B_{p,r}} &\leq C\|u\|_\infty^p\|g_k\|^p_{\mathcal{B}^\alpha}\int_{\mathbb{D}}(1-|z|^2)^{(r-\alpha p)}dA(z)\\
&\quad+\|u\|_\infty^p\epsilon C\|\varphi\|^p_{B^h_{p,r}} + \|u\|_\infty^p 1C\epsilon\\
&\leq \|u\|_\infty^p\|g_k\|^p_{\mathcal{B}^\alpha}\epsilon + \|u\|_\infty^p\epsilon C\|\varphi\|^p_{B^h_{p,r}} + \|u\|_\infty^p 1C\epsilon < C\epsilon.
\end{aligned}
$$

Case $\alpha = 1$: Since $|g_k(\varphi(z))| \leqslant C\|g_k\|_{\mathcal{B}^\alpha}\ln\dfrac{e}{1-|\varphi(z)|^2}$, so

$$
\begin{aligned}
\|uC_\varphi(g_k)\|^p_{B_{p,r}} &\leq C\|u\|_\infty^p\|g_k\|^p_{\mathcal{B}^\alpha}\int_{\mathbb{D}}(1-|z|^2)^{(r-\alpha p)}(\ln\frac{e}{1-|\varphi(z)|^2})^p dA(z)\\
&\quad+\|u\|_\infty^p\epsilon C\|\varphi\|^p_{B^h_{p,r}} + \|u\|_\infty^p 1C\epsilon\\
&\leq C\|u\|_\infty^p\|g_k\|^p_{\mathcal{B}^\alpha}\epsilon + \|u\|_\infty^p\epsilon C\|\varphi\|^p_{B^h_{p,r}} + \|u\|_\infty^p 1C\epsilon < C\epsilon.
\end{aligned}
$$

Case $\alpha > 1$: Since $|g_k(\varphi(z))| \leqslant C\dfrac{\|g_k\|_{\mathcal{B}^\alpha}}{(1-|\varphi(z)|^2)^{\alpha-1}}$, so

$$
\begin{aligned}
\|uC_\varphi(g_k)\|^p_{B_{p,r}} &\leq C\|u\|_\infty^p\|g_k\|^p_{\mathcal{B}^\alpha}\int_{\mathbb{D}}(1-|z|^2)^{(r-\alpha p)}(\frac{1}{(1-|\varphi(z)|^2)^{\alpha-1}})^p dA(z)\\
&\quad+\|u\|_\infty^p\epsilon C\|\varphi\|^p_{B^h_{p,r}} + \|u\|_\infty^p 1C\epsilon\\
&\leq C\|u\|_\infty^p\|g_k\|^p_{\mathcal{B}^\alpha}\epsilon + \|u\|_\infty^p\epsilon C\|\varphi\|^p_{B^h_{p,r}} + \|u\|_\infty^p 1C\epsilon < C\epsilon.
\end{aligned}
$$

Hence, $\|uC_\varphi(g_k)\|^p_{B_p} \to 0$ and $uC_\varphi$ is compact by Lemma 1.5. $\square$

If we take $\alpha = 1$, from Theorems 3.1 and 3.2, we obtain the boundedness and compactness criterions for weighted composition operators from Bloch to Besov-type spaces.

If we take $\alpha = 1$ and $r = p-2$, we obtain the following results about the boundedness and compactness of weighted composition operators from Bloch to Besov spaces.

THEOREM 3.3. *Let $\varphi$ be a holomorphic mapping of $\mathbb{D}$ into itself, $u \in H^\infty(\mathbb{D})$, $1 < p < \infty$. Then*

(1) *If* $\int_{\mathbb{D}}(\ln \frac{e}{(1-|\varphi(z)|^2)})^p d\lambda(z) < \infty$ *(where* $d\lambda(z) = (1-|z|^2)^{-2}dA(z)$ *is the Mobius invariant measure on* $\mathbb{D}$*) and* $\varphi \in B_p^h$, *then* $uC_\varphi : \mathcal{B} \to B_p$ *is a bounded weighted composition operator.*

(2) *If* $uC_\varphi : \mathcal{B} \to B_p$ *is bounded weighted composition operator, then* $\varphi \in B_p^h$.

