Dedicated to the memory of Cabiria Andreian-Cazacu

SPECTRAL STABILITY ESTIMATES OF NEUMANN DIVERGENCE FORM ELLIPTIC OPERATORS

VLADIMIR GOL'DSHTEIN, VALERII PCHELINTSEV, and ALEXANDER UKHLOV

Communicated by Lucian Beznea

We study spectral stability estimates of elliptic operators in divergence form $-\operatorname{div}[A(w)\nabla g(w)]$ with the Neumann boundary condition in domains $\Omega \subset \mathbb{C}$ which satisfy the quasihyperbolic boundary condition. This class of domains includes Lipschitz domains, Hölder singular domains and some fractal type domains like snowflakes. The suggested method is based on connections of quasiconformal mappings and Sobolev spaces with applications to the Poincaré type inequalities.

AMS 2010 Subject Classification: 35P15, 46E35, 30C60.

Key words: elliptic equations, Sobolev spaces, quasiconformal mappings.

1. INTRODUCTION

In this paper we give applications of the quasiconformal mapping theory to spectral stability estimates of the Neumann eigenvalues of A-divergent form elliptic operators:

$$L_A = -\operatorname{div}[A(w)\nabla g(w)], \quad w = (u, v) \in \Omega, \quad \langle A(w)\nabla g, n\rangle\,|_{\partial\Omega} = 0, \tag{1.1}$$

with $A \in M^{2\times 2}(\Omega)$. We denote, by $M^{2\times 2}(\Omega)$, the class of all 2×2 symmetric matrix functions $A(w) = \{a_{kl}(w)\}$, $\det A = 1$, with measurable entries satisfying the uniform ellipticity condition

$$\frac{1}{K}|\xi|^2 \leq \langle A(w)\xi, \xi\rangle \leq K|\xi|^2 \ \text{ a.e. in } \Omega, \tag{1.2}$$

for every $\xi \in \mathbb{C}$, where $1 \leq K < \infty$. Such type of elliptic operators in divergence form arises in various problems of mathematical physics (see, for example, [2]).

The suggested approach is based on the quasiconformal change of variables that allows to reduce the Neumann eigenvalue problem (1.1) in the weak formulation to the weak weighted eigenvalue problem in unit disc $\mathbb{D} \subset \mathbb{C}$:

$$\iint\limits_{\mathbb{D}} \left\langle \nabla f(z), \nabla \overline{g(z)} \right\rangle \mathrm{d}x\mathrm{d}y = \mu \iint\limits_{\mathbb{D}} h(z) f(z)\overline{g(z)}\, \mathrm{d}x\mathrm{d}y, \ \ \forall g \in W^{1,2}(\mathbb{D}), \tag{1.3}$$

where a weight h is a Jacobian of a quasiconformal mapping associated with the matrix A.

On this base we obtain the spectral stability estimates in so-called A-quasiconformal β-regular domains $\Omega \subset \mathbb{C}$, or in another terminology, domains that satisfy to the well known quasihyperbolic boundary condition [24].

In [18] (see also [24, 27]) it was proved that in A-quasiconformal β-regular domains $\Omega \subset \mathbb{C}$ the embedding operator

$$i_\Omega : W^{1,2}(\Omega, A) \hookrightarrow L^2(\Omega)$$

is compact.

Hence we can conclude that in the A-quasiconformal β-regular domains $\Omega \subset \mathbb{C}$ the Neumann spectrum of the elliptic operators $-\operatorname{div}[A(w)\nabla g(w)]$ is discrete and can be written as a non-decreasing sequence

$$0 = \mu_1[A,\Omega] < \mu_2[A,\Omega] \leq \ldots \leq \mu_n[A,\Omega] \leq \ldots,$$

where each eigenvalue is repeated as many time as its multiplicity (see, for example, [27]).

Recall that a simply connected domain $\Omega \subset \mathbb{C}$ is called an A-quasiconformal β-regular domain [18, 19] if

$$\iint\limits_{\Omega} |J(w,\varphi)|^{1-\beta}\, dudv < \infty, \;\; \beta > 1,$$

where $J(w,\varphi)$ is a Jacobian of an A-quasiconformal mapping $\varphi : \Omega \to \mathbb{D}$.

It is said that two A-quasiconformal β-regular domains Ω_1, Ω_2 represent an *A-conformal β-regular pair* if there exists a conformal mapping $\psi : \Omega_1 \to \Omega_2$ such that

$$\iint\limits_{\Omega_1} |J(z,\psi)|^{\beta} dxdy < \infty \;\&\; \iint\limits_{\Omega_2} |J(w,\psi^{-1})|^{\beta} dudv < \infty.$$

Note, that two A-quasiconformal β-regular domains $\Omega_1 = \varphi_1^{-1}(\mathbb{D})$ and $\Omega_2 = \varphi_2^{-1}(\mathbb{D})$ represent an A-conformal β-regular pair if and only if

$$\Phi_\beta(\varphi_1,\varphi_2) = \left(\iint\limits_{\mathbb{D}} \max\left\{\frac{|J(z,\varphi_1^{-1})|^{\beta}}{|J(z,\varphi_2^{-1})|^{\beta-1}}, \frac{|J(z,\varphi_2^{-1})|^{\beta}}{|J(z,\varphi_1^{-1})|^{\beta-1}}\right\} dxdy\right)^{\frac{1}{2\beta}} < \infty.$$

Remark, that two Ahlfors type domains represent an A-conformal regular pair.

The spectral stability estimates of the self-adjoint elliptic operators were intensively studied during the last decade. See, for example, [4, 6, 7, 8, 9, 25, 26] where the history of the problem and main results in this area can be found.

In the previous works [10, 11] (using an approach which is based on the conformal theory of composition operators on Sobolev spaces) were established the conformal spectral stability estimates of Dirichlet eigenvalues and Neumann eigenvalues of Laplacian in a large class of domains that includes some fractal type domains. In [19] for this class of domains were obtained spectral stability estimates of Dirichlet eigenvalues for elliptic operators in divergence form.

In the present paper we prove that for any A-conformal β-regular pair $\Omega_1 = \varphi_1^{-1}(\mathbb{D})$, $\Omega_2 = \varphi_2^{-1}(\mathbb{D})$ the following estimate is correct for any $n \in \mathbb{N}$:

$$|\mu_n[A,\Omega_1] - \mu_n[A,\Omega_2]| \leq c_n B^2_{\frac{4\beta}{\beta-1},2}(\mathbb{D},h)\Phi_\beta(\varphi_1,\varphi_2) \cdot \|J^{\frac{1}{2}}_{\varphi_1^{-1}} - J^{\frac{1}{2}}_{\varphi_2^{-1}} \mid L^2(\mathbb{D})\|.$$

Here $c_n = \max\left\{\mu_n^2[A,\Omega_1], \mu_n^2[A,\Omega_2]\right\}$ and $J_{\varphi_k^{-1}}$ are the Jacobians of the A^{-1}-quasiconformal mappings $\varphi_k^{-1} : \mathbb{D} \to \Omega_k$, $k = 1, 2$.

