Dedicated to the memory of Cabiria Andreian-Cazacu

GEOMETRIC ESTIMATES OF THE FIRST EIGENVALUE OF A QUASILINEAR OPERATOR

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We study the nonlinear eigenvalue problem for quasilinear operator $Lu = -\Delta_p u + V|u|^{p-2}u$ and we give some geometric estimates of the first eigenvalue of the operator L on closed Riemannian manifold with some geometric conditions as integral curvature condition.

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1. INTRODUCTION

In this paper we investigate the principal eigenvalue of quasilinear operator

$$Lu = -\Delta_p u + V|u|^{p-2}u$$

on closed Riemannian manifold (M^n, g), where V is a nonnegative smooth function on M. On a compact Riemannian manifold (M^n, g), for $p \in (1, +\infty)$ the p-Laplace operator is defined as

$$\Delta_p u = div(|\nabla u|^{p-2}\nabla u),$$

and in local coordinate, it is written as

$$\Delta_p u = -\frac{1}{\sqrt{\det g}} \sum_{i,j} \frac{\partial}{\partial x^i}\left(\sqrt{\det g} g^{ij} |\nabla u|^{p-2} \frac{\partial u}{\partial x^j}\right).$$

The p-Laplacian is a second order quasilinear elliptic operator and when $p = 2$, Δ_p reduces to the usual Laplace-Beltrami operator. Corresponding to the operator L, we have two eigenvalue equations

$$\text{(1.1)} \qquad \text{(Dirichlet)} \begin{cases} Lu = \bar{\lambda}|u|^{p-2}u & \text{on } M, \\ u = 0 & \text{on } \partial M, \end{cases}$$

$$\text{(1.2)} \qquad \text{(Neumann)} \begin{cases} Lu = \bar{\lambda}|u|^{p-2}u & \text{on } M, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M, \end{cases}$$

where ν is the outward normal on ∂M. The least eigenvalue of (1.1) and (1.2) on compact manifold are zero with the corresponding eigenfunctions being constants. Hence the first nontrivial Dirichelt eigenvalue (1.1) for M is defined by

$$\lambda_{1,p}(M)=\inf_{0\neq u\in W_0^{1,p}(M)}\frac{\int_M(|\nabla u|^p+V|u|^p)d\mu}{\int_M|u|^pd\mu}, \tag{1.3}$$

and the first Neumann eigenvalue is given by

$$\Lambda_{1,p}(M)=\inf_{0\neq u\in W^{1,p}(M)}\left\{\frac{\int_M(|\nabla u|^p+V|u|^p)d\mu}{\int_M|u|^pd\mu}\Big|\ \int_M|u|^{p-2}u\,d\mu=0\right\}, \tag{1.4}$$

where $d\mu$ is the Riemannian volume element induced by g and the function u is then called the eigenfunction of operator L corresponding to λ (or Λ) on M. Obviously, the infimum does not change when we replace $W_0^{1,p}(M)$ by $C_0^\infty(M)$.

For each point $x\in M^n$, we denote the smallest eigenvalue for the Ricci tensor $Ric:T_xM\to T_xM$ by $\rho(x)$. Let

$$\begin{aligned} Ric_-^K(x) &= ((n-1)K-\rho(x))_+=\max\{0,(n-1)K-\rho(x)\},\\ ||Ric_-^K||_{q,R} &= \sup_{x\in M}\left(\int_{B(x,R)}(Ric_-^K)^qd\mu\right)^{\frac{1}{q}}. \end{aligned}$$

Then $||Ric_-^K||_{q,R}=0$ iff $Ric\geq(n-1)K$. The normalized q-norm of u on the domain Ω we denote by

$$||u||_{q,\Omega}^*=\left(\frac{1}{Vol(\Omega)}\int_\Omega|u|^qd\mu\right)^{\frac{1}{q}}.$$

The first eigenvalue of the geometric operator as Laplace-Beltrami and p-Laplacian associated to a Riemannian metric g on a compact Riemannian manifold M^n has been extensively studied in mathematical literature. For instance, Matei [13] generalized Cheng's first Dirichlet eigenvalue comparison of balls for Laplacian [7] to the p-Laplacian and fund a sharp lower bound for the first nontrivial eigenvalue of p-Laplacian on closed Riemannian manifold with the Ricci curvature bounded below by $(n-1)K$, $K>0$. Then Valtorta [21] and Naber-Valtorta [14] give this result for $K=0$ and $K\in\mathbb{R}$, respectively. L. F. Wang [22] studied the geometric estimate for the first eigenvalue of weighted p-Laplacian on closed Riemannian manifold when the Bakry-Emery curvature has a positive lower bound and fund a lower bound for the first eigenvalue. Abolarinwa [1] obtained classical estimates of Faber-Krahan inequality and Cheeger-type ineqaulity for the first eigenvalue of p-Laplacian on compact manifold with zero boundary condition. Seto and Wei [19] give various estimates of the first eigenvalue of the p-Laplace operator on closed Riemannian manifold with integral curvature conditions. There are other some results on the first eigenvalue of p-Laplacian, see for instance [10, 12, 11, 20, 24].

