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MONOTONICITY OF THE PRINCIPAL EIGENVALUE OF THE p-LAPLACIAN ON AN ANNULUS

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Let $D > 1$ be a positive integer. For each real numbers a and b such that $0 < a < b < \infty$ consider the annulus $A(a,b) := \{x \in \mathbb{R}^D : \ a < |x| < b\}$. Our goal is to give sufficient conditions on a and b such that the function which gives the principal eigenvalue of the p-Laplacian on $A(a,b)$, subject to the homogeneous Dirichlet boundary condition, be monotone or non-monotone with respect to $p \in (1,\infty)$.

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1. INTRODUCTION AND MAIN RESULTS

For each positive integer $D \geq 1$, each bounded and open set $\Omega \subset \mathbb{R}^D$ and each real number $p \in (1,\infty)$ let us recall the classical eigenvalue problem for the p-Laplacian

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u & \text{if} \quad x \in \Omega \\ u = 0 & \text{if} \quad x \in \partial\Omega. \end{cases} \tag{1.1}$$

It is well-known that the principal eigenvalue of the p-Laplacian is a positive real number which has the following variational characterization

$$\lambda_1(p;\Omega) := \min_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx}. \tag{1.2}$$

Moreover, the minimum in (1.2) is achieved by the unique positive solution (up to multiplication by constants) of problem (1.1). Since the corresponding eigenfunctions of $\lambda_1(p;\Omega)$ do not change sign in Ω it is usually referred to as the principal eigenvalue of the p-Laplacian.

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Next, let $\delta_\Omega : \Omega \to [0, \infty)$ be the distance function to the boundary of Ω, given by

$$\delta_\Omega(x) := \inf_{y \in \partial\Omega} |x - y|, \ x \in \Omega\,.$$

Define also

$$\mathbb{P}^D := \{\Omega \subset \mathbb{R}^D \mid \Omega \text{ is an open, bounded, convex domain with smooth boundary } \partial\Omega\}.$$

In the particular case where $D = 1$ and $\Omega = (a, b)$ with $a, b \in \mathbb{R}$, $a < b$, it is well known (see, e.g., [12]) that the principal eigenvalue of the p-Laplacian is given by the explicit formula

$$\lambda_1(p; (a, b)) = (p - 1)\left(\frac{2}{b - a}\right)^p \left(\frac{\pi/p}{\sin(\pi/p)}\right)^p.$$

By [9, Theorem 1.1] we know that when $\frac{b-a}{2} \in (1, \infty)$ there exists $p^\star = p^\star\left(\frac{b-a}{2}\right) \in (1, \infty)$ such that $p \mapsto \lambda_1(p; (a, b))$ is increasing on $(1, p^\star)$ and decreasing on $(p^\star, \infty)$, while when $\frac{b-a}{2} \le 1$ then the map $p \mapsto \lambda_1(p; (a, b))$ is increasing on $(1, \infty)$. Further, when $D > 1$ by [1, Theorem 1] we know that there exists a real number $M \in [e^{-1}, 1]$ such that for each $\Omega \in \mathbb{P}^D$ with $\|\delta_\Omega\|_{L^\infty(\Omega)} \le M$ the function $p \mapsto \lambda_1(p; \Omega)$ is increasing on $(1, \infty)$, and, for each real number $s > M$ there exists $\Omega \in \mathbb{P}^D$ with $\|\delta_\Omega\|_{L^\infty(\Omega)} = s$ for which the function $p \mapsto \lambda_1(p; \Omega)$ is not monotone on $(1, \infty)$. Note that this result does not solve the problem of the monotonicity of $\lambda_1(p; \Omega)$ with respect to $p \in (1, \infty)$ in the case when Ω is not a convex domain. The goal of this paper is to discuss the monotonicity of $\lambda_1(p; \Omega)$ with respect to $p \in (1, \infty)$ in the case when Ω is an annulus, and, consequently $\Omega \notin \mathbb{P}^D$.