Remark 1.1. The constant $B_{\frac{4\beta}{\beta-1},2}(\mathbb{D},h)$ is the best constant for corresponding weighted Poincaré-Sobolev inequalities in the unit disc $\mathbb{D}$.

Our method is based on corresponding Sobolev embedding theorems [15, 20] in connection with the composition operators theory of Sobolev spaces [28, 33].

2. SOBOLEV SPACES AND A-QUASICONFORMAL MAPPINGS

Let $\Omega \subset \mathbb{C}$ be a domain and $h : \Omega \to \mathbb{R}$ be a positive a.e. and locally integrable function i.e. a weight. We consider a two-weighted Sobolev space $W^{1,p}(\Omega, h, 1)$, $1 \leq p < \infty$, defined as a normed space of all locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ endowed with the following norm:

$$\|f \mid W^{1,p}(\Omega, h, 1)\| = \|f \mid L^p(\Omega, h)\| + \|\nabla f \mid L^p(\Omega)\|.$$

The weighted seminormed Sobolev space $L^{1,2}(\Omega, A)$ (associated with the matrix A) defined as the space of all locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ with the finite seminorm given by:

$$\|f \mid L^{1,2}(\Omega, A)\| = \left(\iint\limits_{\Omega} \langle A(z)\nabla f(z), \nabla f(z)\rangle \, \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}.$$

In the case $A = I$, where I is an identical matrix, we have the standard seminormed Sobolev space $L^{1,2}(\Omega) = L^{1,2}(\Omega, I)$.

The corresponding normed Sobolev space $W^{1,2}(\Omega, A)$ is defined as the normed space of all locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ endowed with the following norm:

$$\|f \mid W^{1,2}(\Omega, A)\| = \|f \mid L^2(\Omega)\| + \|f \mid L^{1,2}(\Omega, A)\|.$$

Recall that a homeomorphism $\varphi : \Omega \to \Omega'$, $\Omega, \Omega' \subset \mathbb{C}$, is called a K-quasiconformal mapping if $\varphi \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and there exists a constant $1 \leq K < \infty$ such that

$$|D\varphi(w)|^2 \leq K|J(w,\varphi)| \text{ for almost all } w \in \Omega.$$

Now we give a construction of A-quasiconformal mappings connected with the A-divergent form elliptic operators. Let $A \in M^{2\times 2}(\Omega)$. Consider the Beltrami equation:

$$\varphi_{\overline{w}}(w) = \mu(w)\varphi_w(w), \text{ a.e. in } \Omega, \tag{2.1}$$

with the complex dilatation $\mu(w)$ is given by

$$\mu(w) = \frac{a_{22}(w) - a_{11}(w) - 2ia_{12}(w)}{\det(I + A(w))}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.2}$$

Then the uniform ellipticity condition (1.2) can be written as

$$|\mu(w)| \leq \frac{K-1}{K+1}, \text{ a.e. in } \Omega. \tag{2.3}$$

Conversely we can obtain from (2.2) (see, for example, [2], p. 412) the corresponding matrix

$$A(w) = \begin{pmatrix} \frac{|1-\mu|^2}{1-|\mu|^2} & \frac{-2\operatorname{Im}\mu}{1-|\mu|^2} \\ \frac{-2\operatorname{Im}\mu}{1-|\mu|^2} & \frac{|1+\mu|^2}{1-|\mu|^2} \end{pmatrix}, \text{ a.e. in } \Omega. \tag{2.4}$$

So, for given any $A \in M^{2\times 2}(\Omega)$ can be produced, by (2.3), the complex dilatation $\mu(w)$, for which, in turn, the Beltrami equation (2.1) induces a quasiconformal homeomorphism $\varphi : \Omega \to \varphi(\Omega)$ as its solution, by the Riemann measurable mapping theorem (see, for example, [1]). We will say that the matrix function A induces the corresponding A-quasiconformal mappings φ or that A and φ are agreed [23]. Any A-quasiconformal mapping $\psi : \Omega \to \mathbb{D}$ of simply connected domain $\Omega \subset \mathbb{C}$ onto the unit disc $\mathbb{D} \subset \mathbb{C}$ can be obtained as a composition of A-quasiconformal homeomorphism $\varphi : \Omega \to \varphi(\Omega)$ and a conformal mapping $\omega : \varphi(\Omega) \to \mathbb{D}$.

So, by the given A-divergent form elliptic operator defined in a domain $\Omega \subset \mathbb{C}$ we construct so-called a A-quasiconformal mapping $\psi : \Omega \to \mathbb{D}$ with a quasiconformal coefficient

$$K = \frac{1 + \|\mu \mid L^{\infty}(\Omega)\|}{1 - \|\mu \mid L^{\infty}(\Omega)\|},$$

where μ is defined by (2.2).

Note that the inverse mapping to the A-quasiconformal mapping $\psi : \Omega \to \mathbb{D}$ is the A^{-1}-quasiconformal mapping [18].

In [18] was given a connection between composition operators on Sobolev spaces and A-quasiconformal mappings.

Theorem 2.1. *Let Ω, Ω' be domains in $\mathbb{C}$. Then a homeomorphism $\varphi : \Omega \to \Omega'$ is an A-quasiconformal mapping if and only if φ induces, by the composition rule $\varphi^*(f) = f \circ \varphi$, an isometry of Sobolev spaces $L^{1,2}(\Omega, A)$ and $L^{1,2}(\Omega')$ i.e.*

$$\|\varphi^*(f) \mid L^{1,2}(\Omega, A)\| = \|f \mid L^{1,2}(\Omega')\|$$

for any $f \in L^{1,2}(\Omega')$.

Note, it is well known that conformal mappings $\varphi : \Omega \to \Omega'$ (I-quasiconformal mappings), generates an isometry of seminormed Sobolev spaces $L^{1,2}(\Omega)$ and $L^{1,2}(\Omega')$ (see, for example, [13]). Theorem 2.1 generalizes this property of conformal mappings to A-quasiconformal mappings in the case of Sobolev spaces associated with the matrix A.

Remark 2.2. This theorem also refines (in the case $n = 2$) the functional characterization of quasiconformal mappings given in terms of isomorphisms of seminormed Sobolev spaces [30].

3. THE WEIGHTED EIGENVALUE PROBLEM

In this section, using Theorem 2.1, we prove the corresponding weighted Poincaré-Sobolev inequality in the unit disc $\mathbb{D}$ for so-called quasihyperbolic weights which are Jacobians of mappings inverse to A-quasiconformal homeomorphisms.

