To the memory of Professor Raimo Näkki

ON A GENERALIZED QUASIHYPERBOLIC CONDITION IN A BOUNDED DOMAIN

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We study properties of bounded domains when the domains satisfy a generalized quasihyperbolic growth condition.

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1. INTRODUCTION

We use the *quasihyperbolic metric* to study geometric properties of bounded domains in the Euclidean n-space $\mathbf{R}^n$. Our main question is what are sufficient conditions for the quasihyperbolic metric in a given domain so that a (q,p)-Sobolev-Poincaré inequality holds there with some q and p. The question is interesting because the Sobolev-Poincaré inequalities are significant tools for the study of partial differential equations and their boundary problems.

The quasihyperbolic metric k_d is a generalization of the hyperbolic metric on an open disc or a half-plane in $\mathbf{R}^2$ to any proper subdomain in $\mathbf{R}^n$. It was defined by F. W. Gehring and B. P. Palka [3] as

$$k_D(x,y) = \inf_{\gamma} \int_{\gamma} \frac{\mathrm{d}s}{\operatorname{dist}(u,\partial D)}\,,$$

where the infimum is taken over all rectifiable curves γ in D joining $x \in D$ and $y \in D$. Here $\operatorname{dist}(u,\partial D)$ denotes the distance between $u \in D$ and the boundary of D. A curve attaining this infimum exists and it is called the *quasihyperbolic geodesic*, [2, Lemma 1, p. 53]. For every $x, y \in D$ we fix such a geodesic and denote it by $\gamma_{x,y}$.

Let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous, strictly increasing function such that $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. We say that a bounded domain D

satisfies a *φ-quasihyperbolic boundary condition*, if there exists a point $x_0 \in D$ and if there are constants $C_1, C_2 \geqslant 0$ such that

$$k_D(x_0, x) \leqslant C_1 \varphi\left(\frac{1}{\operatorname{dist}(x, \partial D)}\right) + C_2 \quad \text{for all } x \in D. \tag{1.1}$$

This is a generalization of the quasihyperbolic boundary condition used by Gehring and O. Martio [1] when $\varphi(t) = \log(1+t)$. We say briefly that D is a φ-QHBC domain, if (1.1) holds. If $\varphi(t) = \log(1+t)$, we just say that D is a QHBC domain. The quasihyperbolic metric cannot grow any slower than this, [8, p. 190]. The fact that $\operatorname{dist}(x, \partial D)$ has an upper bound means that the choice $C_2 = 0$ is always possible in (1.1) if C_1 is made large enough. Thus we assume that $C_2 = 0$. Whether a domain satisfies the QHBC condition or the φ-QHBC condition is independent of the choice of the point x_0, but the constants may be different. For related conditions we refer to [8], [10], and [20]. Note that φ-QHBC domains are not to be confused with φ-uniform or ψ-uniform domains in [8], [9], [12], and [20].

The properties of the classical QHBC are well studied. More information about the QHBC domains can be found in [7], [11], and [14]. For example, it was shown in [10, Remark 7.11, p. 27] that there is a $p < n$ such that the (p,p)-Sobolev-Poincaré inequality, see (2.1), holds when D is a QHBC domain with a *Whitney cube # -condition.* We recall that a Whitney cube # -condition means that the number of Whitney cubes Q in D with the diameter of Q, $\operatorname{dia}(Q) = 2^{-j}\operatorname{dia}(D)$ is bounded by a constant times $2^{\lambda j}$, with some λ, $n-1 \leqslant \lambda < n$, see [15, p. 19]. Later it was shown that QHBC domains satisfy the Whitney cube # -condition [18, Corollary 1, p. 352]. Thus for QHBC domain we have $p_0 < n$ such that (p,p)-Sobolev-Poincaré inequality holds for all $p \geqslant p_0$.

We recall that if D satisfies a φ-quasihyperbolic boundary condition with $\varphi(t) = t^\alpha$ for some α and D satisfies the Whitney cube # -condition, then the (p,p)-Sobolev-Poincaré inequality holds in D for all $p \geqslant n$, [10, Corollary 7.17]. A sharp bound for α which guarantees the validity of the (q,n)-Sobolev-Poincaré inequality for some $q \geqslant n$ without relying on the Whitney cube # -condition was given in [13, Theorem 1.1, p. 184].

In the present paper we give a summation condition to a given φ which guarantees that the (q,n)-Sobolev-Poincaré inequality holds in D for some $q \in [n, c(D)]$ where $c(D)$ is a constant coming from the summation condition, Theorem 3.5. In Section 4 we construct, by using the Cantor dust set, a φ-QHBC domain F where the Whitney cube # -condition fails but the (q,n)-Sobolev-Poincaré inequality holds for all $q \geqslant n$. In the last Section 5 we construct a mushroom-type domain D that is a φ-QHBC domain and satisfies a Whitney cube # -condition and the (q,p)-Sobolev-Poincaré inequality fails for all $1 \leqslant q \leqslant p < n$.

We point out the following two observations: If φ grows faster than logarithmic, i.e. $\lim_{t\to\infty} \frac{\varphi(t)}{\log(t)} = \infty$, then φ-QHBC domain may or may not satisfy the Whitney cube # -condition. Furthermore, the requirements of the Whitney cube # -condition may be met and even the $(1,p)$-Sobolev-Poincaré inequality needs not to hold for any $p < n$, refer to Section 5.

2. NOTATION AND WHITNEY DECOMPOSITION

We let C denote constants that appear in our estimates and may change from expression to expression. To note that C depends on $a, b, \dots$, we write $C(a, b, \dots)$. For a line segment with endpoints x and y, we use the notation $[x, y]$, and the length of a line segment is denoted by $\ell[x, y]$. The n-dimensional Lebesgue measure of a set E in $\mathbf{R}^n$ is written as $|E|$. We use in the examples the abbreviation $\delta_D(z) := \operatorname{dist}(z, \partial D)$.