With that end in view, for each integer $D > 1$ and each two positive real numbers a and b such that $0 < a < b < \infty$ define

$$A(a, b) := \{x \in \mathbb{R}^D : \ a < |x| < b\}\,.$$

In this paper our goal is to provide sufficient conditions on a and b such that the function $p \mapsto \lambda_1(p; A(a, b))$ is monotone or non-monotone with respect to $p \in (1, \infty)$.

Our main result is the following theorem.

THEOREM 1. *Let $D > 1$ be an integer and a and b be two real numbers such that $0 < a < b < \infty$. If*

$$\frac{b - a}{2}\left(\frac{b}{a}\right)^{D-1} \le e^{-1}\,,$$

then the function $p \mapsto \lambda_1(p; A(a, b))$ is increasing on $(1, \infty)$.

If $a+2<b$ *and*

$$2 < b \leq \frac{j_{D/2-1,1}}{\sqrt{D/2}}\,,$$

where $j_{D/2-1,1}$ *stands for the first zero of the Bessel function* $J_{D/2-1}$*, then the function* $p \mapsto \lambda_1(p; A(a,b))$ *is not monotone on* $(1,\infty)$.

2. PROPERTIES OF $\boldsymbol{\lambda_1(p; A(a,b))}$

In this section we recall some known properties of $\lambda_1(p; A(a,b))$ that will be useful in the sequel. First, recall that by [7, Lemma 1.5] (see also [4]), we have

$$\lim_{p\to\infty} \sqrt[p]{\lambda_1(p; A(a,b))} = \frac{2}{b-a}, \quad \forall\, 0<a<b<\infty\,. \tag{2.1}$$

In particular, this relation implies that

$$\lim_{p\to\infty} \lambda_1(p; A(a,b)) = \begin{cases} +\infty & \text{if } \frac{b-a}{2} < 1 \\ 0 & \text{if } \frac{b-a}{2} > 1. \end{cases}$$

From this simple observation we infer that if function $p \mapsto \lambda_1(p; A(a,b))$ has a monotony on $(1,\infty)$ then it should be increasing if $\frac{b-a}{2} < 1$ or decreasing if $\frac{b-a}{2} > 1$.

Next, by [8, Theorem 2.4] we know the following estimates

$$\left(\frac{a}{b}\right)^{D-1} \lambda_1(p; (a,b)) \leq \lambda_1(p; A(a,b)) \leq \left(\frac{b}{a}\right)^{D-1} \lambda_1(p; (a,b)), \qquad \forall\, p>1,\ \forall\, 0<a<b<\infty\,, \tag{2.2}$$

where $\lambda_1(p;(a,b)) = (p-1)\left(\frac{2}{b-a}\right)^p \left(\frac{\pi/p}{\sin(\pi/p)}\right)^p$ stands for the principal eigenvalue of the p-Laplacian when $D=1$ and $\Omega=(a,b)$.

Moreover, by [11, Theorem 3.2] we have

$$p\sqrt[p]{\lambda_1(p; A(a,b))} \leq q\sqrt[q]{\lambda_1(q; A(a,b))}, \qquad \forall\, 1<p<q<\infty,\ \forall\, 0<a<b<\infty. \tag{2.3}$$

Furthermore, note that for each $R>0$, considering the rescaled domain

$$RA(a,b) = \{Rx \mid x \in A(a,b)\}\,,$$

it is easy to check that

$$RA(a,b) = A(Ra,Rb)\,.$$

By [2, p. 634] we recall that

$$\lambda_1(p; A(Ra,Rb)) = \frac{1}{R^p}\lambda_1(p; A(a,b)) \ \ \forall\, p>1. \tag{2.4}$$