Denote by $h = h(z) := |J(z, \varphi^{-1})|$ the quasihyperbolic (quasiconformal) weight generated by the A^{-1}-quasiconformal mapping $\varphi : \mathbb{D} \to \Omega$ and by

$$f_{\mathbb{D},h} := \frac{1}{m_h(\mathbb{D})} \iint\limits_{\mathbb{D}} f(z)h(z)\mathrm{d}x\mathrm{d}y = g_\Omega := \frac{1}{|\Omega|} \iint\limits_{\Omega} g(w)\mathrm{d}u\mathrm{d}v,$$

$$f(z) = g(\varphi(z)), \;\; w = \varphi(z).$$

Here

$$m_h(\mathbb{D}) = \iint\limits_{\mathbb{D}} h(z)\mathrm{d}x\mathrm{d}y = \iint\limits_{\mathbb{D}} |J(z, \varphi^{-1})|\mathrm{d}x\mathrm{d}y = |\Omega|.$$

Recall that for A-quasiconformal regular domains the Poincaré-Sobolev inequality

$$\|g - g_\Omega \mid L^2(\Omega)\| \leq V^* \|g \mid L^{1,2}(\Omega, A)\|$$

holds for any function $g \in W^{1,2}(\Omega, A)$ with exact constant V^* (see [18]).

Given this inequality we prove:

Theorem 3.1. *Let A be a matrix satisfies the uniform ellipticity condition* (1.2) *and $\Omega \subset \mathbb{C}$ be an A-quasiconformal β-regular domain. Then the weighted embedding operator*

$$i_{\mathbb{D}} : W^{1,2}(\mathbb{D}) \hookrightarrow L^2(\mathbb{D}, h) \tag{3.1}$$

is compact and for any function $f \in W^{1,2}(\mathbb{D},h,1)$ *the inequality*

$$\|f - f_{\mathbb{D},h} \mid L^2(\mathbb{D},h)\| \leq V^* \|f \mid L^{1,2}(\mathbb{D})\|$$

holds.

Proof. Define the complex dilatation $\mu(w)$ agreed with the matrix A by

$$\mu(w) = \frac{a_{22}(w) - a_{11}(w) - 2ia_{12}(w)}{\det(I + A(w))}.$$

Because the matrix A satisfies the uniform ellipticity condition (1.2) then

$$|\mu(w)| \leq \frac{K-1}{K+1} < 1, \text{ a.e. in } \Omega,$$

and by [1] there exists a μ-quasiconformal mapping $\varphi : \Omega \to \mathbb{D}$ agreed with the matrix A i.e. an A-quasiconformal mapping. Hence by Theorem 2.1 the composition operator

$$\varphi^* : L^{1,2}(\mathbb{D}) \to L^{1,2}(\Omega, A), \;\; \varphi^*(f) = f \circ \varphi$$

is an isometry.

Let $f \in L^{1,2}(\mathbb{D})$ be a smooth function. Then the composition $g = f \circ \varphi$ belongs to $L^{1,2}(\Omega, A)$ and because the matrix A satisfies the uniform ellipticity condition (1.2) by the Sobolev embedding theorem we can conclude that $g = f \circ \varphi \in W^{1,2}(\Omega, A)$ [27] and the Poincaré-Sobolev inequality

$$\|g - g_\Omega \mid L^2(\Omega)\| \leq V^* \|g \mid L^{1,2}(\Omega, A)\| \tag{3.2}$$

holds with the exact constant $V^* = \mu_1[A,\Omega]^{-\frac{1}{2}}$.

Now using the "transfer" diagram [15, 20] and the change of variable formula for quasiconformal mappings [31] we obtain

$$\begin{aligned}
&\|f - f_{\mathbb{D},h} \mid L^2(\mathbb{D},h)\| \\
&= \left(\iint\limits_{\mathbb{D}} |f(z) - f_{\mathbb{D},h}|^2 h(z) dxdy \right)^{\frac{1}{2}} = \left(\iint\limits_{\mathbb{D}} |f(z) - f_{\mathbb{D},h}|^2 |J(z,\varphi^{-1})| dxdy \right)^{\frac{1}{2}} \\
&= \left(\iint\limits_{\Omega} |g(w) - g_\Omega|^2 dudv \right)^{\frac{1}{2}} \leq V^* \left(\iint\limits_{\Omega} \langle A(w)\nabla g(w), \nabla g(w) \rangle \, dudv \right)^{\frac{1}{2}} \\
&= V^* \left(\iint\limits_{\mathbb{D}} |\nabla f(z)|^2 dxdy \right)^{\frac{1}{2}} = V^* \|f \mid L^{1,2}(\mathbb{D})\|.
\end{aligned}$$

Approximating an arbitrary function $f \in W^{1,2}(\mathbb{D}, h, 1)$ by smooth functions we have that the weighted Poincaré-Sobolev inequality

$$\|f - f_{\mathbb{D},h} \mid L^2(\mathbb{D}, h)\| \leq V^* \|f \mid L^{1,2}(\mathbb{D})\|$$

holds for any function $f \in W^{1,2}(\mathbb{D}, h, 1)$.

Further we prove that the embedding operator

$$i_{\mathbb{D}} : W^{1,2}(\mathbb{D}) \hookrightarrow L^2(\mathbb{D}, h) \tag{3.3}$$

is compact. By the same "transfer" diagram [15, 20] this operator can be represented as a composition of three operators: the composition operator $\varphi_w^* : W^{1,2}(\mathbb{D}) \to W^{1,2}(\Omega, A)$, the compact embedding operator

$$i_{\Omega} : W^{1,2}(\Omega, A) \hookrightarrow L^2(\Omega)$$

and the composition operator for Lebesgue spaces $(\varphi^{-1})_l^* : L^2(\Omega) \to L^2(\mathbb{D}, h)$.

Firstly we prove that the operator $(\varphi^{-1})_l^*$ is an isometry. By the change of variables formula we obtain:

$$\|f \mid L^2(\mathbb{D}, h)\| = \left(\iint_{\mathbb{D}} |f(z)|^2 h(z)\ dxdy \right)^{\frac{1}{2}}$$

$$= \left(\iint_{\mathbb{D}} |f(z)|^2 |J(z, \varphi^{-1})|\ dxdy \right)^{\frac{1}{2}} = \left(\iint_{\Omega} |f \circ \varphi(w)|^2\ dudv \right)^{\frac{1}{2}}$$

$$= \|g \mid L^2(\Omega)\|.$$

Secondly we prove that the composition operator

$$(\varphi^{-1})_w^* : W^{1,2}(\mathbb{D}) \to W^{1,2}(\Omega, A)$$

is bounded.