The inequality

$$\left(\int_D |u(x) - u_D|^q \,\mathrm{d}x\right)^{1/q} \leqslant C\left(\int_D |\nabla u(x)|^p \,\mathrm{d}x\right)^{1/p} \tag{2.1}$$

is called the (q,p)-*Sobolev-Poincaré inequality*. Here $1 \leqslant q, p < \infty$ and C is a constant independent of $u \in W^{1,p}(D)$ and u_D is the integral average of u over D. If $q = p$, the inequality reduces to the well known Poincaré inequality. By $W^{1,p}(D)$ we denote the Sobolev space of functions $u \in L^p(D)$ whose first weak partial derivatives belong to $L^p(D)$. A bounded domain D in $\mathbf{R}^n, n \geqslant 2$, is said to be a (q,p)-*Sobolev-Poincaré domain*, if there exists C such that inequality (2.1) holds for all $u \in W^{1,p}(D)$.

We will use the following decomposition in our domains.

Definiton 2.2 ([19, Theorem 3, p. 16]). A family $\mathcal{W}$ of closed dyadic cubes Q whose interiors are pairwise disjoint is called the *Whitney decomposition* of D, if the following three conditions hold:

(1) $D = \bigcup_{Q\in\mathcal{W}} Q$;

(2) $1 \leqslant \frac{\operatorname{dist}(Q,\partial D)}{\operatorname{dia}(Q)} \leqslant 4$;

(3) $\frac{1}{4} \leqslant \frac{\operatorname{dia}(Q_1)}{\operatorname{dia}(Q_2)} \leqslant 4$, when $Q_1 \cap Q_2 \neq \emptyset$.

3. SOBOLEV-POINCARÉ INEQUALITY IN φ-QHBC DOMAINS

The next result characterizes Sobolev-Poincaré domains in terms of a capacity-type estimate. The result is originally from V. Maz'ya's book *Sobolev*

Spaces [16]. P. Hajłasz and P. Koskela gave another proof for the result in [4, Theorem 1, p. 429].

THEOREM 3.1. *Let D be a bounded domain in $\mathbf{R}^n, n \geqslant 2$, and let $1 \leqslant p \leqslant q < \infty$. Then D is a (q,p)-Sobolev-Poincaré domain if and only if the following holds: For a cube Q_0 compactly contained in D there exists a constant $C = C(D, Q_0, p, q)$ such that*

$$\int_D |\nabla u(x)|^p \mathrm{d}x \geqslant C|A|^{p/q}$$

whenever A is an admissible subset of D which is disjoint from Q_0 and $u \in C^\infty(D)$ satisfies $u|_A \geqslant 1$ and $u|_{Q_0} = 0$.

Here, a subset $A \subset D$ is admissible if A is open and $D \cap \partial A$ is a smooth submanifold.

Let us consider the Whitney decomposition $\mathcal{W} = \mathcal{W}(D)$ and the quasihyperbolic metric k_D of a domain D in $\mathbf{R}^n$. Let us denote by c_Q the center of a cube $Q \in \mathcal{W}$. In addition, we fix a central cube Q_0 with center point x_0. Following [13] we divide Whitney cubes to the sets

$$\mathcal{W}_j := \{Q \in \mathcal{W} : j \leqslant k_D(c_Q, x_0) < j+1\},$$

where $j \in \mathbf{N}$. For $Q \in \mathcal{W}$, let us set $P(Q) := \{Q' \in \mathcal{W} : Q' \cap \gamma_{c_Q, x_0} \neq \emptyset\}$, and define the *shadow* of a cube $Q \in \mathcal{W}$ by

$$S(Q) := \bigcup_{\substack{\tilde{Q} \in \mathcal{W} \\ Q \in P(\tilde{Q})}} \tilde{Q}.$$

We need several lemmas. Let γ be a quasihyperbolic geodesic in D starting at the central point $x_0 \in D$. Then by [13, Lemma 2.1, p. 185] for each $j \geqslant 0$, we have

$$\#\{Q \in \mathcal{W}_j : Q \cap \gamma \neq \emptyset\} \leqslant C(n). \tag{3.2}$$

The next lemma is a modification of [13, Lemma 2.4, p. 186].

LEMMA 3.3. *Let D in $\mathbf{R}^n$ be a φ-QHBC domain and $j \geqslant 2$. Then, there exists a constant $C > 0$, independent of j, such that*

$$\operatorname{dia}(S(Q)) \leqslant C \sum_{i \geqslant j-1} \frac{1}{\varphi^{-1}\left(\frac{i}{C_1}\right)}$$

for each Whitney cube $Q \in \mathcal{W}_j$. Here C_1 is from (1.1).

Proof. We start with the observation: If Q is a cube in $\mathcal{W}_i$, $i \geqslant 1$, then by the properties of $\mathcal{W}_i$ and the definition of the φ-QHBC domain we have

$$\operatorname{dia}(Q) \leqslant \operatorname{dist}(Q, \partial D) \leqslant \operatorname{dist}(c_Q, \partial D) \leqslant \frac{1}{\varphi^{-1}\left(\frac{k_D(c_Q, x_0)}{C_1}\right)} \leqslant \frac{1}{\varphi^{-1}\left(\frac{i}{C_1}\right)}.$$

Next, fix $j \in \mathbf{N}$, $j \geqslant 2$, and a cube Q in $\mathcal{W}_j$. Let $\tilde{Q} \subset S(Q)$ and let γ be the fixed geodesic joining x_0 to $c_{\tilde{Q}}$. Then, by the definition of the shadow, there exists a point $x_Q \in \gamma \cap Q$. Now if Q' is a cube in $\mathcal{W}$ such that $Q' \cap \gamma_{x_Q, c_{\tilde{Q}}} \neq \emptyset$, then using the triangle inequality, properties of geodesic and the Whitney decomposition we see that $k_D(c_{Q'}, x_0) \geqslant j-1$. Thus Q' belongs to $\cup_{i \geqslant j-1} \mathcal{W}_i$.