Finally, we point out that by [10, Theorem 2] we know the explicit value of the Cheeger constant on an annulus, namely

$$h(A(a,b)) := \lim_{p\to 1^+} \lambda_1(p; A(a,b)) = D\frac{b^{D-1}+a^{D-1}}{b^D - a^D}\,. \tag{2.5}$$

We end this section by recalling a formula which gives the value of the principal eigenvalue of the Laplacian (that is the case when $p = 2$) on a ball. For each $R > 0$ let B_R be a ball of radius R in $\mathbb{R}^D$. It is well-known (see, e.g. [5, p. 11]) that

$$\lambda_1(2; B_R) = \frac{j^2_{D/2-1,1}}{R^2}\,, \tag{2.6}$$

where $j_{D/2-1,1}$ stands for the first zero of the Bessel function $J_{D/2-1}$.

3. PROOF OF THE MAIN RESULT

The proof of Theorem 1 is a simple consequence of the two propositions below.

PROPOSITION 1. *Let $D > 1$ be an integer and a and b be two real numbers such that $0 < a < b < \infty$. If*

$$\frac{b-a}{2}\left(\frac{b}{a}\right)^{D-1} \le e^{-1}\,, \tag{3.1}$$

then the function $p \mapsto \lambda_1(p; A(a,b))$ is increasing on $(1,\infty)$.

Remark. Note that by condition (3.1) it follows that $\frac{b-a}{2} < 1$. This fact and relation (2.1) imply that $\lim_{p\to\infty} \lambda_1(p; A(a,b)) = \infty$. This last relation suggests that if function $p \mapsto \lambda_1(p; A(a,b))$ were to be monotone on $(1,\infty)$ it should be increasing.

Proof. Assume by contradiction that there exists $1 < p < q < \infty$ such that

$$\lambda_1(q; A(a,b)) \le \lambda_1(p; A(a,b))\,. \tag{3.2}$$

By [1, Lemma 1] we deduce that

$$\lambda_1(p; A(a,b)) < e^q\,. \tag{3.3}$$

On the other hand, observing that

$$\lambda_1(q; (-1,1)) = (q-1)\left(\frac{\pi/q}{\sin(\pi/q)}\right)^q, \quad \forall\, q > 1\,,$$

the left-hand side of relation (2.2) reads as

$$\left(\frac{a}{b}\right)^{D-1}\frac{\lambda_1(q;(-1,1))}{\left(\frac{b-a}{2}\right)^q}\le\lambda_1(q;A(a,b)),\quad\forall\, q>1\,.$$

Hence, using the fact that $\lambda_1(\cdot;(-1,1))$ is strictly increasing and that $\lim_{q\to 1^+}\lambda_1(q;(-1,1))=1$ (see [9, Theorem 1.1 (i)] and [6, Theorem 3.3], respectively), we obtain

$$\left(\frac{a}{b}\right)^{D-1}\left(\frac{2}{b-a}\right)^q\le\lambda_1(q;A(a,b)),\quad\forall\, q>1\,. \tag{3.4}$$

Thus, under assumptions (3.2), using (3.4) and (3.3) we get

$$\left(\frac{a}{b}\right)^{D-1}\left(\frac{2}{b-a}\right)^q\le\lambda_1(q;A(a,b))\le\lambda_1(p;A(a,b))<e^q\,.$$

Finally, observing that when $a<b$ and $q\ge 1$ it holds

$$\left(\frac{a}{b}\right)^{(D-1)q}\le\left(\frac{a}{b}\right)^{D-1},$$

we obtain

$$\frac{b-a}{2}\left(\frac{b}{a}\right)^{D-1}>e^{-1},$$

which contradict assumption (3.1). The proof of Proposition 1 is complete. □

Remark. 1) Note that if for some two real numbers a and b such that $0<a<b<\infty$ we have