In this paper, we extended the first eigenvalue estimate for p-Laplacian given in [1, 19, 13] to the operator L. The eigenvalue and generalized eigenvalue of operator L studied in [15, 16] on $\mathbb{R}^n$. Under the assumption that the integral Ricci curvature is sufficiently small, we will prove the following first eigenvalue estimates:

THEOREM 1.1 (Cheng-Type Estimate). *Let (M^n, g) be a complete Riemannian manifold and V is a radial function. Let $\mathbb{M}^n_K$ be the complete simply connected space of constant curvature K, $B_K(r) \subset \mathbb{M}^n_K$ is the ball of radius r. For any $x_0 \in M$, $K \in \mathbb{R}$, $r > 0$, $p > 1$ and $q > \frac{n}{2}$ there exists an $\epsilon = \epsilon(n, p, q, K, r)$ such that if $\partial B(x_0, r) \neq \emptyset$ and $||Ric^K_-||^*_{\bar{q}, B(x_0,r)} < \epsilon$ then*

$$\lambda_{1,p}(B(x_0, r)) \leq p\lambda_{1,p}(B_K(r)) + C(n, p, q, K, r) \left(||Ric^K_-||^*_{\bar{q}, B(x_0,r)}\right)^{\frac{p}{2}}, \tag{1.5}$$

where $\bar{q} = \max\{q, \frac{p}{2}\}$ and $\lambda_{1,p}(B_K(r))$ is the first Dirichlet eigenvalue of the operator L in the model space $\mathbb{M}^n_K$.

THEOREM 1.2 (Lichnerowicz-Type Estimate). *Let (M^n, g) be a complete Riemannian manifold. For $q > \frac{n}{2}$, $p > 2$ and $K > 0$, there exists $\epsilon = \epsilon(n, p, q, K)$ such that if $||Ric^K_-||^*_{\bar{q}, B(x_0,r)} < \epsilon$ then*

$$\begin{aligned} &(p-1)(\Lambda_{1,p}(M))^{\frac{2}{p}} + K_2(\Lambda_{1,p}(M))^{\frac{1}{p}-1} \\ &\qquad \geq \frac{n + \sqrt{n}(p-2)}{n - 1 + \sqrt{n}(p-2)} \left[(n-1)K - 2||Ric^K_-||^*_{\bar{q}}\right] \end{aligned} \tag{1.6}$$

where $K_2 := \max_M |\nabla V|$. In particular, when $Ric \geq (n-1)K$, we get

$$(p-1)(\Lambda_{1,p}(M))^{\frac{2}{p}} + K_2(\Lambda_{1,p}(M))^{\frac{1}{p}-1} \geq (n-1)K. \tag{1.7}$$

When V is constant function the estimate (1.6) implies the theorem 1.2 of [19] for $p > 2$. We recall Faber-Krahn inequality that it asserts that in $\mathbb{R}^n$ balls minimize the first eigenvalue of the Dirichlet eigenvalue problem of Laplace-Beltrami among sets with given volume. In following we give Faber-Krahn-type estimate for operator L.

THEOREM 1.3 (Faber-Krahn-Type Estimate). *Let M^n be a compact Riemannian manifold and there exists a positive constant K such that $Ric \geq (n-1)K$. Let $\Omega \subset M$ be domain and $B_K \subseteq S^n_K$ be a geodesic ball in the Euclidean n-dimensional sphere with radius $\frac{1}{K^2}$ such that $\frac{Vol(\Omega)}{Vol(M)} = \frac{Vol(B_k)}{Vol(S^n_K)}$.*

Then for $p > 1$, $\lambda_{1,p}(\Omega) \geq \lambda_{1,p}(B_k)$. *Equality holds if and only if there exists an isometry from* M *to* S^n_K *which sends* Ω *to* B_K.

THEOREM 1.4 (Lichnerowicz-Obata-Type Estimate). *Let* M^n *be a compact Riemannian manifold. Suppose there exists a positive constant* K *such that* $Ric \geq (n-1)K$. *Then for any* $p > 1$, $\Lambda_{1,p}(M) \geq \Lambda_{1,p}(S^n_K)$. *Equality holds if and only if* M *is isometric* S^n_K.

To state the next result we recall the Cheeger's isoperimetric constant $\mathcal{C}(M)$ [6] which it defined as

$$\mathcal{C}(M) = \inf_N \frac{Vol(\partial N)}{Vol(N)}, \quad N \subset\subset M,$$

where N is a manifold with compact closure in M and ∂N is assumed to be smooth.

THEOREM 1.5 (Cheeger-Type Estimate). *Let* N *be a compact manifold with smooth boundary in a complete Riemannian manifold* M. *Then we have the following estimate*

$$\lambda_{1,p}(N) \geq \left(\frac{1}{p}\mathcal{C}(N)\right)^p. \tag{1.8}$$

Aa an application of Cheeger type estimate we will obtain the following Mckean type estimate.

THEOREM 1.6 (Mckean-Type Estimate). *Let* M *an* n *dimensional complete simply connected Riemannian manifold of negative constant sectional curvature* $-K$. *Let* $B_R(x)$ *be a geodesic ball of radius* R *in* M. *Then*

$$\lim_{R\to\infty} \lambda_{1,p}(B_R(x)) = \lambda_{1,p}(M) \geq \frac{1}{p^p}((n-1)\sqrt{-K})^p.$$

2. PROOF OF RESULTS

Suppose that (M^n, g) is a complete Riemannian manifold. For a given point $x_0 \in M$, let $r(x) = d(x_0, x)$ be distance function and $\psi(x) = (\Delta r - \bar{\Delta}^K r)_+$ where $\bar{\Delta}^K r$ is the Laplace operator on the model space $\mathbb{M}^n_K$. For the operator L of radial function, we have the following comparison.

PROPOSITION 2.1. *If* u *is a radial function such that* $u' = \frac{du}{dr} \leq 0$, *then*

$$Lu \leq \bar{L}u - \psi u'|u'|^{p-2}, \tag{2.1}$$

where $\bar{L}u = \bar{\Delta}_p u + V|u|^{p-2}u$.