By (3.2) the geodesic γ intersects a bounded number of cubes from each $\mathcal{W}_i, i \geqslant j-1$. Therefore,

$$\begin{aligned} \operatorname{dist}(c_Q, c_{\tilde{Q}}) &\leqslant \operatorname{dist}(c_Q, x_Q) + \operatorname{dist}(x_Q, c_{\tilde{Q}}) \leqslant \operatorname{dia}(Q) + \operatorname{dist}(x_Q, c_{\tilde{Q}}) \\ &\leqslant \operatorname{dia}(Q) + \sum_{i \geqslant j-1} \sum_{\substack{Q' \in \mathcal{W}_i \\ Q' \cap \gamma_{x_Q, c_{\tilde{Q}}} \neq \emptyset}} \operatorname{dia}(Q') \leqslant C \sum_{i \geqslant j-1} \frac{1}{\varphi^{-1}\left(\frac{i}{C_1}\right)}. \end{aligned}$$

Now, the lemma is obtained as follows: Take the supremum over all cubes $\tilde{Q} \subset S(Q)$ and use the triangle inequality to find out that the above is an upper bound for the distance between the centers of any two cubes in $S(Q)$. □

Lemma 3.4 ([13, Lemma 2.3, p. 186]). *Let D in $\mathbf{R}^n$ be a domain and $j \geqslant 0$. Then, for each $s \geqslant 1$ and for every measurable subset $E \subset D$,*

$$\sum_{Q \in \mathcal{W}_j} |S(Q) \cap E|^s \leqslant C(n, s)|E|^s.$$

The next theorem is a generalization of [13, Theorem 3.1, p. 187].

Theorem 3.5. *Let D be a bounded domain in $\mathbf{R}^n, n \geqslant 2$. Let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous strictly increasing function with the properties $\varphi(0) = 0, \lim_{t \to \infty} \varphi(t) = \infty$, and*

$$\sum_{j=1}^{\infty} \left(\sum_{i \geqslant j} \frac{1}{\varphi^{-1}\left(\frac{i}{C_1}\right)} \right)^{ns} < \infty$$

where $0 < s \leqslant \frac{1}{n-1}$ is a constant and C_1 is the constant in (1.1).

Suppose that D is a φ-QHBC domain. Then, the domain D is a (q, n)-Sobolev-Poincaré domain, when

$$n \leqslant q \leqslant \frac{n}{s(n-1)}.$$

Proof. We use Theorem 3.1 and the idea of the proof of [13, Theorem 3.1, p. 187]. Let Q_0 be a Whitney cube so that it is disjoint from a set $E \subset D$, which is admissible in D. Assume that $u \in C^\infty(D)$ satisfies $u|_E \geqslant 1$ and $u|_{Q_0} = 0$. Let $0 < s \leqslant \frac{1}{n-1}$. Our aim is to show for $q = \frac{n}{s(n-1)}$ that

$$\int_D |\nabla u|^n \mathrm{d}y \geqslant C|E|^{n/q}. \tag{3.6}$$

Let us first consider the set $E_g := \{x \in Q \in \mathcal{W}, u_Q \leqslant \frac{1}{2}\} \cap E$. Now we estimate, first using the inequality $|a+b|^{n/q} \leqslant |a|^{n/q} + |b|^{n/q}$ with $n \leqslant q$, and then using the fact $u|_E \geqslant 1$ and the definition of E_g,

$$\begin{aligned}
|E_g|^{n/q} &= \left(\sum_{Q \in \mathcal{W}} |E_g \cap Q| \right)^{n/q} \leqslant \sum_{Q \in \mathcal{W}} |E_g \cap Q|^{n/q} \\
&\leqslant \sum_{\substack{Q \in \mathcal{W} \\ Q \cap E_g \neq \emptyset}} \left(\int_{Q \cap E} \left(2 \cdot \frac{1}{2} \right)^q \mathrm{d}y \right)^{n/q} \leqslant 2^n \sum_{\substack{Q \in \mathcal{W} \\ Q \cap E_g \neq \emptyset}} \left(\int_Q |u - u_Q|^q \mathrm{d}y \right)^{n/q}.
\end{aligned}$$

Then, we apply the (q,n)-Sobolev-Poincaré inequality on cubes and obtain

$$|E_g|^{n/q} \leqslant C \sum_{\substack{Q \in \mathcal{W} \\ Q \cap E_g \neq \emptyset}} \int_Q |\nabla u|^n \mathrm{d}y \leqslant C \int_D |\nabla u|^n \mathrm{d}y.$$

Hence inequality (3.6) holds for the set E_g.

Then we consider the set $E_b := \{x \in Q \in \mathcal{W}, u_Q \geqslant \frac{1}{2}\} \cap E$. Let $x \in E_b$, and let $Q(x) \in \mathcal{W}$ be a cube for which $x \in Q(x)$ and $u_{Q(x)} \geqslant 1/2$. We apply a chaining argument [17, Lemma 8, p. 81]. Let $Q_0, Q_1, \dots, Q_m = Q(x)$ be a minimal chain of cubes in $P(Q(x))$ which joins Q_0 to $Q(x)$. Here, the word 'minimal' means that we cannot remove any cube from the chain and still have a chain from Q_0 to $Q(x)$. We obtain

$$\begin{aligned}
1 &\leqslant 2|u_{Q(x)} - u_{Q_0}| \leqslant 2 \sum_{i=1}^m |u_{Q_i} - u_{Q_{i-1}}| \\
&\leqslant 2 \sum_{i=1}^m \Big(|u_{Q_i} - u_{Q_i \cup Q_{i-1}}| + |u_{Q_i \cup Q_{i-1}} - u_{Q_{i-1}}| \Big) \\
&\leqslant 2 \sum_{i=1}^m \left(\int_{Q_i} \frac{|u(y) - u_{Q_i \cup Q_{i-1}}| \mathrm{d}y}{|Q_i|} + \int_{Q_{i-1}} \frac{|u_{Q_i \cup Q_{i-1}} - u(y)| \mathrm{d}y}{|Q_{i-1}|} \right) \\
&\leqslant C \sum_{i=1}^m \frac{1}{|Q_i \cup Q_{i-1}|} \int_{Q_i \cup Q_{i-1}} |u(y) - u_{Q_i \cup Q_{i-1}}| \mathrm{d}y.
\end{aligned}$$

