Dedicated to the memory of Cabiria Andreian-Cazacu

A NOTE ON THE INJECTIVITY OF SOME SOBOLEV MAPPINGS ON NON-CONVEX DOMAINS

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We prove a global injectivity result for certain Sobolev mappings defined on non-convex domains. This extends the classical theorem of M.O. Reade from classical complex univalence.

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1. INTRODUCTION

For $p \geq 1$ and an open $D \subset \mathbb{R}^n$ the Sobolev space $W^{1,p}_{loc}(D, \mathbb{R}^n)$ consists of all mappings which are locally in $L^p(D)$ together with their first order partial derivatives. For a.e. $x \in D$ we denote the differential matrix by f' and the Jacobian determinant by J_f.

Let $D \subset \mathbb{R}^n$ be open. We say that a mapping $f : D \to \mathbb{R}^n$ is open if it takes open sets into open sets. Furthermore, we say that f is discrete if either $f^{-1}(y) = \emptyset$ or $f^{-1}(y)$ is a discrete set for every $y \in \mathbb{R}^n$ and we say that f is light if $\dim f^{-1}(y) = 0$ for every $y \in \mathbb{R}^n$. Here, if $E \subset \mathbb{R}^n$, $\dim E$ is the topological dimension of E. Of course, a discrete mapping is light.

Let $b, c \in \mathbb{R}^n \setminus \{0\}$. We denote by $a(b, c)$ the angle which is less than or equal to π between the line containing 0 and b and the line containing 0 and c in the plane formed by $0, b, c$. If $b \neq 0$ and $\varphi \in (0, \pi)$, we denote by $C_{0,b,\varphi} = \{z \in \mathbb{R}^n \setminus \{0\} | a(z, b) < \varphi\}$ the set of face points of a cone which has the face angle φ and the vertex 0. If $D \subset \mathbb{R}^n$ is a domain and $0 < \varphi \leq \pi$, we say that D is a φ-angular convex domain if for every $z_1, z_2 \in D$ there exists $z_3 \in D$ such that $z_1 \neq z_3$, $z_2 \neq z_3$, $[z_1, z_3] \cup [z_2, z_3] \subset D$ and $a(z_1 - z_3, z_2 - z_3) \geq \varphi$. If D is a π-angular convex domain, the domain D is convex. If $D = (-1, 1)^2 \setminus [-1, 0]^2$, then D is a non-convex $\frac{\pi}{2}$-angular convex domain.

We denote by μ_n the Lebesgue measure in $\mathbb{R}^n$.

If $D \subset \mathbb{R}^n$ is a domain and $f : D \to \mathbb{C}$ is a mapping, $|\lambda| = 1$, we set $M(f, z, \lambda) = \{w \in \overline{\mathbb{C}} |$ there exists $t_m \to 0$ such that $\frac{f(z+t_m\lambda)-f(z)}{t_m\lambda} \to \omega\}$.

For a Sobolev mapping $f : D \to \mathbb{R}^n$ we define the distortion function $K_0(f) : D \to [0, \infty]$ by

$$K_0(f)(x) = \begin{cases} \frac{\|f'(x)\|^n}{J_f(x)}, & \text{if } J_f(x) \neq 0 \\ 1, & \text{if } J_f(x) = 0 \text{ and } f'(x) = 0 \\ \infty, & \text{otherwise.} \end{cases}$$

If $K_0(f) \leq K < \infty$ a.e., we obtain the well-known class of quasiregular mappings. If $K_0(f) \in L^1_{loc}(D)$ when $n = 2$ and $K_0(f) \in L^p_{loc}(D)$ for some $p > n - 1$ when $n \geq 3$, it was shown in [4] and [8] that a non-constant f is open and discrete.