Proof. From the definition of the p-Laplacian we have

$$\Delta_p u = (p-2)|\nabla u|^{p-4} Hess\, u(\nabla u, \nabla u) + |\nabla u|^{p-2}\Delta u.$$

Since $u = u(r)$ is a radial function we get

$$\begin{aligned} Lu &= -\Delta_p u + V|u|^{p-2}u \\ &= -(p-2)u'|u'|^{p-2}u'' - |u'|^{p2}(u'' + u'\Delta r) + V|u|^{p-2}u \\ &= \bar{L}u - (\Delta r - \bar{\Delta}^K r)u'|u'|^{p-2}, \end{aligned}$$

when $u' \leq 0$, we have $(\Delta r - \bar{\Delta}^K r)u'|u'|^{p-2} \geq \psi u'|u'|^{p-2}$ which completes the proof. □

Now we prove theorem 1.1.

Proof of Theorem 1.1. Let $\bar{u}$ be a first eigenfunction for the Dirichlet problem for the operator L on $B_K(r)$ with $\bar{u}(0) = 1$. Hence $0 \leq \bar{u} \leq 1$. Let $r = r(x) = dist(x_0, x)$ and $B = B(x_0, r)$. Therefore $\bar{u}(r) \in W_0^{1,p}(B)$. Let

$$Q = \frac{\int_B (|\nabla \bar{u}|^p + V|\bar{u}|^p) d\mu}{\int_B |\nabla \bar{u}|^p d\mu}.$$

By the definition of the first eigenvalue of (1.4) we have $\lambda_{1,p}(B) \leq Q$. Using integration by parts and proposition 2.1, imply

$$\begin{aligned} Q &= \frac{\int_B \bar{u} L\bar{u} d\mu}{\int_B |\bar{u}|^p d\mu} \leq \frac{\int_B (\bar{u}\bar{L}\bar{u} - \psi \bar{u}'|\bar{u}'|^{p-2}\bar{u}) d\mu}{\int_B |\bar{u}|^p d\mu} \\ &= \lambda(B_K(r)) - \frac{\int_B \psi \bar{u}'|\bar{u}'|^{p-2}\bar{u} d\mu}{\int_B |\bar{u}|^p d\mu} \leq \lambda_{1,p}(B_K(r)) + \frac{\int_B \psi |\bar{u}'|^{p-1} d\mu}{\int_B |\bar{u}|^p d\mu}. \end{aligned} \tag{2.2}$$

Hölder inequality implies

$$\int_B \psi |\bar{u}'|^{p-1} d\mu \leq \left(\int_B \psi^p d\mu\right)^{\frac{1}{p}} \left(\int_B |\bar{u}'|^p d\mu\right)^{1-\frac{1}{p}}.$$

If $r_0 \in (0, r)$ such that $\bar{u}(r_0) = \frac{1}{2}$ then the decreasing of $\bar{u}$ yields $\bar{u} \geq \frac{1}{2}$ on $[0, r_0]$. Hence

$$\begin{aligned} \int_B |\bar{u}|^p d\mu &\geq \left(\int_B |\bar{u}|^p d\mu\right)^{1-\frac{1}{p}} \left(\int_{B(x_0,r_0)} |\bar{u}|^p d\mu\right)^{\frac{1}{p}} \\ &\geq \left(\int_B |\bar{u}|^p d\mu\right)^{1-\frac{1}{p}} (Vol(B(x_0, r_0))2^{-p})^{\frac{1}{p}}. \end{aligned}$$

Thus

$$\frac{\int_B \psi |\bar{u}'|^{p-1} d\mu}{\int_B |\bar{u}|^p d\mu} \leq 2 \left(\frac{\int_B \psi^p d\mu}{Vol(B(x_0, r))}\right)^{\frac{1}{p}} \left(\frac{\int_B |\bar{u}'|^p d\mu}{\int_B |\bar{u}|^p d\mu}\right)^{1-\frac{1}{p}}$$

$$= 2\left(\frac{\int_B \psi^p d\mu}{\int_B d\mu}\right)^{\frac{1}{p}}\left(\frac{Vol(B(x_0,r))}{Vol(B(x_0,r_0))}\right)^{\frac{1}{p}}\left(\frac{\int_B(|\bar{u}'|^p+V|\bar{u}|^p)d\mu}{\int_B |\bar{u}|^p d\mu}\right)^{1-\frac{1}{p}}$$

$$= 2Q^{1-\frac{1}{p}}\left(\fint_B \psi^p d\mu\right)^{\frac{1}{p}}\left(\frac{Vol(B(x_0,r))}{Vol(B(x_0,r_0))}\right)^{\frac{1}{p}}$$

$$\leq 2Q^{1-\frac{1}{p}}||\psi||^*_{\bar{q},B(x_0,r)}\left(\frac{Vol(B(x_0,r))}{Vol(B(x_0,r_0))}\right)^{\frac{1}{p}}.$$

On the other hand from [17, 18], we have

$$||\psi||^*_{\bar{q},B(x_0,r)} \leq C(n,\bar{q})\left(||Ric^K_-||^*_{\bar{q},B(x_0,r)}\right)^{\frac{1}{2}},$$

hence

$$\frac{\int_B \psi|\bar{u}'|^{p-1}d\mu}{\int_B |\bar{u}|^p d\mu} \leq C(n,p,\bar{q},K,r)\left(||Ric^K_-||^*_{\bar{q},B(x_0,r)}\right)^{\frac{1}{2}} Q^{1-\frac{1}{p}},$$

and

$$Q \leq \lambda_{1,p}(B_K(r)) + C(n,p,q,K,r)\left(||Ric^K_-||^*_{\bar{q},B(x_0,r)}\right)^{\frac{1}{2}} Q^{1-\frac{1}{p}}.$$

