*Dedicated to the memory of Cabiria Andreian Cazacu*

# BESOV-MORREY SPACES AND DIFFERENCES

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*Communicated by Peter Hästö*

We study the Besov-Morrey spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ and show that under certain conditions on the parameters these spaces can be characterised in terms of higher-order differences. Furthermore we prove that some of the mentioned conditions are also necessary

*AMS 2010 Subject Classification:* 46E35.

*Key words:* Besov space, Morrey space, Besov-Morrey space, higher-order differences.

## 1. INTRODUCTION AND MAIN RESULTS

Nowadays the Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ are a well-established tool to describe the regularity of functions and distributions. These function spaces have been introduced by Nikol'skij and Besov between 1951 and 1961, see [12] and [1]. Later the spaces $B^s_{p,q}(\mathbb{R}^d)$ have been investigated in detail in the famous books of Triebel, see [20] for example. In the recent years a growing number of authors worked with a generalisation of the Besov spaces where the $L_p(\mathbb{R}^d)$ - quasinorm was replaced by a Morrey - quasinorm. These function spaces are called Besov-Morrey spaces and have the symbol $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ with $0 < p \leq u < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. The Besov-Morrey spaces have been introduced by Kozono and Yamazaki in 1994, see [10]. This paper has two main goals. The first one is to prove an equivalent characterisation in terms of higher-order differences for the spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$. More exactly we will answer the question under which restrictions on the parameters $s, u, p, q$ and $d$ the spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ can be described by using only $\Delta^N_h f(x)$. When we solve this problem we will obtain some sufficient conditions concerning the parameter $s$. Because of this our second main goal is to investigate whether these conditions are also necessary. For the original Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ characterisations in terms of differences are known since many years. So for example a result concerning that topic can be found in a well-known book of Triebel from 1983, see chapter 2.5.12. in [20]. For the more general Besov-Morrey spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ we can prove the following.

THEOREM 1. *Let $0 < p \le u < \infty$ and $0 < q \le \infty$. Let $0 < v \le \infty$ and $N \in \mathbb{N}$. We assume $d \max(0, \frac{1}{p} - 1, \frac{1}{p} - \frac{1}{v}) < s < N$. Then a function $f \in L_p^{loc}(\mathbb{R}^d)$ belongs to $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ if and only if $f \in L_v^{loc}(\mathbb{R}^d)$ and*

$$\underbrace{\|f|\mathcal{M}^u_p(\mathbb{R}^d)\| + \Big( \int_0^\infty t^{-sq-d\frac{q}{v}} \Big\| \Big( \int_{B(0,t)} |\Delta_h^N f(x)|^v dh \Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\|^q \frac{dt}{t} \Big)^{\frac{1}{q}}}_{\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(v,\infty)} :=} < \infty$$

*with modifications if $q = \infty$ and/or $v = \infty$. The quasinorms $\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|$ and $\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(v,\infty)}$ are equivalent for $f \in L_p^{loc}(\mathbb{R}^d)$.*

As already mentioned the second main goal of this paper is to investigate whether the conditions concerning the parameter $s$ that you can find in Theorem 1 are also necessary. For that purpose by $\mathbf{N}^{s,N,\infty}_{u,p,q,v}(\mathbb{R}^d)$ we define the collection of all $f \in L^{loc}_{\max(p,v)}(\mathbb{R}^d)$ such that $\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(v,\infty)}$ is finite. Using this notation we can formulate the following Theorem which is our second main result.