If $x = (x_1, ..., x_n) \in \mathbb{R}^n$, we set $|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$ and if $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $y = (y_1, ..., y_n) \in \mathbb{R}^n$, we denote by $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$ the usual scalar product on $\mathbb{R}^n$. If $A \in L(\mathbb{R}^n, \mathbb{R}^n)$, we say that A is positive definite if $\langle A(x), x\rangle \geq 0$ for every $x \in \mathbb{R}^n$. Here $L(\mathbb{R}^n, \mathbb{R}^n) = \{A : \mathbb{R}^n \to \mathbb{R}^n$ linear mapping$\}$.

We shall use the following generalization of the classical Hurwitz's theorem [1]:

Theorem A. *Let $n \geq 1$, $D \subset \mathbb{R}^n$ be a domain, $f_m : D \to f_m(D)$ be a sequence of homeomorphisms such that $f_m \to f$ uniformly on the compact subsets of D and f is a light map. Then f is injective.*

We shall also use the following chain rule result from [11], page 9:

Theorem B. *Let $n \geq 2$, $U, V \subset \mathbb{R}^n$ be open, $g \in C^2(V, U)$, $p \geq 1$ and let $f \in W^{1,p}_{loc}(U, \mathbb{R}^n) \cap C(U, \mathbb{R}^n)$. Then $f \circ g \in W^{1,p}_{loc}(V, \mathbb{R}^n) \cap C(V, \mathbb{R}^n)$ and we have that $\frac{\partial_k(f \circ g)}{\partial x_i}(x) = \sum_{j=1}^{n} \frac{\partial f_k}{\partial y_j}(g(x))\frac{\partial g_j}{\partial x_i}(x)$ for a.e. $x \in V$ and $i, k = 1, ..., n$.*

The classical univalence theorem of Alexander and Warschawski tells us that if $D \subset \mathbb{C}$ is a convex domain, $f : D \to \mathbb{C}$ is analytic and $Ref'(z) > 0$ for every $z \in D$, then f is injective on D.

A generalization of this theorem for plane quasiregular mappings was given in [5]. It was shown that if $D \subset \mathbb{C}$ is a convex domain and a quasiregular $f : D \to \mathbb{C}$ satisfies $\frac{\partial f_1}{\partial x}(z) + \frac{\partial f_2}{\partial y}(z) = 0$ for a.e. $z \in D$ then f is injective on D.

M. O. Reade proved in [10] a generalization of the theorem of Alexander and Warschawski for non-convex domains. Let $D \subset \mathbb{C}$ be a 2φ-angular convex domain with $\varphi \in (0, \frac{\pi}{2}]$. Then an analytic function $f : D \to \mathbb{C}$ such that

$f'(z) \neq 0$ for every $z \in \mathbb{C}$ and $|\arg f'(z)| < \varphi$ for every $z \in D$, is injective on D.

P. T. Mocanu generalized in [9] this result for C^1 mappings, showing that if $\varphi \in (0, \frac{\pi}{2}]$, $D \subset \mathbb{C}$ is a 2φ-angular convex domain with $\varphi \in (0, 2\pi]$, $f \in C^1(D, \mathbb{R}^2)$ is such that $J_f(z) \neq 0$ for every $z \in D$ and $a(f'(z)(h), h) < \varphi$ for every $z \in D$ and every $h \in \mathbb{R}^2 \setminus \{0\}$, then f is injective on D. G. Kohr extended Mocanu's theorem to mappings in $\mathbb{C}^n$, see [6].

A topological version of Mocanu's result is given in [2]. It states that a continuous $f : D \to \mathbb{C}$ whose domain $D \subset \mathbb{C}$ is a 2φ-angular domain with $\varphi \in (0, \frac{\pi}{2})$ and $M(f, z, \lambda)$ is a compact subset of some fixed cone $C_{a,b,\varphi}$ for every $z \in D$ and every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, is injective on D. A similar several complex variables version is proved by G. Kohr in [6].