Using Young's inequality we get

$$Q \leq \lambda_{1,p}(B_K(r)) + \frac{1}{p}C(n,p,q,K,r)\left(||Ric^K_-||^*_{\bar{q},B(x_0,r)}\right)^{\frac{p}{2}} + \frac{p-1}{p}Q.$$

it gives (1.5). □

Proof of Theorem 1.2. Aubry's diameter estimate [2] implies M is closed. The p-Bochner formula as follows

$$\frac{1}{p}\Delta(|\nabla u|^p) = |\nabla u|^{p-2}\left\{(p-2)|\nabla|\nabla u||^2 + |Hessu|^2 + \langle\nabla u, \nabla\Delta u\rangle + Ric(\nabla u,\nabla u)\right\}. \tag{2.3}$$

Integrating of (2.3) on M we have

$$0 = \fint_M |\nabla u|^{p-2}\left\{(p-2)|\nabla|\nabla u||^2 + |Hessu|^2 + \langle\nabla u, \nabla\Delta u\rangle + Ric(\nabla u,\nabla u)\right\} d\mu. \tag{2.4}$$

For the Hessian term we have

$$\fint_M |\nabla u|^{p-2}|Hessu|^2 d\mu \geq \fint_M |\nabla u|^{p-2}\frac{(\Delta u)^2}{n}d\mu = \frac{1}{n}\fint_M \Delta u\Delta_p u d\mu - \frac{p-2}{n}\fint_M \Delta u|\nabla u|^{p-4}Hessu(\nabla u,\nabla u)d\mu$$

$$\geq \frac{1}{n}\fint_M \Delta u \Delta_p u d\mu - \frac{p-2}{\sqrt{n}}\fint_M |\nabla u|^{p-2}|Hessu|^2 d\mu. \tag{2.5}$$

Therefore

$$\fint_M |\nabla u|^{p-2}|Hessu|^2 d\mu \geq \frac{1}{n+\sqrt{n}(p-2)}\frac{1}{n}\fint_M \Delta u \Delta_p u d\mu. \tag{2.6}$$

For the curvature term we obtain

$$\begin{aligned}
&\fint_M |\nabla u|^{p-2} Ric(\nabla u, \nabla u) d\mu \\
&\geq (n-1)K\fint_M |\nabla u|^p d\mu - \fint_M |Ric_-^K||\nabla u|^p d\mu \\
&\geq (n-1)K\fint_M |\nabla u|^p d\mu - ||Ric_-^K||_q^* \left(\fint_M |\nabla u|^{\frac{pq}{q-1}} d\mu\right)^{\frac{q-1}{q}}.
\end{aligned} \tag{2.7}$$

By Sobolev inequality [8] for $|\nabla u|^{\frac{p}{2}}$ there is a constant $C_s = C_s(n,q,K)$ such that

$$\left(\fint_M |\nabla u|^{\frac{pq}{q-1}} d\mu\right)^{\frac{q-1}{q}} \leq C_s \frac{p^2}{4}\fint_M |\nabla u|^{p-2}|\nabla|\nabla u||^2 d\mu + 2\fint_M |\nabla u|^p d\mu$$

and combining it with (2.7), we conclude that

$$\begin{aligned}
\fint_M |\nabla u|^{p-2} Ric(\nabla u, \nabla u) d\mu &\geq (n-1)K\fint_M |\nabla u|^p d\mu \\
&- ||Ric_-^K||_q^* \left(C_s\frac{p^2}{4}\fint_M |\nabla u|^{p-2}|\nabla|\nabla u||^2 d\mu + 2\fint_M |\nabla u|^p d\mu\right).
\end{aligned} \tag{2.8}$$

Plugging (2.6) and (2.8) into (2.4), we get

$$\begin{aligned}
0 \quad \geq \quad & \left[(p-2) - C_s||Ric_-^K||_q^* \frac{p^2}{4}\right]\fint_M |\nabla u|^{p-2}|\nabla|\nabla u||^2 d\mu \\
& -\frac{n-1+\sqrt{n}(p-2)}{n+\sqrt{n}(p-2)}\fint_M \Delta u \Delta_p u d\mu \\
& + \left[(n-1)K - 2||Ric_-^K||_q^*\right]\fint_M |\nabla u|^p d\mu.
\end{aligned} \tag{2.9}$$

For $p > 2$, let $||Ric_-^K||_q^*$ be chosen sufficiently small such that

$$\left[(p-2) - C_s||Ric_-^K||_q^* \frac{p^2}{4}\right] > 0.$$

Then we can rewrite (2.9) as

$$\begin{aligned}
0 \geq -\frac{n-1+\sqrt{n}(p-2)}{n+\sqrt{n}(p-2)}\fint_M \Delta u \Delta_p u d\mu \\
+ \left[(n-1)K - 2||Ric_-^K||_q^*\right]\fint_M |\nabla u|^p d\mu.
\end{aligned} \tag{2.10}$$