By Theorem 2.1 A-quasiconformal mappings $\varphi : \Omega \to \mathbb{D}$ generate a bounded composition operator on seminormed Sobolev spaces

$$\varphi^* : L^{1,2}(\mathbb{D}) \to L^{1,2}(\Omega, A).$$

Since the matrix A satisfies to the uniform ellipticity condition (1.2) then the norm of Sobolev space $W^{1,2}(\Omega, A)$ is equivalent to the norm of Sobolev space $W^{1,2}(\Omega)$ and by [17] we obtain that the composition operator on normed Sobolev spaces

$$\varphi_w^* : W^{1,2}(\mathbb{D}) \to W^{1,2}(\Omega, A)$$

is bounded.

Hence the embedding operator $i_{\mathbb{D}}$ is compact as a composition of the compact operator i_{Ω} and bounded operators φ_w^* and $(\varphi^{-1})_l^*$. $\square$

According to Theorem 3.1 the weighted embedding operator is compact. By standard arguments we conclude that the spectrum of the weighted eigenvalue problem (1.3) with quasihyperbolic (quasiconformal) weights h is discrete and can be written in the form of a non-decreasing sequence

$$0=\mu_1[h,\mathbb{D}]<\mu_2[h,\mathbb{D}]\leq\ldots\leq\mu_n[h,\mathbb{D}]\leq\ldots,$$

where each eigenvalue is repeated as many time as its multiplicity (see, for example, [3, 14]). The weighted eigenvalue problem in the unit disc $\mathbb{D}$ is equivalent to the eigenvalue problem in the domain Ω and

$$\mu_n[h,\mathbb{D}]=\mu_n[A,\Omega],\quad n\in\mathbb{N}. \tag{3.4}$$

For weighted eigenvalues we have the following properties [11]:

(*i*) $\lim\limits_{n\to\infty}\mu_n[h,\mathbb{D}]=\infty$.

(*ii*) for each $n\in\mathbb{N}$

$$\mu_n[A,\Omega]=\inf_{\substack{L\subset W^{1,2}(\Omega,A)\\ \dim L=n}}\sup_{\substack{g\in L\\ g\neq 0}}\frac{\iint\limits_{\Omega}\langle A(w)\nabla g,\nabla g\rangle\,\mathrm{d}u\mathrm{d}v}{\iint\limits_{\Omega}|g|^2\mathrm{d}u\mathrm{d}v} \tag{3.5}$$

$$=\inf_{\substack{L\subset W^{1,2}(\mathbb{D},h,1)\\ \dim L=n}}\sup_{\substack{f\in L\\ f\neq 0}}\frac{\iint\limits_{\mathbb{D}}|\nabla f|^2\mathrm{d}x\mathrm{d}y}{\iint\limits_{\mathbb{D}}|f|^2h(z)\mathrm{d}x\mathrm{d}y}=\mu_n[h,\mathbb{D}]$$

(Min-Max Principle), and

$$\mu_n[h,\mathbb{D}]=\sup_{\substack{f\in M_n\\ f\neq 0}}\frac{\iint\limits_{\mathbb{D}}|\nabla f|^2\mathrm{d}x\mathrm{d}y}{\iint\limits_{\mathbb{D}}|f|^2h(z)\mathrm{d}x\mathrm{d}y} \tag{3.6}$$

where

$$M_n=\operatorname{span}\{\varphi_1[h],\ldots,\varphi_n[h]\}$$

and $\{\varphi_n[h]\}_{n=1}^{\infty}$ is an orthonormal (in the space $W^{1,2}(\mathbb{D},h,1)$) set of eigenfunctions corresponding to the eigenvalues $\{\mu_n[h,\mathbb{D}]\}_{n=1}^{\infty}$.

(*iii*) $\mu_1[h,\mathbb{D}]=0$ and $\varphi_1=\frac{1}{\sqrt{m_h(\mathbb{D})}}$. For $n\geq 2$ alongside with (3.6) we have

$$\mu_n[h,\mathbb{D}]=\sup_{\substack{f\in M_n\\ f\neq 0}}\frac{\iint\limits_{\mathbb{D}}|\nabla f|^2\mathrm{d}x\mathrm{d}y}{\iint\limits_{\mathbb{D}}|f-f_{\mathbb{D},h}|^2h(z)\mathrm{d}x\mathrm{d}y}. \tag{3.7}$$

(It may happen that the above fraction takes the form $\frac{0}{0}$. In this case we assume that $\frac{0}{0}=0$.)

4. THE $L^{1,2}$-SEMINORM ESTIMATES

In this section we estimate variation of Neumann eigenvalues of two weighted eigenvalue problems in the unit disc $\mathbb{D}$:

$$\iint\limits_{\mathbb{D}} \left\langle \nabla f(z), \nabla \overline{g(z)} \right\rangle \, dxdy = \mu \iint\limits_{\mathbb{D}} h_1(z) f(z) \overline{g(z)} \, dxdy \,, \quad \forall g \in W^{1,2}(\mathbb{D}, h, 1)$$

and

$$\iint\limits_{\mathbb{D}} \left\langle \nabla f(z), \nabla \overline{g(z)} \right\rangle \, dxdy = \mu \iint\limits_{\mathbb{D}} h_2(z) f(z) \overline{g(z)} \, dxdy \,, \quad \forall g \in W^{1,2}(\mathbb{D}, h, 1).$$

The following result in the case of hyperbolic (conformal) weights was proved in ([11], Lemma 3.1). In the present paper we formulate this lemma in the case of quasihyperbolic (quasiconformal) weights.

LEMMA 4.1. *Let $\mathbb{D} \subset \mathbb{C}$ be the unit disc and let h_1, h_2 be quasiconformal weights on $\mathbb{D}$. Suppose that there exists a constant $B > 0$ such that*

$$\max \left\{ \iint\limits_{\mathbb{D}} |h_1(z) - h_2(z)| |f - f_{\mathbb{D},h_1}|^2 \, dxdy, \iint\limits_{\mathbb{D}} |h_1(z) - h_2(z)| |f - f_{\mathbb{D},h_2}|^2 \, dxdy \right\} \leq B \iint\limits_{\mathbb{D}} |\nabla f|^2 \, dxdy, \ \forall f \in L^{1,2}(\mathbb{D}). \tag{4.1}$$

Then for any $n \in \mathbb{N}$

$$|\mu_n[h_1, \mathbb{D}] - \mu_n[h_2, \mathbb{D}]| \leq \frac{B\tilde{c}_n}{1 + B\sqrt{\tilde{c}_n}} < B\tilde{c}_n \,, \tag{4.2}$$

where

$$\tilde{c}_n = \max\{\mu_n^2[h_1, \mathbb{D}], \mu_n^2[h_2, \mathbb{D}]\} \,. \tag{4.3}$$

Further we estimate the constant B in Lemma 4.1 in terms of an L^s-distance between weights. Similarly to Theorem 3.1 we have

$$\|f - f_{\mathbb{D},h_k} \mid L^r(\mathbb{D}, h_k)\| \leq B_{r,2}(\mathbb{D}, h_k) \|\nabla f \mid L^2(\mathbb{D})\| \tag{4.4}$$

for $r \geq 1$ and any function $f \in W^{1,2}(\mathbb{D}, h_k, 1)$, where $h_k = |J(z, \varphi_k^{-1})|$, $k = 1, 2$, are the quasiconformal weights defined by inverse mappings to A-quasiconformal homeomorphisms $\varphi_k : \Omega_k \to \mathbb{D}$. Here $B_{r,2}(\mathbb{D}, h_k)$, $k = 1, 2$, are best positive constants in these inequalities.