Kovalev and Onninen proved in [7] a result concerning the local injectivity of open, discrete mappings $f \in W^{1,n}_{loc}(D, \mathbb{R}^n) \cap C(D, \mathbb{R}^n)$ under certain integrable assumptions on the distortion function $K_0(f) : D \to [0, \infty]$, namely $K_0(f) \in L^1_{loc}(D)$ when $n = 2$, $K_0(f) \in L^p_{loc}(D)$ for some $p > n-1$ when $n \geq 3$. If further there exists $\delta \in (-1, 1]$ such that

$$\langle f'(x)(h), h\rangle \geq \delta |h||f'(x)(h)| \text{ for a.e. } x \in D \text{ and every } h \in \mathbb{R}^n \tag{1}$$

then f is a local homeomorphism.

They also showed in [7] that if a Sobolev mapping $f \in W^{1,1}_{loc}(D, \mathbb{R}^n)$, where $D \subset \mathbb{R}^n$ is a convex domain, satisfies the relation (1) with $\delta \in [0, 1]$ then f is injective on D. We remark that if $x \in D$ is such that $J_f(x) \neq 0$ and $\epsilon = \arccos \delta$, the relation (1) tells us that $a(f'(x)(h), h) \leq \epsilon$.

Our main result extends the theorem of M. O. Reade for light Sobolev mappings $f \in W^{1,1}_{loc}(D, \mathbb{R}^n)$. Here D is a φ-angular convex domain with $\varphi \in (0, \pi)$.

THEOREM 1. *Let $n \geq 2$, $\delta \in [0, 1]$, $D \subset \mathbb{R}^n$ a domain, $f \in W^{1,1}_{loc}(D, \mathbb{R}^n) \cap C(D, \mathbb{R}^n)$ be light such that $\langle f'(x)(h), h\rangle \geq \delta |h||f'(x)(h)|$ for a.e. $x \in D$ and every $h \in \mathbb{R}^n$, let $\epsilon = \arccos \delta$ and suppose that D is a 2ϵ-angular convex domain. Then f is injective on D.*

The lightness assumption in Theorem 1 is necessary as the constant mapping shows.

Remark 1. In Theorem 1 we can take $D = (-1, 1)^2 \setminus [-1, 0]^2$ and $\delta = \frac{1}{\sqrt{2}}$. Indeed, $\arccos \frac{1}{\sqrt{2}} = \frac{\pi}{4}$ and D is a $\frac{\pi}{2}$-angular convex domain.

We also prove an extension of the theorem of Alexander and Warschanski for Sobolev mappings.

THEOREM 2. *Let $n \geq 2$, $D \subset \mathbb{R}^n$ be a convex domain, $f \in W^{1,1}_{loc}(D, \mathbb{R}^n) \cap C(D, \mathbb{R}^n)$ be light such that $f'(x)$ is positive definite for a.e $x \in D$. Then f is* *injective on D.*

Also the lightness condition in Theorem 2 is necessarily as the constant mapping shows. We immediately obtain:

THEOREM 3. *Let $n \geq 2$, $D \subset \mathbb{R}^n$ be a domain, $\delta \in [0,1]$, $f \in W^{1,1}_{loc}(D, \mathbb{R}^n) \cap C(D, \mathbb{R}^n)$ be light, $g \in C^2(D, \mathbb{R}^n)$ be injective such that $J_f(x) \neq 0$ a.e. on D, $\langle f'(x)(h), g'(x)(h)\rangle \geq \delta |f'(x)(h)||g'(x)(h)|$ for a.e. $x \in D$ and every $h \in \mathbb{R}^n$, let $\epsilon = \arccos\delta$ and suppose that $g(D)$ is a 2ϵ-angular convex domain. Then f is injective on D.*