Let u be the first eigenfunction for operator L. Then

$$\begin{aligned}
-\fint_M \Delta u\Delta_p u d\mu &= \fint_M Lu\Delta u d\mu - \fint_M V|u|^{p-2}u\Delta u d\mu \\
&= \Lambda_{1,p}(M)\fint |u|^{p-2}u\Delta u d\mu - \fint_M V|u|^{p-2}u\Delta u d\mu \\
&= -\Lambda_{1,p}(M)\fint \langle\nabla(|u|^{p-2}u),\nabla u\rangle d\mu \\
&\quad +\fint_M \langle\nabla(V|u|^{p-2}u),\nabla u\rangle d\mu \\
&= -(p-1)\Lambda_{1,p}(M)\fint |u|^{p-2}|\nabla u|^2 d\mu \\
&\quad +(p-1)\fint V|u|^{p-2}|\nabla u|^2 d\mu + \fint_M |u|^{p-2}u\langle\nabla V,\nabla u\rangle d\mu \\
&\geq -(p-1)\Lambda_{1,p}(M)\left(\fint_M |u|^p d\mu\right)^{1-\frac{2}{p}}\left(\fint_M |\nabla u|^p d\mu\right)^{\frac{2}{p}} \\
&\quad -\fint_M |u|^{p-1}|\nabla V||\nabla u| d\mu \\
&\geq \left(-(p-1)(\Lambda_{1,p}(M))^{\frac{2}{p}} - K_2(\Lambda_{1,p}(M))^{\frac{1}{p}-1}\right)\fint_M |\nabla u|^p d\mu.
\end{aligned}$$

This gives

$$\begin{aligned}
(2.11)\quad 0 \geq &-\frac{n-1+\sqrt{n}(p-2)}{n+\sqrt{n}(p-2)}\left((p-1)(\Lambda_{1,p}(M))^{\frac{2}{p}} + K_2(\Lambda_{1,p}(M))^{\frac{1}{p}-1}\right) \\
&+\left[(n-1)K - 2||Ric_-^K||_q^*\right],
\end{aligned}$$

which is (1.6). □

Proof of Theorem 1.3. Let u be the eigenfunction associated with $\lambda_{1,p}(\Omega)$. Let $\Omega_t := \{x \in \Omega|\, u > t\}$, then $\partial\Omega_t = \{x \in \Omega|\, u = t\}$ and symmetrise them by considering a family of concentric geodesic ball Ω_t^* in S_K^n with the same center as B_K satisfying $Vol(\Omega_t) = \alpha Vol(\Omega_t^*)$ where $\alpha = \frac{Vol(M)}{Vol(S_K^n)}$. We construct a radially decreasing function $u^* : B_K \to \mathbb{R}$ by setting $u^*|_{\partial\Omega_t^*} = t$, therefore $\partial\Omega_t^* = \{x \in B_K^n : u^*(x) = t\}$ and $u^* \in W_0^{1,p}(B_K)$. Also, similar to [3], we construct a radialy increasing $u_* : B_K \to \mathbb{R}$ by

$$u_*(x) = \inf\{t \geq 0 ||\Omega_t| < |\Omega| - C_n|x|^n\},$$

where C_n is the volume S_K^n. Similarly, V^* and V_* are defined. We denote by $d(\partial\Omega_t)$ the $(n-1)$-dimensional Riemannian meaure of the set $\partial\Omega_t$. By the co-area formula [5], we have

$$\frac{\partial}{\partial t}(Vol(\Omega_t)) = \frac{\partial}{\partial t}\int_{\{u>t\}} d\mu = \frac{\partial}{\partial t}\int_t^{+\infty}\left(\int_{\{u=t\}}\frac{1}{|\nabla u|}d(\partial\Omega_t)\right)dt$$
$$= -\int_{\{u=t\}}\frac{1}{|\nabla u|}dt$$

Again, by the co-area formula

$$\begin{aligned}\int_\Omega |u|^p d\mu &= \int_0^\infty t^p \int_{\partial\Omega_t}\frac{1}{|\nabla u|}d(\partial\Omega_t)dt = \int_0^\infty \frac{d}{dt}(t^p)\int_0^\infty\int_{\partial\Omega_t}\frac{1}{|\nabla u|}d(\partial\Omega_t)dt\\ &= \int_0^\infty pt^{p-1}Vol(\Omega_t)dt = p\alpha\int_0^\infty t^{p-1}Vol(B_K)dt\\ &= \alpha\int_{B_K}|u^*|^p d\mu.\end{aligned}$$

Similar method of [3, Theorem 3.8] we have

$$\int_\Omega V|u|^p d\mu \geq \alpha\int_{B_K} V_*|u^*|^p d\mu.$$

Since u^* is a radial function, we get $|\nabla u^*| = |\frac{\partial u^*}{\partial r}|$ which is a constant on $\partial\Omega_t^*$. Applying Hölder inequality we obtain

$$\begin{aligned}Vol(\{u=t\}) &= \int_{\{u=t\}} d\mu = \int_{\{u=t\}}\frac{|\nabla u|^{\frac{p-1}{p}}}{|\nabla u|^{\frac{p-1}{p}}}d\mu\\ &\leq \left(\int_{\{u=t\}}\frac{1}{|\nabla u|}d\mu\right)^{\frac{p-1}{p}}\left(\int_{\{u=t\}}|\nabla u|^{p-1}d\mu\right)^{\frac{1}{p}}.\end{aligned}$$

On the other hand

$$Vol(\{u^*=t\}) = \int_{\{u^*=t\}} d\mu = \left(\int_{\{u^*=t\}}\frac{1}{|\nabla u^*|}d\mu\right)^{\frac{p-1}{p}}\left(\int_{\{u^*=t\}}|\nabla u^*|^{p-1}d\mu\right)^{\frac{1}{p}}.$$