THEOREM 2. *Let $s \in \mathbb{R}$, $0 < p \le u < \infty$, $0 < q \le \infty$, $0 < v \le \infty$ and $N \in \mathbb{N}$. Then we have $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{N}^{s,N,\infty}_{u,p,q,v}(\mathbb{R}^d)$ if we are in one of the following cases.*

(i) *We have $s \le 0$.*

(ii) *We have $0 < p < 1$ and either $s < d\frac{p}{u}(\frac{1}{p} - 1)$ or $s = d\frac{p}{u}(\frac{1}{p} - 1)$ with $q > 1$.*

(iii) *We have $s < d\frac{p}{u}(\frac{1}{p} - \frac{1}{v})$ with $0 < p < v < \infty$.*

(iv) *It is either $N < s$ or $N = s$ with $q \neq \infty$ or $N = s$ with $q = \infty$, $u = p$ and $v \ge 1$.*

If you compare this result with Theorem 1 it turns out that there are still some open questions. Let us look at the special case $v = 1$ and $0 < p < 1$. Then for $d\frac{p}{u}(\frac{1}{p} - 1) < s \le d(\frac{1}{p} - 1)$ it is not clear whether it is possible to describe the spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ in terms of differences. At least we know that there is the continuous embedding $\mathbf{N}^{s,N,\infty}_{u,p,q,1}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$, see Proposition 4. Notice that in the special case $p = u$ we recover the original Besov spaces $B^s_{p,q}(\mathbb{R}^d)$. Here the gap we just described disappears.

In this article we will use the following notation. As usual $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0$ the natural numbers including 0, $\mathbb{Z}$ the integers and $\mathbb{R}$ the real numbers. $\mathbb{R}^d$ denotes the $d$-dimensional Euclidean space. For $x \in \mathbb{R}^d$ and $t > 0$ we put $B(x,t) := \{y \in \mathbb{R}^d : \quad |x - y| < t\}$. For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$

we define the dyadic cube $Q_{j,k} = 2^{-j}([0,1)^d + k)$. By $\chi_{j,k}$ we denote the characteristic function of the cube $Q_{j,k}$. For $0 < u < \infty$ we put $\chi_{j,k}^{(u)} = 2^{\frac{jd}{u}} \chi_{j,k}$. Let $\mathcal{S}(\mathbb{R}^d)$ be the collection of all Schwartz functions on $\mathbb{R}^d$ endowed with the usual topology and denote by $\mathcal{S}'(\mathbb{R}^d)$ its topological dual. The symbol $\mathcal{F}$ refers to the Fourier transform, $\mathcal{F}^{-1}$ to its inverse transform, both defined on $\mathcal{S}'(\mathbb{R}^d)$. By $C_0^\infty(\mathbb{R}^d)$ we mean the set of all infinitely often differentiable functions on $\mathbb{R}^d$ with compact support. For two quasi-Banach spaces $X$ and $Y$ we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of $X$ into $Y$ is continuous. For all $p \in (0,\infty)$ we write $\sigma_p := d \max(0, \frac{1}{p} - 1)$. The symbols $C, C_1, c, c_1 \ldots$ denote positive constants that depend only on the fixed parameters $d, s, u, p, q$ and probably on auxiliary functions. In this paper one important tool will be differences of higher order. Let $f : \mathbb{R}^d \to \mathbb{C}$ be a function. For $x, h \in \mathbb{R}^d$ we define the difference of the first order by $\Delta_h^1 f(x) := f(x+h) - f(x)$. For $N \in \mathbb{N}$ we define the difference of order $N$ by $\Delta_h^N f(x) := (\Delta_h^1(\Delta_h^{N-1} f))(x)$.

## 2. DEFINITION AND BASIC PROPERTIES OF BESOV-MORREY SPACES

The Besov-Morrey spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ are function spaces that are built upon Morrey spaces. Because of this at first we want to recall the definition of the Morrey spaces $\mathcal{M}^u_p(\mathbb{R}^d)$.

*Definition* 1. Let $0 < p \le u < \infty$. Then the Morrey space $\mathcal{M}^u_p(\mathbb{R}^d)$ is defined to be the set of all functions $f \in L_p^{loc}(\mathbb{R}^d)$ such that

$$\|f|\mathcal{M}^u_p(\mathbb{R}^d)\| := \sup_{y\in\mathbb{R}^d, r>0} |B(y,r)|^{\frac{1}{u}-\frac{1}{p}} \Big( \int_{B(y,r)} |f(x)|^p dx \Big)^{\frac{1}{p}} < \infty.$$