THEOREM 4. *Let $n \geq 2$, $D \subset \mathbb{R}^n$ be a domain, $f \in W^{1,1}_{loc}(D, \mathbb{R}^n) \cap C(D, \mathbb{R}^n)$ be light, $g \in C^2(D, \mathbb{R}^n)$ be injective such that $J_f(x) \neq 0$ a.e. on D, $\langle f'(x)(h), g'(x)(h)\rangle \geq 0$ for a.e. $x \in D$ and every $h \in \mathbb{R}^n$ and suppose that $g(D)$ is convex. Then f is injective on D.*

2. PROOFS OF THE RESULTS

Proof of Theorem 1. Let $a, x \in D$ be such that $[a, x] \subset D$, let $h : [0,1] \to \mathbb{R}^n$, $h(t) = f(a + t(x - a))$ for $t \in [0,1]$ and suppose that h is absolutely continuous on $[0,1]$ and f has first partial derivatives at $a + t(x-a)$ for μ_1 a.e. $t \in [0,1]$. We show that

$$\langle f(x) - f(a), x - a\rangle \geq \int_0^1 \langle f'(a + t(x-a))(x-a), x-a\rangle \mathrm{d}t \tag{2}$$

Indeed,

$$\begin{aligned}
\langle f(x) - f(a), x - a\rangle &= \sum_{j=1}^n (f_j(x) - f_j(a))(x_j - a_j) \\
&= \sum_{j=1}^n (h_j(1) - h_j(0))(x_j - a_j) = \sum_{j=1}^n \int_0^1 h_j'(t)\mathrm{d}t(x_j - a_j) \\
&= \sum_{j=1}^n \Big(\int_0^1 \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}(a + t(x-a))(x_i - a_i)\mathrm{d}t\Big)(x_j - a_j) \\
&= \int_0^1 \sum_{i=1}^n \Big(\sum_{j=1}^n \frac{\partial f_j}{\partial x_i}(a + t(x-a))(x_j - a_j)\Big)(x_i - a_i)\mathrm{d}t \\
&= \int_0^1 \langle f'(a + t(x-a))(x-a), x-a\rangle \mathrm{d}t.
\end{aligned}$$

Using (2), we find that

$$\langle f(x)-f(a), x-a\rangle \geq \int_0^1 \langle f'(a+t(x-a))(x-a), x-a\rangle \mathrm{d}t$$
$$\geq \delta|x-a| \int_0^1 |f'(a+t(x-a)|dt \geq \delta|x-a||f(x)-f(a)|.$$

Since $f \in W^{1,1}_{loc}(D, \mathbb{R}^n \cap C(D, \mathbb{R}^n)$, we have

$$\langle f(x)-f(a), x-a\rangle \geq \delta|x-a||f(x)-f(a)| \text{ if } [a,x] \subset D \tag{3}$$

Let $f_\lambda : D \to \mathbb{R}^n$, $f_\lambda = f + \lambda \mathrm{Id}_D$ for $\lambda > 0$. Let $b \in D$ and $r > 0$ be such that $\overline{B}(b,r) \subset D$ and let $[a,x] \subset D$. Using (2), we find that

$$\langle f_\lambda(x) - f_\lambda(a), x-a\rangle \geq \lambda|x-a| > 0$$

for every $a, x \in B(b,r)$ such that $a \neq x$ and every $\lambda > 0$. It follows that each mapping f_λ is injective on $B(b,r)$ and since $f_\lambda \to f$ uniformly on the compact subsets of $B(b,r)$ and f is a light mapping, using Hurwitz's Theorem A, we deduce that f is injective on $B(b,r)$. Therefore, we proved that f is local homeomorphism on D.