LEMMA 4.2. *Let h_1, h_2 be quasiconformal weights on $\mathbb{D}$ such that*

$$d_s(h_1, h_2) := \left\| (h_1 - h_2) \left(\min\{h_1, h_2\}\right)^{\frac{1-s}{s}} \mid L^s(\mathbb{D}) \right\| < \infty \tag{4.5}$$

for some $1 < s \leq \infty$.

Then inequality (4.1) *holds with the constant*

$$B = B^2_{\frac{2s}{s-1},2}(\mathbb{D}, h)\, d_s(h_1, h_2), \tag{4.6}$$

where

$$B_{\frac{2s}{s-1},2}(\mathbb{D}, h) = \max\left\{B_{\frac{2s}{s-1},2}(\mathbb{D}, h_1), B_{\frac{2s}{s-1},2}(\mathbb{D}, h_2)\right\}.$$

Proof. By the Hölder inequality and Poincaré-Sobolev inequality (4.4) we get for $k = 1, 2$

$$\iint\limits_{\mathbb{D}} |h_1(z) - h_2(z)||f(z) - f_{\mathbb{D},h_k}|^2 \, dxdy$$

$$= \iint\limits_{\mathbb{D}} |h_1(z) - h_2(z)| h_k^{\frac{1-s}{s}} h_k^{\frac{s-1}{s}} |f(z) - f_{\mathbb{D},h_k}|^2 \, dxdy$$

$$\leq \left\| (h_1 - h_2) h_k^{\frac{1-s}{s}} \mid L^s(\mathbb{D}) \right\| \left(\iint\limits_{\mathbb{D}} h_k(z) |f(z) - f_{\mathbb{D},h_k}|^{\frac{2s}{s-1}} \, dxdy \right)^{\frac{s-1}{s}}$$

$$\leq B^2_{\frac{2s}{s-1},2}(\mathbb{D}, h)\, d_s(h_1, h_2) \iint\limits_{\mathbb{D}} |\nabla f(z)|^2 \, dxdy.$$

□

By the two previous lemmas we have the following result for variations of the weighted eigenvalues:

THEOREM 4.3. *Let h_1, h_2 be quasiconformal weights on $\mathbb{D}$. Assume that $d_s(h_1, h_2) < \infty$ for some $s > 1$.*

Then for any $n \in \mathbb{N}$

$$|\mu_n[h_1, \mathbb{D}] - \mu_n[h_2, \mathbb{D}]| \leq \tilde{c}_n B^2_{\frac{2s}{s-1},2}(\mathbb{D}, h) d_s(h_1, h_2).$$

5. ON "DISTANCE" $d_s(h_1, h_2)$ FOR QUASICONFORMAL WEIGHTS $h_1(z)$, $h_2(z)$

Let Ω_k, $k = 1, 2$, be bounded simply connected domains in $\mathbb{C}$ and $A \in M^{2\times 2}(\Omega_k)$. Assume that there exist A-quasiconformal mappings $\varphi_1 : \Omega_1 \to \mathbb{D}$

and $\varphi_2 : \Omega_2 \to \mathbb{D}$. Recall that for quasiconformal mappings $\varphi^{-1} : \mathbb{D} \to \Omega$

$$J_{\varphi^{-1}}(z) := \lim_{r\to 0} \frac{|\varphi^{-1}(B(z,r))|}{|B(z,r)|} = |J(z,\varphi^{-1})|$$

for almost all $z \in \mathbb{D}$.

Now we estimate the quantity $d_s(h_1,h_2)$ using the L^2-norms of weights.

Lemma 5.1. *Let $\varphi_1 : \Omega_1 \to \mathbb{D}$ and $\varphi_2 : \Omega_2 \to \mathbb{D}$ be A-quasiconformal homeomorphisms and h_1, h_2 be the corresponding quasihyperbolic weights. Assume that for some $\beta > 1$*

$$\Phi_\beta(\varphi_1,\varphi_2) = \left(\iint\limits_{\mathbb{D}} \max\left\{ \frac{|J(z,\varphi_1^{-1})|^\beta}{|J(z,\varphi_2^{-1})|^{\beta-1})}, \frac{|J(z,\varphi_2^{-1})|^\beta}{|J(z,\varphi_1^{-1})|^{\beta-1}} \right\} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2\beta}} < \infty.$$

Then for $s = \frac{2\beta}{\beta+1}$

$$d_s(h_1,h_2) \leq 2\Phi_\beta(\varphi_1,\varphi_2) \cdot \left(\iint\limits_{\mathbb{D}} \left(|J(z,\varphi_1^{-1})|^{\frac{1}{2}} - |J(z,\varphi_2^{-1})|^{\frac{1}{2}} \right)^2 \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}},$$

where $J(z,\varphi_1^{-1})$, $J(z,\varphi_2^{-1})$ are Jacobians inverse mappings to A-quasiconformal homeomorphisms $\varphi_1 : \Omega_1 \to \mathbb{D}$ and $\varphi_2 : \Omega_2 \to \mathbb{D}$ respectively.