Then we use $(1,1)$-Sobolev-Poincaré inequality. Now, the constant $C(Q_i \cup$

Q_{i-1}) is comparable to $\operatorname{dia}(Q_i)$, [17, Lemma 6, p. 81]. We obtain

$$\begin{aligned}1 &\leqslant C\sum_{i=1}^{m}\frac{C(Q_i\cup Q_{i-1})}{|Q_i\cup Q_{i-1}|}\int_{Q_i\cup Q_{i-1}}|\nabla u(y)|\mathrm{d}y\\ &\leqslant C\sum_{i=1}^{m}\frac{1}{\operatorname{dia}(Q_i)^{n-1}}\int_{Q_i\cup Q_{i-1}}|\nabla u(y)|\mathrm{d}y\leqslant C\sum_{Q\in P(Q(x))}\operatorname{dia}(Q)⨍_{Q}|\nabla u(y)|\mathrm{d}y.\end{aligned}$$

Now we have

$$1\leqslant C\sum_{Q\in P(Q(x))}\operatorname{dia}(Q)⨍_{Q}|\nabla u(y)|\mathrm{d}y,$$

which we then integrate over E_b and use Hölder's inequality to obtain

$$|E_b|\leqslant C\int_{E_b}\sum_{Q\in P(Q(x))}\operatorname{dia}(Q)\left(⨍_{Q}|\nabla u(y)|^n\mathrm{d}y\right)^{1/n}\mathrm{d}x.$$

Interchanging the order of summation and integration and applying Hölder's inequality with $\left(\frac{n}{n-1},n\right)$ give

$$\begin{aligned}|E_b| &\leqslant C\sum_{Q\in\mathcal{W}}\int_{E_b}\chi_{S(Q)}(x)\left(\int_{Q}|\nabla u(y)|^n\mathrm{d}y\right)^{1/n}\mathrm{d}x\\ &\leqslant C\left(\sum_{Q\in\mathcal{W}}|S(Q)\cap E_b|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}\left(\sum_{Q\in\mathcal{W}}\int_{Q}|\nabla u(y)|^n\mathrm{d}y\right)^{1/n}\\ &\leqslant C\left(\sum_{Q\in\mathcal{W}}|S(Q)\cap E_b|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}\left(\int_{D}|\nabla u(y)|^n\mathrm{d}y\right)^{1/n}.\end{aligned}\tag{3.7}$$

Let us estimate the first part of product (3.7). Since $0<s\leqslant\frac{1}{n-1}$, we use Lemma 3.4 for $\frac{n}{n-1}-s\geqslant 1$ to obtain

$$\begin{aligned}\sum_{Q\in\mathcal{W}}|S(Q)\cap E_b|^{\frac{n}{n-1}} &\leqslant\sum_{j=0}^{\infty}\max_{Q\in\mathcal{W}_j}\left(|S(Q)\cap E_b|^s\right)\sum_{Q\in\mathcal{W}_j}|S(Q)\cap E_b|^{\frac{n}{n-1}-s}\\ &\leqslant C|E_b|^{\frac{n}{n-1}-s}\sum_{j=0}^{\infty}\max_{Q\in\mathcal{W}_j}\left(\operatorname{dia}(S(Q))^{ns}\right).\end{aligned}$$

Let us continue by using Lemma 3.3. We have by the assumption that

$$
\begin{aligned}
&C|E_b|^{\frac{n}{n-1}-s}\sum_{j=1}^{\infty}\max_{Q\in\mathcal{W}_j}\left(\operatorname{dia}(S(Q))^{ns}\right)\\
&\leqslant C|E_b|^{\frac{n}{n-1}-s}\left(2\operatorname{dia}(D)^{ns}+\sum_{j=2}^{\infty}\left(\sum_{i\geqslant j}\frac{1}{\varphi^{-1}\left(\frac{i}{C_1}\right)}\right)^{ns}\right)\leqslant C|E_b|^{\frac{n}{n-1}-s}.
\end{aligned}
$$

This and (3.7) yield

$$
|E_b|\leqslant C|E_b|^{1-\frac{s(n-1)}{n}}\left(\int_D|\nabla u(y)|^n\mathrm{d}y\right)^{1/n},
$$

and thus

$$
|E_b|^{s(n-1)}\leqslant C\int_D|\nabla u(y)|^n\mathrm{d}y.
$$

So (3.6) holds for the set E_b. This follows by choosing $s=\frac{n}{q(n-1)}$. □

Example 3.8. We apply Theorem 3.5 to the function $\varphi(t)=t^{\alpha}\log^{\beta}(1+t)$.

(a) The case $\alpha\in(0,1),\beta=0$ is the one studied in [13]. Let us now look at how our Theorem 3.5 reconstructs that result. Indeed, if $\varphi(t)=t^{\alpha},\alpha\in(0,1)$, we have

$$
\sum_{j=1}^{\infty}\left(\sum_{i\geqslant j}\frac{1}{\varphi^{-1}\left(\frac{i}{C_1}\right)}\right)^{ns}=\sum_{j=1}^{\infty}\left(\sum_{i\geqslant j}\left(\frac{i}{C_1}\right)^{-1/\alpha}\right)^{ns}\leqslant C\sum_{j=1}^{\infty}\left(j^{1-1/\alpha}\right)^{ns},
$$

where the inside sum in the middle has been estimated by the Riemann integral. The last sum converges, when $\left(1-\frac{1}{\alpha}\right)ns<-1$, in other words, when $s>\frac{\alpha}{n(1-\alpha)}$. On the other hand, we require that $s\leqslant\frac{1}{n-1}$, which implies $\frac{\alpha}{n(1-\alpha)}<\frac{1}{n-1}$, or, equivalently, $\alpha<\frac{n}{2n-1}$. This is the upper bound for α given by [13, Theorem 3.4, p. 190]. Furthermore, according to our Theorem 3.5, the domain is a (q,n)-Sobolev-Poincaré domain, when q is less than or equal to $\frac{n}{s(n-1)}$. Since the convergence requires that $s>\frac{\alpha}{n(1-\alpha)}$, we obtain $q<\frac{n^2(1-\alpha)}{\alpha(n-1)}$, which is the upper bound for q in [13, Theorem 3.4, p. 190].