Since $Vol(\Omega_t) = Vol(\{u^* > t\})$ we get $\int_{\{u=t\}}\frac{1}{|\nabla u|}d\mu = \int_{\{u^*=t\}}\frac{1}{|\nabla u^*|}d\mu$. By Gromov's isoperimetric inequality [9] we have $Vol(\{u=t\}) \geq \alpha Vol(\{u^*=t\})$. It implies that

$$\int_{\{u=t\}}|\nabla u|^{p-1}d(\partial\Omega_t) \geq \alpha\int_{\{u^*=t\}}|\nabla u^*|^{p-1}d(\partial\Omega_t^*).$$

Now using the co-area formula for gradient

$$\int_\Omega |\nabla u|^p d\mu = \int_0^\infty\left(\int_{\{u=t\}}|\nabla u|^{p-1}d(\partial\Omega_t)\right)dt$$

$$\geq \alpha \int_0^\infty \left(\int_{\{u^*=t\}} |\nabla u^*|^{p-1} d(\partial\Omega_t^*) \right) dt = \alpha \int_{B_K} |\nabla u^*|^p d\mu.$$

By Rayleigh quotient, we obtain

$$\frac{\int_\Omega (|\nabla u|^p + V|u|^p) d\mu}{\int_\Omega |u|^p d\mu} \geq \frac{\int_{B_k} (|\nabla u^*|^p + V_*|u^*|^p) d\mu}{\int_{B_k} |u^*|^p d\mu} \geq \lambda_{1,p}(B_k).$$

Hence $\lambda_{1,p}(\Omega) \geq \lambda_{1,p}(B_k)$. Now, we consider the equality

$$\lambda_{1,p}(\Omega) = \lambda_{1,p}(B_k) \tag{2.12}$$

holds. Let u be a eigenfunction corresponding to eigenvalue $\lambda_{1,p}(\Omega)$ and $\{u_n\}_{n\in\mathbb{N}} \in W_0^{1,p}(\Omega)$ be a sequence of Mors function which converges to u in $W_0^{1,p}(\Omega)$. Suppose $\{u_n^*\}_{n\in\mathbb{N}}$ is their symmetrization. Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$ then $\{u_n^*\}_{n\in\mathbb{N}}$ is bounded too and there is a $u^* \in W_0^{1,p}(B_K)$ such that $\{u_n^*\}_{n\in\mathbb{N}}$ and $\{|\nabla u_n^*|\}_{n\in\mathbb{N}}$ are converge to u^* and $|\nabla u^*|$ weakly in $W_0^{1,p}(B_K)$ and strongly in $L^p(B_K)$, respectively. It follows that $||u^*||_{W^{1,p}(B_K)} \leq \liminf ||u_n^*||_{W^{1,p}(B_K)}$. Therefore

$$\begin{aligned}
\int_\Omega |u|^p d\mu &= \lim_{n\to\infty} \int_\Omega |u_n|^p d\mu = \alpha \lim_{n\to\infty} \int_{B_K} |u_n^*|^p d\mu = \alpha \int_{B_K} |u^*|^p d\mu \\
\int_\Omega |\nabla u|^p d\mu &= \lim_{n\to\infty} \int_\Omega |\nabla u_n|^p d\mu \geq \alpha \lim_{n\to\infty} \int_{B_K} |\nabla u_n^*|^p d\mu = \alpha \int_{B_K} |\nabla u^*|^p d\mu.
\end{aligned}$$

Similarly $\int_\Omega V|u|^p d\mu \geq \alpha \int_{B_K} V_*|u^*|^p d\mu$. Hence

$$\lambda_{1,p}(\Omega) = \frac{\int_\Omega (|\nabla u|^p + V|u|^p) d\mu}{\int_\Omega |u|^p d\mu} \geq \frac{\int_{B_K} (|\nabla u^*|^p + V_*|u^*|^p) d\mu}{\int_{B_K} |u^*|^p d\mu} \geq \lambda_{1,p}(B_K).$$

Equality (2.12) implies that

$$\lambda_{1,p}(\Omega) = \frac{\int_\Omega (|\nabla u|^p + V|u|^p) d\mu}{\int_\Omega |u|^p d\mu} = \frac{\int_{B_K} (|\nabla u^*|^p + V_*|u^*|^p) d\mu}{\int_{B_K} |u^*|^p d\mu} = \lambda_{1,p}(B_K),$$

and

$$\int_\Omega |\nabla u|^p d\mu \leq \alpha \int_{B_K} |\nabla u^*|^p d\mu. \tag{2.13}$$

Therefore u^* is a first eigenfunction for L on B_K. Since M is compact, $\{u_n\}_{n\in\mathbb{N}}$ and $\{u_n^*\}_{n\in\mathbb{N}}$ converge uniformly to u and u^*, respectively. It follows that u^* is the symmetrization of u. Let (a, b) be an interval of regular values of u, we get

$$\int_{u^{-1}((a,b))} |\nabla u|^p d\mu \geq \alpha \int_{(u^*)^{-1}((a,b))} |\nabla u^*|^p d\mu. \tag{2.14}$$

On the other hand, for any $n \in \mathbb{N}$, we have

$$\int_{\Omega\setminus u^{-1}((a,b))} |\nabla u_n|^p d\mu \geq \alpha \int_{B_K\setminus (u_n^*)^{-1}((a,b))} |\nabla u^*|^p d\mu, \tag{2.15}$$

hence

$$\int_{\Omega\setminus u^{-1}((a,b))} |\nabla u|^p d\mu \geq \alpha \int_{B_K\setminus (u^*)^{-1}((a,b))} |\nabla u^*|^p d\mu. \tag{2.16}$$