Proof. By the definitions of h_1, h_2 and $d_s(h_1,h_2)$

$$[d_s(h_1,h_2)]^s = \iint\limits_{\mathbb{D}} |h_1(z) - h_2(z)|^s (\min\{h_1(z), h_2(z)\})^{1-s} \, \mathrm{d}x\mathrm{d}y$$

$$= \iint\limits_{\mathbb{D}} ||J(z,\varphi_1^{-1})| - |J(z,\varphi_2^{-1})||^s (\min\{|J(z,\varphi_1^{-1})|, |J(z,\varphi_2^{-1})|\})^{1-s} \, \mathrm{d}x\mathrm{d}y$$

$$= \iint\limits_{\mathbb{D}} \left| |J(z,\varphi_1^{-1})|^{\frac{1}{2}} + |J(z,\varphi_2^{-1})|^{\frac{1}{2}} \right|^s (\min\{|J(z,\varphi_1^{-1})|, |J(z,\varphi_2^{-1})|\})^{1-s}$$

$$\times \left| |J(z,\varphi_1^{-1})|^{\frac{1}{2}} - |J(z,\varphi_2^{-1})|^{\frac{1}{2}} \right|^s \, \mathrm{d}x\mathrm{d}y.$$

Applying to the last integral the Hölder inequality with $q = \frac{2}{s}$ ($1 \leq q < 2$ because $1 < s \leq 2$) and $q' = \frac{2}{2-s}$ we obtain

$$[d_s(h_1,h_2)]^s = \left(\iint\limits_{\mathbb{D}} \left| |J(z,\varphi_1^{-1})|^{\frac{1}{2}} + |J(z,\varphi_2^{-1})|^{\frac{1}{2}} \right|^{\frac{2s}{2-s}} \right.$$

$$\left. \times (\min\{|J(z,\varphi_1^{-1})|, |J(z,\varphi_2^{-1})|\})^{\frac{2(1-s)}{2-s}} \mathrm{d}x\mathrm{d}y \right)^{\frac{2-s}{2}}$$

$$\times\left(\iint\limits_{\mathbb{D}}\left(|J(z,\varphi_1^{-1})|^{\frac{1}{2}}-|J(z,\varphi_2^{-1})|^{\frac{1}{2}}\right)^2 dxdy\right)^{\frac{1}{2}}$$

$$\leq 2^s\left(\iint\limits_{\mathbb{D}}\frac{\max\left\{|J(z,\varphi_1^{-1})|^{\frac{1}{2}},|J(z,\varphi_2^{-1})|^{\frac{1}{2}}\right\}^{\frac{2s}{2-s}}}{\min\left\{|J(z,\varphi_1^{-1})|^{\frac{1}{2}},|J(z,\varphi_2^{-1})|^{\frac{1}{2}}\right\}^{\frac{4(s-1)}{2-s}}}dxdy\right)^{\frac{2-s}{2}}$$

$$\times\left(\iint\limits_{\mathbb{D}}\left(|J(z,\varphi_1^{-1})|^{\frac{1}{2}}-|J(z,\varphi_2^{-1})|^{\frac{1}{2}}\right)^2 dxdy\right)^{\frac{1}{2}}.$$

Since $s=\frac{2\beta}{\beta+1}$ we have

$$d_s(h_1,h_2)\leq 2\left(\iint\limits_{\mathbb{D}}\frac{\max\left\{|J(z,\varphi_1^{-1})|^{\frac{1}{2}},|J(z,\varphi_2^{-1})|^{\frac{1}{2}}\right\}^{2\beta}}{\min\left\{|J(z,\varphi_1^{-1})|^{\frac{1}{2}},|J(z,\varphi_2^{-1})|^{\frac{1}{2}}\right\}^{2(\beta-1)}}dxdy\right)^{\frac{1}{2\beta}}$$

$$\times\left(\iint\limits_{\mathbb{D}}\left(|J(z,\varphi_1^{-1})|^{\frac{1}{2}}-|J(z,\varphi_2^{-1})|^{\frac{1}{2}}\right)^2 dxdy\right)^{\frac{1}{2}}$$

$$=2\left(\iint\limits_{\mathbb{D}}\max\left\{\frac{|J(z,\varphi_1^{-1})|^{\beta}}{|J(z,\varphi_2^{-1})|^{\beta-1}},\frac{|J(z,\varphi_2^{-1})|^{\beta}}{|J(z,\varphi_1^{-1})|^{\beta-1}}\right\}dxdy\right)^{\frac{1}{2\beta}}$$

$$\times\left(\iint\limits_{\mathbb{D}}\left(|J(z,\varphi_1^{-1})|^{\frac{1}{2}}-|J(z,\varphi_2^{-1})|^{\frac{1}{2}}\right)^2 dxdy\right)^{\frac{1}{2}}.$$

□

Let us mention some results for quasiconformal mappings (see, for example, [2]). Let Ω_1, Ω_2 and Ω_3 be bounded simply connected domains on the complex plane $\mathbb{C}$.

If mapping $\psi:\Omega_1\to\Omega_2$ is quasiconformal, then $|J(z,\psi)|=|J(w,\psi^{-1})|^{-1}$ for almost all $z\in\Omega_1$ and for almost all $w=\psi(z)\in\Omega_2$.

If mappings $\psi_1:\Omega_1\to\Omega_2$ and $\psi_2:\Omega_2\to\Omega_3$ are quasiconformal, then

$$J(z,\psi)=J(z,\psi_1)\cdot J(\psi_1(z),\psi_2),\ \text{a.e. in } \Omega_1\ (\psi=\psi_2\circ\psi_1(z)).$$

If two quasiconformal mappings $\varphi:\Omega_1\to\Omega_2$ and $\psi:\Omega_1\to\Omega_3$, defined on Ω_1, have the same Beltrami coefficient, then the mapping $\psi\circ\varphi^{-1}:\Omega_2\to\Omega_3$ is conformal.

Given these results we prove the following assertion.

Proposition 5.2. *Two A-quasiconformal β-regular domains $\Omega_1 = \varphi_1^{-1}(\mathbb{D})$ and $\Omega_2 = \varphi_2^{-1}(\mathbb{D})$ represent an A-conformal β-regular pair if and only if*

$$\Phi_\beta(\varphi_1, \varphi_2) = \left(\iint\limits_{\mathbb{D}} \max \left\{ \frac{|J(z,\varphi_1^{-1})|^\beta}{|J(z,\varphi_2^{-1})|^{\beta-1}}, \frac{|J(z,\varphi_2^{-1})|^\beta}{|J(z,\varphi_1^{-1})|^{\beta-1}} \right\} dxdy \right)^{\frac{1}{2\beta}} < \infty.$$

Proof. The condition $\Phi_\beta(\varphi_1, \varphi_2) < \infty$ means that domains $\Omega_1 = \varphi_1^{-1}(\mathbb{D})$, $\Omega_2 = \varphi_2^{-1}(\mathbb{D})$ represent a A-conformal β-regular pair. Indeed, conformal mapping $\psi : \Omega_1 \to \Omega_2$ can be written as a composition

$$\psi(w) = \varphi_2^{-1}\left(\varphi_1(w)\right).$$

Then

$$J(w, \psi) = J(w, \varphi_1(w)) \cdot J(\varphi_1(w), \varphi_2^{-1}), \text{ a.e. in } \Omega_1.$$

Using the change of variable formula we obtain

$$\begin{aligned}
\iint\limits_{\Omega_1} |J(w,\psi)|^\beta \, dudv &= \iint\limits_{\mathbb{D}} |J(\varphi_1^{-1}(z),\psi)|^\beta |J(z,\varphi_1^{-1})| \, dxdy \\
&= \iint\limits_{\mathbb{D}} |J(\varphi_1^{-1}(z),\varphi_1)|^\beta |J(z,\varphi_2^{-1})|^\beta |J(z,\varphi_1^{-1})| \, dxdy \\
&= \iint\limits_{\mathbb{D}} |J(z,\varphi_1^{-1})|^{-\beta} |J(z,\varphi_2^{-1})|^\beta |J(z,\varphi_1^{-1})| \, dxdy \\
&= \iint\limits_{\mathbb{D}} |J(z,\varphi_2^{-1})|^\beta |J(z,\varphi_1^{-1})|^{1-\beta} \, dxdy.
\end{aligned}$$

The similar calculation is correct for the inverse mapping $\psi^{-1} : \Omega_2 \to \Omega_1$. □

Now we are ready to prove the main result of this work.