(b) Assume then that $\alpha=0$ and $\beta\geqslant 1$. Then, the domain D is a (q,n)-Sobolev-Poincaré domain for all $q\geqslant n$. Namely, $\varphi^{-1}(t)=\exp\left(t^{1/\beta}\right)-1$. Because the exponential growth is faster than the polynomial one, we estimate

$$
\sum_{i\geqslant j}\frac{1}{\varphi^{-1}\left(\frac{i}{C_1}\right)}=\sum_{i\geqslant j}\frac{1}{\exp\left(\left(\frac{i}{C_1}\right)^{1/\beta}\right)-1}\leqslant C\sum_{i\geqslant j}\frac{1}{i^{\tau}}\leqslant Cj^{1-\tau},
$$

where we can choose $\tau > 1$ to be as large as we want to. Then,

$$\sum_{j=1}^{\infty}\left(\sum_{i\geqslant j}\frac{1}{\varphi^{-1}\left(\frac{i}{C_1}\right)}\right)^{ns}\leqslant C\sum_{j=1}^{\infty}j^{ns(1-\tau)}.$$

The last sum converges, when $s > \frac{1}{n(\tau-1)}$. Because τ can be made large, s can be as close to zero as we want to. Considering Theorem 3.5, this means that there will be no upper bound for q.

(c) Assume then that $\alpha > 0$ and $\beta > 0$. It is essential to note that we can choose $\varepsilon > 0$ as close to zero as we want to and have $t^{\alpha}\log^{\beta}(1+t)\leqslant Ct^{\alpha+\varepsilon}$, provided that C is large enough. Therefore, we obtain an upper bound for q by following the same procedure as in case (a), but this time with the exponent $\alpha+\varepsilon$ instead of α. The upper bound for q will be

$$q < \frac{n^2(1-\alpha-\varepsilon)}{(\alpha+\varepsilon)(n-1)} \xrightarrow{\varepsilon\to 0+} \frac{n^2(1-\alpha)}{\alpha(n-1)}.$$

4. CANTOR DUST FRACTAL DOMAIN

In this section, we construct a bounded domain F in $\mathbf{R}^2$ that has the following properties:

(a) F is a φ-QHBC domain with $\varphi(t)=\log^{k+1}(1+t)$, $k\geqslant 1$;

(b) F does not satisfy the Whitney cube # -condition;

(c) F supports a (q,n)-Sobolev-Poincaré inequality for all $q\geqslant n$.

QHBC domains satisfy the Whitney cube # -condition [18, Corollary 1, p. 352]. Here we show that a φ-QHBC domain, in which the quasihyperbolic metric has only slightly faster growth than in QHBC domains, does not necessarily satisfy the Whitney cube # -condition. The domain will be a Cantor dust fractal domain in $\mathbf{R}^2$ having a φ-QHBC property with $\varphi(t)=\log^{k+1}(1+t)$, $k\geqslant 1$. In [10, Remark 7.18, p. 31] a special case has been studied. We generalize this and [7, Theorem 3.1, p. 3]. We need the following lemma, which tells us the quasihyperbolic length of the Euclidean line segment $[x,c]$.

LEMMA 4.1 ([7, Lemma 2.5, p. 3]). *Let $G=\mathbf{R}^n\backslash\{a,b\}$ where $a\neq b$. Let $c=(a+b)/2$, the line l be the perpendicular bisector of $[a,b]$, and $x\in l$. Then*

$$\int_{[x,c]}\frac{\mathrm{d}s}{\delta_G(z)}=\log\left(2\left(|x-c|+\sqrt{|a-b|^2/4+|x-c|^2}\right)\right)-\log|a-b|.$$

Construction of the domain

Let Q_0 be a closed square in the plane with side length 1 and centered at the origin. We make a Cantor construction in Q_0. Let y_n be the width of the strip taken away in the n^{th} case and x_n be the edge length of a cube left in the n^{th} case and E_n the union of cubes which are left in the n^{th} case. Define $G := \cap_{j=1}^{\infty} E_j \cap Q_0$ and set $F := B(0,2)\backslash G$, where $B(0,2)$ is an open ball centered at the origin and with a radius 2. Now F is a bounded domain in $\mathbf{R}^2$. We choose

$$x_n = \frac{2^{1-n}}{n^k + 2}, \quad y_n = \frac{2x_n\left(n^k - (n-1)^k\right)}{(n-1)^k + 2},$$

where $n, k \geqslant 1$. A calculation shows that $2x_n + y_n = x_{n-1}$, as it should be. Our choice for the fixed point of F is $z_0 = (0,0)$. We let Q_n denote the cube which is left in the n^{th} case and which lies in the upper right corner for every step $1, 2, ..., n$. Moreover, let z_n be the midpoint of Q_n.