We have to prove that f is injective on D. Suppose that this is not the case. Then we can find $z_1, z_2 \in D$, $z_1 \neq z_2$ such that $f(z_1) = f(z_2)$. Since D is a 2ϵ-angular convex domain, we can find $z_3 \in D$ such that $z_1 \neq z_3$, $z_2 \neq z_3$, $[z_1, z_3] \cup [z_2, z_3] \subset D$ and $a(z_1 - z_3, z_2 - z_3) \geq 2\epsilon$ and using the fact that f is a local homeomorphism on D, we can suppose that $f(z_1) \neq f(z_3)$, $f(z_2) \neq f(z_3)$. Using (2), we find that if $d = f(z_1) - f(z_3) = f(z_2) - f(z_3)$, then $d \in C_{0, z_1 - z_3, \epsilon}$, $d \in C_{0, z_2 - z_3, \epsilon}$ and on the other side we have that $a(z_1 - z_3, z_2 - z_3) \geq 2\epsilon$.

We reached a contradiction which proves that f is injective on D. □

Proof of Theorem 2. Let $a, x \in D$ be such that $[a,x] \subset D$, let $h : [0,1] \to \mathbb{R}^n$ be defined by $h(t) = f(a + t(x-a))$ for $t \in [0,1]$ and suppose that h is absolutely continuous and that there exists $f'(a + t(x-a))$ and is positive definite for μ_1 a.e. $t \in [0,1]$. Using (1) we have

$$\langle f(x)-f(a), x-a\rangle \geq \int_0^1 \langle f'(a+t(x-a))(x-a), x-a\rangle \mathrm{d}t \geq 0 \tag{4}$$

Since $f \in W^{1,1}_{loc}(D, \mathbb{R}^n) \cap C(D, \mathbb{R}^n)$, we see that relation (4) is valid for every $a, x \in D$ such that $[a,x] \subset D$. Let $f_\lambda : D \to \mathbb{R}^n$ be given by $f_\lambda = f + \lambda Id_D$ for $\lambda > 0$. Then $\langle f_\lambda(x) - f_\lambda(a), x-a\rangle \geq \lambda|x-a| > 0$ if $a \neq x$, $[a,x] \subset D$ and every $\lambda > 0$. It follows that each mapping f_λ is injective on D and since

$f_\lambda \to f$ uniformly on the compact subsets of D and f is a light mapping, we use again Hurwitz's Theorem A to obtain that f is injective on D. $\square$

Proof of Theorem 3. Let $a \in D$ be such that f has first partial derivatives at a, $J_g(a) \neq 0$ and $\langle f'(a)(u), g'(a)(u)\rangle \geq \delta|f'(a)(u)||g'(a)(u)|$ for every $u \in \mathbb{R}^n$. Taking $u = g'(a)^{-1}(v)$, we have $\langle f'(a) \circ g'(a)^{-1}(v), v\rangle \geq \delta|f'(a) \circ g'(a)^{-1}(v)||v|$ for every $v \in \mathbb{R}^n$. Let $b = g(a)$. Using Theorem B, we see that $(f \circ g^{-1})'(b) = f'(a) \circ g'(a)^{-1}$ and hence $\langle (f \circ g^{-1})'(b)(v), v\rangle \geq \delta|(f \circ g^{-1})'(b)(v)||v|$ for every $v \in \mathbb{R}^n$ and a.e. $b \in g(D)$. We apply Theorem 1 and we see that $f \circ g^{-1}$ is injective on $g(D)$ and hence that f is injective on D. $\square$

Proof of Theorem 4. As in Theorem 3 we see that $\langle (f \circ g^{-1})'(b)(v), v\rangle \geq 0$ for a.e. $b \in g(D)$ and every $v \in \mathbb{R}^n$. We apply Theorem 2 and we see that $f \circ g^{-1}$ is injective on $g(D)$ and hence that f is injective on D. $\square$

Remark 2. If in the preceding theorems the mapping f is such that there exists $q > n - 1$ such that $f \in W^{1,q}_{loc}(D, \mathbb{R}^n)$ and a set $E \subset D$ with $\dim E = 0$ such that f has first partial derivatives on $D \setminus E$ and $J_f(x) \neq 0$ for every $x \in D \setminus E$, then f is a light mapping. Indeed, this follows immediately from Lemma 5.10 page 116 in [3] and Theorem 5.21 page 129 in [3].

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