Theorem 5.3. *Let A be a matrix satisfies the uniform ellipticity condition (1.2) and $\Omega_1 = \varphi_1^{-1}(\mathbb{D})$, $\Omega_2 = \varphi_2^{-1}(\mathbb{D})$ be an A-conformal β-regular pair.*

Then for any $n \in \mathbb{N}$

$$|\mu_n[A,\Omega_1] - \mu_n[A,\Omega_2]| \leq 2c_n B^2_{\frac{4\beta}{\beta-1},2}(\mathbb{D},h) \Phi_\beta(\varphi_1,\varphi_2) \cdot \| J^{\frac{1}{2}}_{\varphi_1^{-1}} - J^{\frac{1}{2}}_{\varphi_2^{-1}} \mid L^2(\mathbb{D})\|,$$

where $c_n = \max\left\{\mu_n^2[A,\Omega_1], \mu_n^2[A,\Omega_2]\right\}$ and $J_{\varphi_1^{-1}}$, $J_{\varphi_2^{-1}}$ are Jacobians inverse mappings to A-quasiconformal homeomorphisms $\varphi_1 : \Omega_1 \to \mathbb{D}$ and $\varphi_2 : \Omega_2 \to \mathbb{D}$ respectively.

Proof. According to Theorem 4.3 we have

$$|\mu_n[h_1,\mathbb{D}] - \mu_n[h_2,\mathbb{D}]| \leq \tilde{c}_n B^2_{\frac{2s}{s-1},2}(\mathbb{D},h) d_s(h_1,h_2)\,.$$

By Lemma 5.1 we obtain the estimate of quality $d_s(h_1,h_2)$. For $s=\frac{2\beta}{\beta+1}$ we have

$$d_s(h_1,h_2) \leq 2\Phi_\beta(\varphi_1,\varphi_2)\cdot \|J^{\frac{1}{2}}_{\varphi_1^{-1}} - J^{\frac{1}{2}}_{\varphi_2^{-1}} \mid L^2(\mathbb{D})\|.$$

Finally, using equality (3.4) and given that $\frac{2s}{s-1}=\frac{4\beta}{\beta-1}$ for $s=\frac{2\beta}{\beta+1}$ we get the required inequality

$$|\mu_n[A,\Omega_1] - \mu_n[A,\Omega_2]| \leq 2c_n B^2_{\frac{4\beta}{\beta-1},2}(\mathbb{D},h)\Phi_\beta(\varphi_1,\varphi_2)\cdot \|J^{\frac{1}{2}}_{\varphi_1^{-1}} - J^{\frac{1}{2}}_{\varphi_2^{-1}} \mid L^2(\mathbb{D})\|.$$

□

Remark 5.4. According to the work [17] the constant $B_{\frac{4\beta}{\beta-1},2}(\mathbb{D},h)$ can be estimated as

$$B_{\frac{4\beta}{\beta-1},2}(\mathbb{D},h) \leq K^{\frac{1}{2}} B_{q,2}(\mathbb{D}) \max\left\{\|J_{\varphi_1^{-1}} \mid L^2(\mathbb{D})\|^{\frac{\beta-1}{4\beta}}, \|J_{\varphi_2^{-1}} \mid L^2(\mathbb{D})\|^{\frac{\beta-1}{4\beta}}\right\}$$

where K is a quasiconformal coefficient of φ_1, φ_2 and

$$B_{q,2}(\mathbb{D}) \leq \left(2^{-1}\pi\right)^{\frac{2-q}{2q}} (q+2)^{\frac{q+2}{2q}}\,,\ q=\left(\frac{2\beta}{\beta-1}\right)^2.$$

6. ON ISOSPECTRAL OPERATORS

Let us remind that two linear operators are called isospectral if they have the same spectrum. It is known that there are distinct domains such that all the eigenvalues of the linear operator (in the case of the Laplace operator, see, for instance, [12]) coincide. For this reason, these domains are called isospectral domains.

Let $\varphi:\Omega\to\mathbb{D}$ be A-quasiconformal mappings for which $|J(w,\varphi)|=1$ for almost all $w\in\Omega$. In this case quasihyperbolic weights $h(z)=|J(z,\varphi^{-1})|=1$ for almost all $z\in\mathbb{D}$. Hence, by formula (3.5) we have $\mu_n[A,\Omega]=\mu_n[\mathbb{D}]$. So, we conclude that if $|J(w,\varphi)|=1$ for almost all $w\in\Omega$, then the Neumann eigenvalues for the elliptic operator in divergence form $-\textrm{div}[A(w)\nabla g(w)]$ in a domain Ω are equal to the Neumann eigenvalues for the Laplace operator in the unit disc $\mathbb{D}$.

Acknowledgments. The first author was supported by the United States-Israel Binational Science Foundation (BSF Grant No. 2014055). The second author was supported by the Ministry of Science and Higher Education of Russia (agreement No. 075-02-2020-1479/1) and RFBR Grant No. 18-31-00011.

REFERENCES

[1] L. Ahlfors, *Lectures on quasiconformal mappings.* Second edition. With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard. University Lecture Series, 38. American Mathematical Society, Providence, RI, 2006.

[2] K. Astala, T. Iwaniec, and G. Martin, *Elliptic partial differential equations and quasiconformal mappings in the plane.* Princeton University Press, Princeton and Oxford, 2008.

[3] C. Bandle, *Isoperimetric inequality for some eigenvalues of an inhomogeneous, free membrane.* SIAM J. Appl. Math. **22** (1972), 142–147.

[4] G. Barbatis, V. I. Burenkov, and P.D. Lamberti, *Stability estimates for resolvents, eigenvalues, and eigenfunctions of elliptic operators on variable domains.* In: *Around research of Vladimir Maz'ya. II*, pp. 23–60. International Mathematical Series 11, Springer, New York, 2009.