(a) φ-QHBC property

By the geometry of F, it suffices to find an upper bound for the distance $k_F(z_n, z_0)$. This is because every x, that lies in the strips taken away from Q_n, can be connected to z_n by line segments. The case $x \in B(0,2)\backslash Q_0$ leaves x completely outside the Cantor construction, thus this situation is uninteresting and does not need any closer study. We connect z_n and z_0 by a curve which

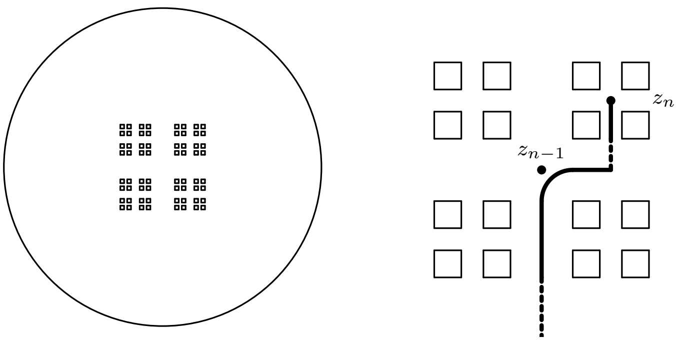

Fig. 1 – Cantor dust fractal domain. The left figure is from [7].

is partly presented in Figure 1. Note that there are circle arcs near the corner points. We estimate $k_F(z_n, z_0)$ in several parts. Consider first the dotted part of the curve in Q_{n-1}. We write $p_1 = y_{n+1}/2$ and $p_2 = p_1 + \bar{e}_2 \cdot y_n/2$. By Lemma

4.1, we obtain

$$\begin{aligned}\int_{[p_1,p_2]}\frac{ds}{\delta_F(z)} &= \log\left(y_n+\sqrt{y_{n+1}^2+y_n^2}\right)-\log y_{n+1}\leqslant \log 3y_n-\log y_{n+1}\\ &\leqslant \log\frac{6n^k\left((n+1)^k+2\right)}{(n-1)^k+2}\leqslant \log\left(3n^k\left((n+1)^k+2\right)\right)\\ &\leqslant \log\left(6(n+1)^{2k}\right)=2k\log(n+1)+\log 6.\end{aligned}$$

There are two line segments inside the cube Q_{n-1}. The longer line segment has the length x_n and the shorter one $x_n/2$. In both parts the distance to the boundary is at least $y_n/2$. Hence, for the part of these line segments, we have an upper bound

$$\frac{\frac{3}{2}x_n}{\frac{1}{2}y_n}=\frac{3\left((n-1)^k+2\right)}{2\left(n^k-(n-1)^k\right)}\leqslant 3\left((n-1)^k+2\right)$$

for the quasihyperbolic length. For the quarter of the circle inside the cube Q_{n-1}, the radius is $y_n/2$ and hence the quasihyperbolic length of this circle arc is

$$\frac{\frac{\pi}{2}\cdot\frac{1}{2}y_n}{\frac{1}{2}y_n}=\frac{\pi}{2}.$$

The first and the last part of our curve need extra attention. Inside the cube Q_n, there is the line segment which has the length $x_n/2$. The quasihyperbolic length of this part is less than

$$\frac{\frac{1}{2}x_n}{\frac{1}{2}y_{n+1}}=\frac{(n+1)^k+2}{(n+1)^k-n^k}\leqslant (n+1)^k+2.$$

On the other hand, close to z_0 there is the line segment with the length $1/3$. Since the distance to the boundary is at least $1/6$, this part of the curve has the quasihyperbolic length less than 2.

Now we collect our piecewise results, put them together and derive a sufficient estimate for $k_F(z_n,z_0)$. Two constants depending on k appear in the following chain of inequalities. They are $C_1=21+4k+2^k$ and $C_2=C_1 2^{2k+2}$. We obtain

$$\begin{aligned}k_F(z_n,z_0) &\leqslant 2+n(2k\log(n+1)+\log 6)\\ &\quad+(n-1)\left(3\left((n-1)^k+2\right)+\pi/2\right)+(n+1)^k+2\\ &\leqslant 2nk(n+1)+6n+3n\left((n-1)^k+2\right)+2n+(n+1)^k+4\\ &\leqslant 2n^2k+2nk+3n^{k+1}+14n+2^kn^k+4\leqslant C_1n^{k+1}.\end{aligned}\tag{4.2}$$

We continue by estimating

$$C_1 n^{k+1} \leqslant C_1(2n-1)^{k+1} = C_2 2^{-k-1}\left(n-\frac{1}{2}\right)^{k+1}. \tag{4.3}$$

Let us then consider the distance of z_n to the boundary. We have $\delta_F(z_n) = \sqrt{2}/2 \cdot y_{n+1}$, and therefore

$$\begin{aligned}\log\frac{1}{\delta_F(z_n)} &= \log\frac{\left((n+1)^k+2\right)(n^k+2)}{2^{\frac{1}{2}-n}\left((n+1)^k-n^k\right)}\\ &= \log 2^{n-\frac{1}{2}} + \log\frac{\left((n+1)^k+2\right)(n^k+2)}{(n+1)^k-n^k} \geqslant \frac{1}{2}\left(n-\frac{1}{2}\right).\end{aligned} \tag{4.4}$$

Combining (4.2), (4.3) and (4.4) leads to the conclusion that

$$k_F(z_n, z_0) \leqslant C_2 \log^{k+1}\frac{1}{\delta_F(z_n)}.$$

Let us now consider an arbitrary point x in a strip. A line segment connects x and w which is a point in the middle of the strip. Another line segment connects w and z_n. The calculations are as before. Thus F is a φ-QHBC domain with $\varphi(t) = \log^{k+1}(1+t)$.

(b) Whitney cube # -condition fails, and (c) Sobolev-Poincaré inequality holds.

The special case of this domain in [10, Remark 7.18, p. 31] does not satisfy the Whitney cube # -condition. This follows from [15, Corollary 4.3, p. 26] because the Hausdorff dimension of that domain is 2. Similar argument applies also in this generalization: The Hausdorff dimension remains the same because the exponential term still dominates in the edge length x_n and the number of cubes in the n^{th} case is unchanged at 4^n. Hence, F does not satisfy the Whitney cube # -condition.

By Example 3.8 (b) F is a (q,n)-Sobolev-Poincaré domain for all $q \geqslant n$.