Therefore equation (2.13) implies

$$\int_{u^{-1}((a,b))} |\nabla u|^p d\mu = \alpha \int_{(u^*)^{-1}((a,b))} |\nabla u^*|^p d\mu. \tag{2.17}$$

It follows then that $Vol(u^{-1}(t)) = Vol(u^*)^{-1}(t))$ for any regular volume t of u. But inequality in Gromov's isoperimetric inequality implies that there exists an isometry from M to S_K^n which sends Ω_t to Ω_t^*. Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of regular values for u, with $\lim\limits_{n\to\infty} t_n = 0$. Then Ω_{t_n} are concentric geodesic balls included in Ω with $\lim\limits_{n\to\infty} Vol(\Omega_{t_n}) = Vol(\Omega)$. Let Ω_0 be the geodesic ball in M with the same center as Ω_{t_n} and with $Vol(\Omega_0) = Vol(\Omega)$. the geodesic ball Ω_0 is included in Ω. Therefore $\Omega = \Omega_0$ and there exists an isometry from M to S_K^n which sends Ω to B_K. Conversly is obvious. □

LEMMA 2.2. *Let u be the first eigenfunction for the operator L on M and A be a nodal domain of u. Then*

$$\Lambda_{1,p}(M) = \frac{\int_A (|\nabla u|^p + V|u|^p) d\mu}{\int_A |u|^p d\mu}.$$

Proof. By classical density theorems, u is the limit of a sequence $\{u_n\}_{n\in\mathbb{N}}$ of C^∞ functions such that zero is regular value for each u_n. Let $A = u^{-1}(\mathbb{R}_+)$ and $A_n = u_n^{-1}(\mathbb{R}_+)$. The boundary of A_n is the closed regular hypersurface $u_n^{-1}(0)$. For any $n \in \mathbb{N}$ we have

$$\begin{aligned}
\Lambda_{1,p}(M) \int_{A_n} |u|^{p-2} u u_n d\mu &= \int_{A_n} u_n Lu\, d\mu \\
&= -\int_{A_n} u_n \Delta u\, d\mu + \int_{A_n} V|u|^{p-2} u u_n d\mu \\
&= \int_{A_n} \left[|\nabla u|^{p-2} \langle \nabla u_n, \nabla u\rangle + V|u|^{p-2} u u_n \right] d\mu.
\end{aligned}$$

Thus

$$\Lambda_{1,p}(M) = \frac{\int_{A_n} \left[|\nabla u|^{p-2} \langle \nabla u_n, \nabla u\rangle + V|u|^{p-2} u u_n \right] d\mu}{\int_{A_n} |u|^{p-2} u u_n d\mu}.$$

By extracting a subsequence if necessary and we can assume that $\{u_n\}_{n\in\mathbb{N}}$ and $\{|\nabla u_n|\}_{n\in\mathbb{N}}$ converge almost every where to u and $|\nabla u|$, respectively. Therefore the two sequences

$$\frac{\int_{A_n}\left[|\nabla u|^{p-2}\langle\nabla u_n,\nabla u\rangle+V|u|^{p-2}uu_n\right]d\mu}{\int_{A_n}|u|^{p-2}uu_nd\mu}\quad\text{and}\quad\frac{\int_{A_n}(|\nabla u_n|^p+V|u_n|^p)d\mu}{\int_{A_n}|u_n|^pd\mu}$$

converge to

$$\Lambda_{1,p}(M)=\frac{\int_A(|\nabla u|^p+V|u|^p)d\mu}{\int_A|u|^pd\mu}.$$

□

Corollary 2.3. *Let u be a first eigenfunction for L on M, $p>1$ and Let $A_1=u^{-1}(\mathbb{R}_+)$ and $A_2=u^{-1}(\mathbb{R}_-)$ be the nodal domains of u. Then $\Lambda_{1,p}(M)=\lambda_{1,p}(A_1)=\lambda_{1,p}(A_2)$.*

Proof. For each $i=1,2$, we have

$$\lambda_{1,p}(A_i)\le\frac{\int_{A_i}(|\nabla u|^p+V|u|^p)d\mu}{\int_{A_i}|u|^pd\mu}=\Lambda_{1,p}(M).\tag{2.18}$$

Hence $\Lambda_{1,p}(M)\ge\max\{\lambda_{1,p}(A_1),\lambda_{1,p}(A_2)\}$. Let u_1 and u_2 be the first eigenfunctions for the Dirichlet problem for L in A_1 and A_2, respectively. Extend u_i by zero outside A_i and β be a real constant satisfying $\int_M|u_1+\beta u_2|^{p-2}(u_1+\beta u_2)d\mu=0$. Then

$$\begin{aligned}\Lambda_{1,p}(M)\int_{A_1}&|u_1|^pd\mu+\Lambda_{1,p}(M)\int_{A_2}|\beta u_2|^pd\mu=\Lambda_{1,p}(M)\int_M|u_1+\beta u_2|^pd\mu\\&\le\int_M\left(|\nabla(u_1+\beta u_2)|^p+V|u_1+\beta u_2|^p\right)d\mu\\&=\int_{A_1}\left(|\nabla u_1|^p+V|u_1|^p\right)d\mu+\int_{A_2}\left(|\nabla(\beta u_2)|^p+V|\beta u_2|^p\right)d\mu\\&=\lambda_{1,p}(A_1)\int_{A_1}|u_1|^pd\mu+\lambda_{1,p}(A_2)\int_{A_2}|\beta u_2|^pd\mu.\end{aligned}$$

From (2.18) conclude that $\Lambda_{1,p}(M)=\lambda_{1,p}(A_1)=\lambda_{1,p}(A_2)$. □