$$\lambda_1(p;A(a,b))\le\lambda_1(q;A(a,b)),\quad\forall\, 1<p<q<\infty\,,$$

then by relation (2.4) we deduce that for each $R\in(0,1)$ we have

$$\lambda_1(p;A(Ra,Rb))\le\lambda_1(q;A(Ra,Rb)),\quad\forall\, 1<p<q<\infty\,,$$

2) Assume that D, a and b satisfy the hypothesis of Proposition 1. Then for each $R\in(0,1)$ we have

$$\frac{Rb-Ra}{2}\left(\frac{Rb}{Ra}\right)^{D-1}=R\frac{b-a}{2}\left(\frac{b}{a}\right)^{D-1}<\frac{b-a}{2}\left(\frac{b}{a}\right)^{D-1}<e^{-1}\,.$$

Consequently, the function $p\mapsto\lambda_1(p;A(Ra,Rb))$ is increasing on $(1,\infty)$.

PROPOSITION 2. *Let $D>1$ be an integer and a and b be two real numbers such that $0<a<b<\infty$. If $a+2<b$ and*

$$2<b\le\frac{j_{D/2-1,1}}{\sqrt{D/2}}\,, \tag{3.5}$$

where $j_{D/2-1,1}$ stands for the first zero of the Bessel function $J_{D/2-1}$, then the function $p\mapsto\lambda_1(p;A(a,b))$ is not monotone on $(1,\infty)$.

Proof. If $a+2<b$ then $\frac{b-a}{2}>1$ and by (2.1) we deduce that

$$\lim_{p\to\infty}\lambda_1(p;A(a,b))=0.$$

Consequently, if we assume by contradiction that the function $p\mapsto\lambda_1(p;A(a,b))$ is monotone on $(1,\infty)$ it follows that it should be decreasing. In particular, taking into account that fact and using relation (2.5) we deduce that

$$\lambda_1(2;A(a,b))\le\lim_{p\to1^+}\lambda_1(p;A(a,b))=h(A(a,b))=D\frac{b^{D-1}+a^{D-1}}{b^D-a^D}\,.$$

On the other hand, since $A(a,b)\subset B_b$ by relation (2.6) we get

$$\lambda_1(2;A(a,b))>\lambda_1(2;B_b)=\frac{j^2_{D/2-1,1}}{b^2}\,.$$

Taking into account all the above pieces of information we find

$$D\frac{b^{D-1}+a^{D-1}}{b^D-a^D}\ge\frac{j^2_{D/2-1,1}}{b^2}\,,$$

which implies

$$b^2\ge\frac{j^2_{D/2-1,1}}{D}\,\frac{b^D-a^D}{b^{D-1}+a^{D-1}}\,.$$

Finally, since $\frac{b-a}{2}>1$ we deduce that

$$\frac{b^D-a^D}{b^{D-1}+a^{D-1}}=2\frac{b-a}{2}\,\frac{b^{D-1}+b^{D-2}a+\ldots+ba^{D-2}+a^{D-1}}{b^{D-1}+a^{D-1}}>2\,,$$

and, consequently,

$$b>\frac{j_{D/2-1,1}}{\sqrt{D/2}}\,,$$

a contradiction with (3.5). It follows that the conclusion of Proposition 2 holds true. □

Remark. Note that for each integer $D>1$ it holds true that

$$2D<j^2_{D/2-1,1}\,.$$

Indeed, since $j^2_{r,1}<j^2_{s,1}-(s^2-r^2)$ for all $0\le r<s$ (see, e.g. [3, p. 64]) taking $r=0$ and $s=D/2-1$ we get

$$j^2_{D/2-1,1}>(D/2-1)^2+j^2_{0,1}\,.$$

Recalling that $j_{0,1}>2.404$ (see, e.g. [5, p. 11]) we deduce that

$$j^2_{D/2-1,1}>D^2/4-D+6.779>2D\,,$$

if $D\le3$ or $D\ge8$. Moreover, in the case when $D\in\{4,5,6,7\}$ it can be checked easily (using the approximating values of j from tables) that the inequality still holds true.

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