5. MUSHROOM DOMAIN

Let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous strictly increasing function with the properties $\varphi(0) = 0$ and $\varphi(t)/\log(1+t)$ is non-decreasing. In this section we construct for every φ, that satisfies the previous properties, a bounded domain D in $\mathbf{R}^n, n \geqslant 2$, that has the following properties:

(a) D is a φ-QHBC domain;

(b) D satisfies the Whitney cube # -condition;

(c) D does not support a (q,p)-Sobolev-Poincaré inequality for any $1 \leqslant q < \infty$ and $1 \leqslant p < n$.

Applying this construction to $\varphi(t) := \log^{\beta}(1+t), \beta > 1$, we obtain a concrete domain that satisfies (a), (b) and (c) and which supports (q,n)-Sobolev-Poincaré inequality for all $q \geqslant n$ by Example 3.8 (b). Examples of mushrooms type domains can be found for example in [6], [5] , [14] , [16].

Construction of the domain

Let $Q_0 := [-1/2, 1/2]^n$ and let $r_m = 2^{-m}$. For $m = 1, 2, ...,$ let Q_m be a closed cube with side length $2r_m$ and P_m a closed rectangle which has side length $4\exp(-\varphi(r_m^{-1}))$ for one side and $2\exp(-\varphi(r_m^{-1}))$ for the remaining $n-1$ sides. We attach Q_m and P_m together so that the bottom face of P_m is contained in the boundary of Q_m and the top face of P_m is contained in the boundary of Q_0 that lies in the hyperplane $x_2 = -1/2$. All the cubes Q_m and Q_0 have to be pairwise disjoint. Let Q_m^* and P_m^* be the images of the sets Q_m and P_m, respectively, under a reflection across the hyperplane $x_2 = 0$. We set

$$D := \operatorname{int}\left(Q_0 \cup \bigcup_{m=1}^{\infty} (Q_m \cup P_m \cup Q_m^* \cup P_m^*)\right).$$

In other words, we have put similar "mushrooms" to the side of Q_0 that lies in the hyperplane $x_2 = 1/2$, see Figure 2 for the planar case.

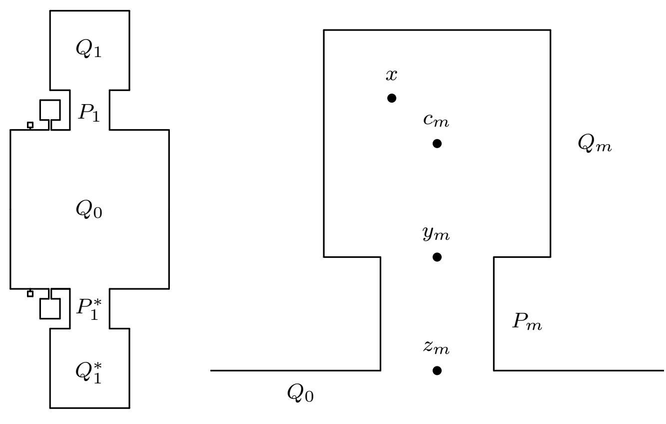

Fig. 2 – A mushroom domain. The left figure is from [5].

(a) φ-QHBC property

Next we will show that D is a φ-QHBC domain. For this end let y_m and z_m denote the midpoints of $Q_m \cap P_m$ and $P_m \cap Q_0$, respectively, and let c_m denote the midpoint of Q_m, see Figure 2. Our fixed point needed for the definition of φ-QHBC domain is c_0. By the triangle inequality, we have

$$k_D(c_m, c_0) \leqslant k_D(c_m, y_m) + k_D(y_m, z_m) + k_D(z_m, c_0). \tag{5.1}$$

The line segment $[y_m, z_m]$ is the optimal way to connect those points, and hence

$$k_D(y_m, z_m) = 4\exp(-\varphi(r_m^{-1})) \cdot (\exp(-\varphi(r_m^{-1})))^{-1} = 4. \tag{5.2}$$

To get the estimate for the distance $k_D(z_m, c_0)$, let $a \in [z_m, c_0]$ so that $|a - z_m| = \delta_D(z_m)$. Hence, $|z - z_m| \leqslant \delta_D(z_m)$ when $z \in [z_m, a]$. We have

$$k_D(z_m, c_0) \leqslant \int_{[z_m, c_0]} \frac{\mathrm{d}s}{\delta_D(z)} = \int_{[z_m, a]} \frac{\mathrm{d}s}{\delta_D(z)} + \int_{[a, c_0]} \frac{\mathrm{d}s}{\delta_D(z)}.$$

By elementary geometry, we see for the first term that

$$\int_{[z_m, a]} \frac{\mathrm{d}s}{\delta_D(z)} \leqslant \int_{[z_m, a]} \frac{\mathrm{d}s}{\frac{1}{2}\delta_D(z_m)} = \frac{2|a - z_m|}{\delta_D(z_m)} = 2,$$

and for the latter term

$$\begin{aligned}\int_{[a, c_0]} \frac{\mathrm{d}s}{\delta_D(z)} &\leqslant \int_{[a, c_0]} \frac{\mathrm{d}s}{\frac{1}{2}\ell([z_m, z])} = 2\int_{|a - z_m|}^{|c_0 - z_m|} \frac{\mathrm{d}t}{t} = 2\log\frac{|c_0 - z_m|}{|a - z_m|} \\ &\leqslant 2\log\frac{2}{\delta_D(z_m)} = 2\log\frac{1}{\delta_D(z_m)} + 2\log 2.\end{aligned}$$

Combining the upper bounds of the two terms we obtain

$$\begin{aligned}k_D(z_m, c_0) &\leqslant 2\log\frac{1}{\delta_D(z_m)} + 2\log 2 + 2 \\ &\leqslant 2\varphi(r_m^{-1}) + 2\log 2 + 2.\end{aligned} \tag{5.3}$$