Proof of Theorem 1.4. Let u be a first eigenfunction of L on M and $\mathfrak{B}=u^{-1}((0,\infty))$. If necessary we can replace u by $-u$ and we may assume that $Vol(\mathfrak{B})\le\frac{1}{2}Vol(M)$. From corollary 2.3, we have

$$\Lambda_{1,p}(M)=\lambda_{1,p}(\mathfrak{B}).\tag{2.19}$$

Let $\mathfrak{B}^*$ be a geodesic ball of center x_0 in S^n_K with $\frac{Vol(\mathfrak{B})}{Vol(M)}=\frac{Vol(\mathfrak{B}^*)}{Vol(S^n_K)}$. From theorem 1.3 we get

$$\lambda_{1,p}(\mathfrak{B})\ge\lambda_{1,p}(\mathfrak{B}^*).\tag{2.20}$$

Since $Vol(\mathfrak{B}) \leq \frac{1}{2}Vol(M)$ we get $Vol(\mathfrak{B}^*) \leq \frac{1}{2}Vol(S^n_K)$ and $\mathfrak{B}^*$ is contained in the hemisphere of center x_0 which we denote by S^n_{K+}. Also, since the first eigenfunction of L on S^n_K is radial with respect toa point x_0, then the nodal domains of this function are geodesic balls centered at x_0 and $-x_0$, respectively. Thus

$$\Lambda_{1,p}(S^n_K) = \lambda_{1,p}(S^n_{K+}) = \lambda_{1,p}(S^n_{K-}) \leq \lambda_{1,p}(\mathfrak{B}^*). \tag{2.21}$$

From (2.19), (2.20) and (2.21) we obtain $\Lambda_{1,p}(M) \geq \Lambda_{1,p}(S^n_K)$. Equality implies that we have $\lambda_{1,p}(\mathfrak{B}) = \lambda_{1,p}(\mathfrak{B}^*)$ and theorem 1.3 implies that M is isometric to S^n_K. □

Proof of Theorem 1.5. Let for some constant $c(N)$ depending to the geometry of Ω,

$$\int_N |\nabla\phi| d\mu \geq \int_N \phi d\mu, \quad \forall \phi \in C_0^\infty(N).$$

Setting $\phi = u^p$, we can write

$$\begin{aligned}
c(N)\int_N u^p d\mu &\leq \int_N |\nabla u^p| d\mu = p\int_N |u^{p-1}\nabla u| d\mu \\
&\leq p\left(\int_N |u|^p d\mu\right)\left(\frac{\int_N |\nabla u|^p d\mu}{\int_N |u|^p d\mu}\right)^{\frac{1}{p}} \\
&\leq p\left(\int_N |u|^p d\mu\right)\left(\frac{\int_N (|\nabla u|^p + V|u|^p) d\mu}{\int_N |u|^p d\mu}\right)^{\frac{1}{p}} \\
&\leq p(\lambda_{1,p}(N))^{\frac{1}{p}} \int_N |u|^p d\mu.
\end{aligned}$$

Hence $\lambda_{1,p}(N) \geq \left(\frac{1}{p}c(N)\right)^p$. By applying co-area formula for ϕ we estimate the last inequality from below.

$$\begin{aligned}
\int_N |\nabla\phi| d\mu &= \int_{-\infty}^{+\infty}\left(\int_{\Omega_t} d\Omega_t\right) dt = \int_{-\infty}^{+\infty} Vol(\partial\Omega_t) dt \\
&= \int_{-\infty}^{+\infty} \frac{Vol(\partial\Omega_t)}{Vol(\Omega_t)} Vol(\Omega_t) dt \\
&\geq \inf_t \left(\frac{Vol(\partial\Omega_t)}{Vol(\Omega_t)}\right) \int_{-\infty}^{+\infty} Vol(\Omega_t) dt \\
&= \inf_t \left(\frac{Vol(\partial\Omega_t)}{Vol(\Omega_t)}\right) \int_N V\phi d\mu = c(N)\int_N |u|^p d\mu.
\end{aligned}$$

In the follows that $\lambda_{1,p}(N) \geq \left(\frac{1}{p}\mathcal{C}(N)\right)^p$. □

Proof of Theorem 1.6. By inequality (1.8),

$$\lambda_{1,p}(B_R(x)) \geq \left(\frac{1}{p}\mathcal{C}(B_R(x))\right)^p.$$

Fixe a point $y \notin B_R(x)$ such that $\eta(y) = d(x, y)$. The function η is differentiable with $||\nabla\eta|| = 1$ and by Laplacian comparison theorem we have

$$\Delta\eta \geq \sqrt{-K}\coth\sqrt{-K}\eta \geq (n-1)\sqrt{-K}.$$

On the other hand

$$Vol(\partial B_R(x)) = \int_{\partial B_R} dA \geq \int_{\partial B_R}\langle\nabla\eta, \omega\rangle dA = \int_{B_R}\Delta\eta d\mu,$$

where ω is the outward normal on $\partial B_R(x)$. Therefore

$$Vol(\partial B_R) \geq (n-1)\sqrt{-K}\int_{B_R} dA = (n-1)\sqrt{-K}Vol(B_R),$$

hence

$$\mathcal{C}(B_R(x)) = \frac{Vol(\partial B_R(x))}{Vol(B_R(x))} \geq (n-1)\sqrt{-K}.$$

We conclude that $\lambda_{1,p}(B_R(x)) \geq (\frac{(n-1)\sqrt{-K}}{p})^p$. $\square$

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