The Morrey spaces $\mathcal{M}^u_p(\mathbb{R}^d)$ are quasi-Banach spaces and Banach spaces for $p \ge 1$. They have many connections to the Lebesgue spaces $L_p(\mathbb{R}^d)$. So for $p \in (0,\infty)$ we have $\mathcal{M}^p_p(\mathbb{R}^d) = L_p(\mathbb{R}^d)$. Moreover for $0 < p_2 \le p_1 \le u < \infty$ we have $L_u(\mathbb{R}^d) = \mathcal{M}^u_u(\mathbb{R}^d) \hookrightarrow \mathcal{M}^u_{p_1}(\mathbb{R}^d) \hookrightarrow \mathcal{M}^u_{p_2}(\mathbb{R}^d)$. To define the spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ we need a so-called smooth dyadic decomposition of the unity. Let $\varphi_0 \in C_0^\infty(\mathbb{R}^d)$ be a non-negative function such that $\varphi_0(x) = 1$ if $|x| \le 1$ and $\varphi_0(x) = 0$ if $|x| \ge \frac{3}{2}$. For $k \in \mathbb{N}$ we define $\varphi_k(x) := \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x)$. Because of $\sum_{k=0}^\infty \varphi_k(x) = 1$ and $\operatorname{supp} \varphi_k \subset \big\{ x \in \mathbb{R}^d : \; 2^{k-1} \le |x| \le 3 \cdot 2^{k-1} \big\}$ for every $k \in \mathbb{N}$ we call the system $(\varphi_k)_{k\in\mathbb{N}_0}$ a smooth dyadic decomposition of the unity on $\mathbb{R}^d$. For $k \in \mathbb{N}_0$, because of the Paley-Wiener-Schwarz theorem, $\mathcal{F}^{-1}[\varphi_k \mathcal{F} f]$ is a smooth function for all $f \in \mathcal{S}'(\mathbb{R}^d)$.

*Definition* 2. Let $0 < p \le u < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. $(\varphi_k)_{k\in\mathbb{N}_0}$ is a smooth dyadic decomposition of the unity. Then the Besov-Morrey space $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ is defined to be the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\| := \Big(\sum_{k=0}^{\infty} 2^{ksq}\|\mathcal{F}^{-1}[\varphi_k\mathcal{F}f]|\mathcal{M}^u_p(\mathbb{R}^d)\|^q\Big)^{\frac{1}{q}} < \infty.$$

In the case $q = \infty$ the usual modifications are made.

In what follows we want to collect some basic properties of the Besov-Morrey spaces. Most of them will be used later.

Lemma 1. *Let $0 < p \le u < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Then the following assertions are true.*

- (i) *The spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ are independent of the chosen smooth dyadic decomposition of unity in the sense of equivalent quasi-norms.*
- (ii) *The spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ are quasi-Banach spaces. For $p \ge 1$ and $q \ge 1$ they are Banach spaces.*
- (iii) *We have $\|f+g|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^\tau \le \|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^\tau + \|g|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^\tau$ for all $f, g \in \mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ and $\tau = \min(1,p,q)$.*
- (iv) *It holds $\mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$.*
- (v) *We have $\mathcal{N}^s_{p,p,q}(\mathbb{R}^d) = B^s_{p,q}(\mathbb{R}^d)$.*

*Proof.* (i) was proved in [19], see Theorem 2.8. The proofs of (ii) and (iii) are standard, see Corollary 2.6. in [10]. (iv) was proved in [16], see Theorem 3.2. (v) is obvious, see Proposition 3.6. in [16]. □

It is interesting to know that in some cases the spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ also contain singular distributions. The following result is a combination of Theorem 3.3 and Theorem 3.4. from [3] and Theorem 3.4. from [4].

Lemma 2. *Let $s \in \mathbb{R}$, $0 < p \le u < \infty$ and $0 < q \le \infty$. Then $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \not\subset L^{loc}_1(\mathbb{R}^d)$ if and only if we have either $s < \frac{p}{u}\sigma_p$ or $s = \frac{p}{u}\sigma_p$ with $q > \min(\max(p,1),2)$.*

The following inequality is a very important tool. Let $M$ denote the Hardy-Littlewood maximal operator. Then in [2] there is the following result.