[5] B. Bojarski, V. Gutlyanskiĭ, O. Martio, and V. Ryazanov, *Infinitesimal geometry of quasiconformal and bi-Lipschitz mappings in the plane.* EMS, Zurich, 2013.

[6] V. I. Burenkov and P. D. Lamberti, *Spectral stability of Dirichlet second order uniformly elliptic operators.* J. Differential Equations **244** (2008), 1712–1740.

[7] V. I. Burenkov and P. D. Lamberti, *Spectral stability of higher order uniformly elliptic differential operators.* In: *Sobolev spaces in mathematics II*, pp. 69–102. International Mathematical Series 9, Springer, New York, 2009.

[8] V. I. Burenkov and P. D. Lamberti, *Sharp spectral stability estimates via the Lebesgue measure of domains for higher order elliptic operators.* Revista Matemática Compultense **25** (2012), 435–457.

[9] V. I. Burenkov, P. D. Lamberti, and M. Lanza de Cristoforis, *Spectral stability of nonnegative self-adjoint operators.* Sovrem. Mat. Fundam. Napravl. **15** (2006), 76–111. Translation in J. Math. Sci. **149** (2008), 1417–1452.

[10] V. I. Burenkov, V. Gol'dshtein, and A. Ukhlov, *Conformal spectral stability for the Dirichlet–Laplacian.* Math. Nachr. **288** (2015), 1822–1833.

[11] V. I. Burenkov, V. Gol'dshtein, and A. Ukhlov, *Conformal spectral stability for the Neumann–Laplacian.* Math. Nachr. **289** (2016), 2133–2146.

[12] P. Buser, J. Conway, P. Doyle, and K.-D. Semmler, *Some planar isospectral domains.* Int. Math. Res. Not. **9** (1994), 391–400.

[13] R. Courant, *Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces.* Springer-Verlag: Berlin-Heidelberg-New York, 1977.

[14] A. Girouard and I. Polterovich, *Shape optimization for low Neumann and Steklov eigenvalues.* Math. Methods Appl. Sci. **33** (2010), 501–516.

[15] V. Gol'dshtein and L. Gurov, *Applications of change of variables operators for exact embedding theorems.* Integral Equ. Oper. Theory **19** (1994), 1–24.

[16] V. Gol'dshtein, V. Pchelintsev, and A. Ukhlov, *Integral estimates of conformal derivatives and spectral properties of the Neumann-Laplacian.* J. Math. Anal. Appl. **463** (2018), 19–39.

[17] V. Gol'dshtein, V. Pchelintsev, and A. Ukhlov, *Spectral properties of the Neumann-Laplace operator in quasiconformal regular domains.* In: *Differential Equations, Mathematical Physics, and Applications. Selim Grigorievich Krein Centennial,* pp. 129–145. Contemporary Mathematics, AMS 734, 2019.

[18] V. Gol'dshtein, V. Pchelintsev, and A. Ukhlov, *Composition operators on Sobolev spaces and eigenvalues of divergent elliptic operators.* To appear in Complex Var. Elliptic Equ., https://doi.org/10.1080/17476933.2021.1921752.

[19] V. Gol'dshtein, V. Pchelintsev, and A. Ukhlov, *Spectral stability estimates of Dirichlet divergence form elliptic operators.* Anal. Math. Phys. **10** (2020), 74.

[20] V. Gol'dshtein and A. Ukhlov, *Weighted Sobolev spaces and embedding theorems.* Trans. Am. Math. Soc. **361** (2009), 3829–3850.

[21] V. Gol'dshtein and A. Ukhlov, *Sobolev homeomorphisms and Brennan's conjecture.* Comput. Methods Funct. Theory **14** (2014), 247–256.

[22] V. Gutlyanskiĭ and O. Martio, *Rotation estimates and spirals.* Conform. Geom. Dyn. **5** (2001), 6–20.

[23] V. Gutlyanskiĭ, O. Nesmelova, and V. Ryazanov, *On quasiconformal maps and semilinear equations in the plane.* J. Math. Sci. **229** (2018), 7–29.

[24] P. Koskela, J. Onninen, and J. T. Tyson, *Quasihyperbolic boundary conditions and capacity: Poincaré domains.* Math. Ann. **323** (2002), 811–830.

[25] P. D. Lamberti and M. Perin, *On the sharpneess of a certain spectral stability estimate for the Dirichlet Laplacian.* Eurasian Math. J. **1** (2010), 111–122.

[26] A. A. Lemenant, E. Milakis, and L. V. Spinolo, *Spectral stability estimates for the Dirichlet and Neumann Laplacian in rough domains.* J. Funct. Anal. **264** (2013), 2097–2135.

[27] V. Maz'ya, *Sobolev spaces: with applications to elliptic partial differential equations.* Springer: Berlin/Heidelberg, 2010.

[28] A. Ukhlov, *On mappings, which induce embeddings of Sobolev spaces.* Siberian Math. J. **34** (1993), 185–192.

[29] A. Ukhlov and S. K. Vodop'yanov, *Mappings with bounded (P,Q)-distortion on Carnot groups.* Bull. Sci. Math. **134** (2010), 605–634.

[30] S. K. Vodop'yanov and V. M. Gol'dstein, *Lattice isomorphisms of the spaces W^1_n and quasiconformal mappings.* Siberian Math. J. **16** (1975), 224–246.

[31] S. K. Vodop'yanov, V. M. Gol'dstein, and Yu. G. Reshetnyak, *On geometric properties of functions with generalized first derivatives.* Uspekhi Mat. Nauk **34** (1979), 17–65.

[32] S. K. Vodop'yanov and A. D. Ukhlov, *Sobolev spaces and (P,Q)-quasiconformal mappings of Carnot groups.* Siberian Math. J. **39** (1998), 665–682.

[33] S. K. Vodop'yanov and A. D. Ukhlov, *Superposition operators in Sobolev spaces.* Izvestiya VUZ **46** (2002), 11–33.

[34] S. K. Vodop'yanov and A. D. Ukhlov, *Set functions and their applications in the theory of Lebesgue and Sobolev spaces.* Siberian Adv. in Math. **14** (2004), 78–125.

Ben-Gurion University of the Negev
Department of Mathematics
P.O. Box 653, 8410501 Beer Sheva, Israel
`vladimir@math.bgu.ac.il`

Tomsk Polytechnic University
Division for Mathematics and Computer Sciences
Lenin Ave. 30, 634050 Tomsk, Russia
and
Tomsk State University
Regional Scientific and Educational Mathematical Center
Lenin Ave. 36, 634050 Tomsk, Russia
`vpchelintsev@vtomske.ru`

Ben-Gurion University of the Negev
Department of Mathematics
P.O. Box 653, 8410501 Beer Sheva, Israel
`ukhlov@math.bgu.ac.il`