The calculations for $k_D(c_m, y_m)$ are similar. Since $r_m = \delta_D(c_m)$, estimates (5.1)-(5.3) lead to the conclusion that

$$k_D(c_m, c_0) \leqslant C\varphi\left(\frac{1}{\delta_D(c_m)}\right) + C.$$

Next we will show that a growth condition like this holds for any point $x \in D$. Since a cube is a QHBC domain and $\varphi(t)/\log(1+t)$ is non-decreasing, the case $x \in Q_0 \cap D$ is already covered. Let $x \in Q_m \cap D$. Again, since cubes are QHBC domains and thus $k_D(x, c_m)$ satisfies a logarithmic growth condition, we have

$$\begin{aligned} k_D(x, c_0) &\leqslant k_D(x, c_m) + k_D(c_m, c_0) \leqslant k_D(x, c_m) + C\varphi\left(\frac{1}{\delta_D(c_m)}\right) + C \\ &\leqslant C \log \frac{1}{\delta_D(x)} + C\varphi\left(\frac{1}{\delta_D(x)}\right) + C \leqslant C\varphi\left(\frac{1}{\delta_D(x)}\right) + C. \end{aligned}$$

Suppose then $x \in P_m \cap D$. We connect x to the line segment $[y_m, z_m]$ by using the line segment $[x, u]$ perpendicular to the line segment $[y_m, z_m]$ so that $u \in [y_m, z_m]$. Now, since $k_D(x, u)$ satisfies a logarithmic growth condition, we have

$$\begin{aligned} k_D(x, c_0) &\leqslant k_D(x, u) + k_D(u, c_0) \leqslant k_D(x, u) + k_D(c_m, c_0) \\ &\leqslant k_D(x, u) + C\varphi\left(\frac{1}{\delta_D(c_m)}\right) + C \\ &\leqslant k_D(x, u) + C\varphi\left(\frac{1}{\delta_D(x)}\right) + C \leqslant C\varphi\left(\frac{1}{\delta_D(x)}\right) + C. \end{aligned}$$

By the symmetry, all the possible cases are covered, and thus the domain D is a φ-QHBC domain.

(b) Whitney cube # -property

Let $\mathcal{W}$ be a Whitney decomposition of D. We need to estimate how many cubes there are in the set $\mathcal{W}^j := \{Q \in \mathcal{W} : \mathrm{dia}(Q) = 2^{-j}\}$. In Q_0 of the mushroom domain construction there are at most $C(n)2^{j(n-1)}$ of those cubes. Here, it is good to notice that certain size cubes are in annulus defined by property (2) of the Whitney decomposition. To the room Q_m (or Q_m^*) that size cube can fit only if $2^{-j} \leqslant r_m = 2^{-m}$. So only when $m \leqslant j$ it is possible that cubes from $\mathcal{W}^j$ fit in the rooms. Thus in $\cup_m Q_m$ (and similarly in $\cup_m Q_m^*$) there are at most $C(n)j2^{j(n-1)}$ cubes from $\mathcal{W}^j$. Since P_m's and P_m^*'s are even smaller, the same upper bound holds for them i.e. there are at most $C(n)j2^{j(n-1)}$ cubes from $\mathcal{W}^j$ in $\cup_m P_m$ (and similarly in $\cup_m P_m^*$). Thus we have Whitney cube # -property for D since

$$\#\mathcal{W}^j \leqslant C(n)j2^{j(n-1)} \leqslant C(n,\lambda)2^{\lambda j} \text{ for any } \lambda > n-1.$$

(c) The Sobolev-Poincaré inequality does not hold.

Let us now show that D does not support a (q,p)-Sobolev-Poincaré inequality for any $1 \leqslant p < n$ and $1 \leqslant q < \infty$ provided that φ grows faster than logarithm i.e $\frac{\varphi(t)}{\log(t)} \to \infty$ as $t \to \infty$. For this end, we define a sequence of piecewise linear continuous functions. For $k = 1, 2, ...,$ we set

$$u_k(x) := \begin{cases} v_k & \text{in } Q_k; \\ -v_k & \text{in } Q_k^*; \\ 0 & \text{in } D\backslash(P_k \cup Q_k \cup P_k^* \cup Q_k^*), \end{cases}$$

where $v_k := 2^{\frac{2p-n-2}{p}} e^{\frac{n-p}{p}\varphi(r_k^{-1})}$, and u_k is defined linearly in sets P_k and P_k^* so that it is continuous. We have

$$\begin{aligned} \int_D |\nabla u_k(x)|^p \mathrm{d}x &= \int_{P_k \cup P_k^*} \left(\frac{v_k}{4e^{-\varphi(r_k^{-1})}} \right)^p \mathrm{d}x \\ &= 2 \cdot 4e^{-\varphi(r_k^{-1})} \cdot \left(2e^{-\varphi(r_k^{-1})}\right)^{n-1} \cdot \frac{2^{2p-n-2} e^{(n-p)\varphi(r_k^{-1})}}{4^p e^{-p\varphi(r_k^{-1})}} = 1 \end{aligned}$$

for every k. In addition, since the integral average of u_k over D is zero, we estimate

$$\begin{aligned} \left(\int_D |u_k(x) - (u_k)_D|^q \mathrm{d}x \right)^{1/q} &\geqslant \left(\int_{Q_k} |u_k(x)|^q \mathrm{d}x \right)^{1/q} \\ &= \left((2r_k)^n \cdot \left(2^{\frac{2p-n-2}{p}} e^{\frac{n-p}{p}\varphi(r_k^{-1})} \right)^q \right)^{1/q} \\ &= 2^{\frac{2p-n-2}{p} + \frac{n}{q}} e^{\frac{n-p}{p}\varphi(r_k^{-1})} \cdot r_k^{n/q} \to \infty, \end{aligned}$$

as $k \to \infty$, whenever $1 \leqslant p < n$ and $1 \leqslant q < \infty$, since our assumption was that $\varphi(t)$ has faster growth than logarithm when t is large.

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