Lemma 3. *Let $1 < p \le u < \infty$. $f$ is a locally Lebesgue-integrable function on $\mathbb{R}^d$. Then there is a constant $C > 0$ independent of $f$ such that*

$$\|Mf|\mathcal{M}^u_p(\mathbb{R}^d)\| \le C\|f|\mathcal{M}^u_p(\mathbb{R}^d)\|.$$

It is also possible to define the Besov-Morrey spaces on domains. Here we want to work with smooth and bounded domains only. We use the following definition, whereby $\mathcal{D}'(\Omega)$ denotes the space of distributions on $\Omega \subset \mathbb{R}^d$ as usual.

*Definition* 3. Let $0 < p \le u < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Let $\Omega \subset \mathbb{R}^d$ be a bounded $C^\infty$ domain. Then we define

$$\mathcal{N}^s_{u,p,q}(\Omega) = \left\{ f \in \mathcal{D}'(\Omega) \ : \ f = g \text{ in } \ \Omega \ \text{ for some } g \in \mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \right\}$$

and put $\|f|\mathcal{N}^s_{u,p,q}(\Omega)\| = \inf\{\|g|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\| \ : \ f = g \text{ in } \ \Omega \ \text{ for } g \in \mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\}$.

For our purposes the following result of Haroske and Skrzypczak that can be found in [6] is of special interest.

Lemma 4. *Let $0 < p \le u < \infty$, $0 < q \le \infty$ and $1 \le v < \infty$ with $p < v$. Let $s \in \mathbb{R}$. Then the embedding $\mathcal{N}^s_{u,p,q}(\Omega) \hookrightarrow L_v(\Omega)$ implies $s \ge d\frac{p}{u}(\frac{1}{p} - \frac{1}{v})$.*

*Proof.* This result can be found in [6], see Proposition 5.3. The special case $p = u$ can be found in Corollary 2 of chapter 2.2.4 in [15], see also [18]. □

## 3. A CHARACTERISATION OF BESOV-MORREY SPACES IN TERMS OF DIFFERENCES

In this chapter we want to prove characterisations of the Besov-Morrey spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ that only use differences. For that purpose we want to use some ideas from Hedberg and Netrusov, see [7]. They developed an abstract theory to describe different function spaces. It turns out that the spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ fit into this theory.

Proposition 1. *Let $0 < p \le u < \infty$ and $0 < q \le \infty$. Let $0 < v \le \infty$ and $s > d\max(0, \frac{1}{p} - 1, \frac{1}{p} - \frac{1}{v})$. Let $N \in \mathbb{N}$ with $N > s$. Then a function $f \in L^{loc}_p(\mathbb{R}^d)$ belongs to $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ if and only if $f \in L^{loc}_v(\mathbb{R}^d)$ and*

$$\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\clubsuit)} := \Bigg( \Big\| \Big( \int_{B(x,1)} |f(y)|^v dy \Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\|^q + \sum_{j=1}^{\infty} 2^{jq(s+\frac{d}{v})} \Big\| \Big( \int_{B(0,2^{-j})} |\Delta^N_h f(x)|^v dh \Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\|^q \Bigg)^{\frac{1}{q}}$$

*is finite. In the cases $q = \infty$ and/or $v = \infty$ the usual modifications are made. The quasinorms $\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|$ and $\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\clubsuit)}$ are equivalent for $f \in L^{loc}_p(\mathbb{R}^d)$.*