THEOREM 3.4. *Let* $\varphi$ *be a holomorphic mapping of* $\mathbb{D}$ *into itself,* $u \in H^\infty(\mathbb{D})$, $1 < p < \infty$. *If* $\int_{\mathbb{D}}(\ln \frac{e}{(1-|\varphi(z)|^2)})^p d\lambda(z) = 0$ *and* $\varphi \in B_p^h$, *then* $uC_\varphi : \mathcal{B} \to B_p$ *is a compact weighted composition operator.*

## 4. RELATION BETWEEN $B^{\#}_{P,R}$ AND $B^{H}_{P,R}$

Since analytic functions $f$ in the unit disc $\mathbb{D}$ are also meromorphic in $\mathbb{D}$, we can study the class of meromorphic functions, provided that the ordinary derivative of $f$ is replaced by the spherical derivative $f^{\#}$, where

$$f^{\#}(z) = \frac{|f'(z)|}{1+|f(z)|^2}, \qquad z \in \mathbb{D}.$$

Take $0 < \alpha < \infty$. The family of $\alpha-$*normal meromorphic* functions in $\mathbb{D}$ is denoted by $\mathcal{N}^\alpha$ and is defined by

$$\mathcal{N}^\alpha = \{f \text{ meromorphic in } \mathbb{D} : \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha f^{\#}(z) < \infty\}.$$

We define

$$\|f\|_{\mathcal{N}^\alpha} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha f^{\#}(z). \tag{5}$$

When $\alpha = 1$, we obtain the family of *normal meromorphic* functions in $\mathbb{D}$, simply denoted by $\mathcal{N}$.

*Definition* 4.1. For $1 < p < \infty$ and $-1 < r < \infty$, we define the *meromorphic (or spherical) Besov-type class*, $B^{\#}_{p,r}$ by

$$B^{\#}_{p,r} = \{f \text{ meromorphic in } \mathbb{D} : \int_{\mathbb{D}}(f^{\#}(z))^p(1-|z|^2)^r dA(z) < \infty\}.$$

We define

$$\|f\|_{B^{\#}_{p,r}} = \int_{\mathbb{D}}(f^{\#}(z))^p(1-|z|^2)^r dA(z). \tag{6}$$

When $1 < p < \infty$ and $r = p-2$, we obtain the *meromorphic (or spherical) Besov class*, simply denoted by $B^{\#}_p$ (see [1]).

PROPOSITION 4.2. *Take* $1 < p < \infty$, $-1 < r < \infty$ *and* $0 < \alpha < \infty$. *Let* $f$ *be a* $\alpha-$*normal meromorphic function in* $\mathbb{D}$ *and* $\sup_{z\in\mathbb{D}}(1-|\varphi(z)|^2)^{p(1-\alpha)} < \infty$. *If* $\varphi \in B^h_{p,r}$, *then* $f \circ \varphi \in B^{\#}_{p,r}$.

*Proof.* Since $\varphi \in B^h_{p,r}$, we have

$$\|\varphi\|^p_{B^h_{p,r}} = \int_{\mathbb{D}} (1-|z|^2)^r (\varphi^*(z))^p dA(z) < \infty.$$

Then

$$\begin{aligned}
\|f\circ\varphi\|_{B^{\#}_{p,r}} &= \int_{\mathbb{D}} (1-|z|^2)^r ((f\circ\varphi)^{\#}(z))^p dA(z) \\
&= \int_{\mathbb{D}} (1-|z|^2)^r \frac{|\varphi'(z)|^p |f'(\varphi(z))|^p}{(1+|f(\varphi(z))|)^p} dA(z) \\
&= \int_{\mathbb{D}} (1-|z|^2)^r \frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^p} (1-|\varphi(z)|^2)^p \frac{|f'(\varphi(z))|^p}{(1+|f(\varphi(z))|)^p} dA(z) \\
&\leqslant \|f\|^p_{\mathcal{N}_\alpha} \int_{\mathbb{D}} (1-|z|^2)^r (\varphi^*(z))^p (1-|\varphi(z)|^2)^{p(1-\alpha)} dA(z) \\
&\leqslant C\|f\|^p_{\mathcal{N}_\alpha} \int_{\mathbb{D}} (1-|z|^2)^r (\varphi^*(z))^p dA(z) \\
&\leqslant C\|f\|^p_{\mathcal{N}_\alpha} \|\varphi\|^p_{B_{p,r}} < \infty.
\end{aligned}$$

Thus, $f \circ \varphi \in B^{\#}_{p,r}$. □

If we take $\alpha = 1$, we obtain the following proposition:

PROPOSITION 4.3. *Take* $1 < p < \infty$ *and* $-1 < r < \infty$. *Let* $f$ *be a normal meromorphic function in* $\mathbb{D}$. *If* $\varphi \in B^h_{p,r}$, *then* $f \circ \varphi \in B^{\#}_{p,r}$.

If we take $\alpha = 1$ and $r = p - 2$, we get the following proposition, which was stated in [10].

PROPOSITION 4.4. *Take* $1 < p < \infty$. *Let* $f$ *be a normal meromorphic function in* $\mathbb{D}$. *If* $\varphi \in B^h_p$, *then* $f \circ \varphi \in B^{\#}_p$.

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