*Proof.* To prove this result we proceed like it is described in chapter 3 of [8]. Here a characterisation in terms of differences for the Triebel-Lizorkin-Morrey spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ is proved. For $s \in \mathbb{R}$, $0 < p \leq u < \infty$ and $0 < q \leq \infty$ we define

$$\ell^s_q(\mathcal{M}^u_p(\mathbb{R}^d)) = \Big\{ \{f_j\}_{j=0}^{\infty} : \|\{f_j\}_{j=0}^{\infty}\|_{\ell^s_q(\mathcal{M}^u_p(\mathbb{R}^d))} := \Big( \sum_{j=0}^{\infty} 2^{jsq} \|f_j|\mathcal{M}^u_p(\mathbb{R}^d)\|^q \Big)^{\frac{1}{q}} < \infty \Big\}$$

with modifications if $q = \infty$. It is not difficult to see that under some restrictions on the parameters the pair $(\ell^s_q(\mathcal{M}^u_p(\mathbb{R}^d)), \|\cdot\|_{\ell^s_q(\mathcal{M}^u_p(\mathbb{R}^d))})$ has all the properties that are written down in Definition 1.1.1. from [7]. The main tool to prove this is Lemma 3. Next we investigate the space $Y(\ell^s_q(\mathcal{M}^u_p(\mathbb{R}^d)))$, see Definition 1.1.6. from [7]. We can use Theorem 2.4. from [16] or Theorem 2.7 from [19] to prove $Y(\ell^s_q(\mathcal{M}^u_p(\mathbb{R}^d))) = \mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$. Now we apply Proposition 1.1.12. and Theorem 1.1.14. from [7] to obtain the desired result. Notice that a result similar to Proposition 1 also can be found in [21], see Corollary 4.12. □

*Remark* 4. In the formulation of Proposition 1 it is possible to replace the quasinorm $\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\clubsuit)}$ by (with the usual modifications in the case $q = \infty$)

$$\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)} = \Big( \|f|\mathcal{M}^u_p(\mathbb{R}^d)\|^q + \sum_{j=1}^{\infty} 2^{jq(s+\frac{d}{v})} \Big\| \Big( \int_{B(0,2^{-j})} |\Delta^N_h f(x)|^v dh \Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\|^q \Big)^{\frac{1}{q}}.$$

To prove this we can use the ideas from Remark 3.1. in [8].

THEOREM 3. *Let $0 < p \leq u < \infty$ and $0 < q \leq \infty$. Let $0 < v \leq \infty$, $1 \leq a \leq \infty$ and $N \in \mathbb{N}$. We assume $d \max(0, \frac{1}{p} - 1, \frac{1}{p} - \frac{1}{v}) < s < N$. Then a function $f \in L^{loc}_p(\mathbb{R}^d)$ belongs to $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ if and only if $f \in L^{loc}_v(\mathbb{R}^d)$ and*

$$\underbrace{\|f|\mathcal{M}^u_p(\mathbb{R}^d)\| + \Big( \int_0^a t^{-sq-d\frac{q}{v}} \Big\| \Big( \int_{B(0,t)} |\Delta^N_h f(x)|^v dh \Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\|^q \frac{dt}{t} \Big)^{\frac{1}{q}}}_{\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(v,a)} :=} < \infty$$

*with modifications if $q = \infty$ and/or $v = \infty$. The quasinorms $\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|$ and $\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(v,a)}$ are equivalent for $f \in L^{loc}_p(\mathbb{R}^d)$.*

*Remark* 5. Let $0 < v \leq \infty$ and $1 \leq a \leq \infty$. Then the letters $v$ and $a$ in the abbreviation $(v, a)$ indicate the dependence of the concrete quasinorm $\|\cdot|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(v,a)}$ on these parameters.

*Proof.* To prove Theorem 3 we can proceed as in [8], see the proofs of Proposition 4.1 and Theorem 6. Only some minor modifications have to be made.

*Step* 1. At first we will deal with the case $a = 1$.

*Substep* 1.1. We prove that there is a constant $C > 0$ independent of $f \in L_p^{loc}(\mathbb{R}^d)$ such that $\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)} \le C\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(v,1)}$. But this is just a consequence of the monotonicity of $\int_{B(0,t)} |\Delta_h^N f(x)|^v dh$ in $t$.

*Substep* 1.2. Next we will prove that for $f \in \mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ there is a constant $C > 0$ independent of $f$ such that $\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(v,1)} \le C\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)}$. To prove this again at first we use the monotonicity of $\int_{B(0,t)} |\Delta_h^N f(x)|^v dh$ in $t$. Moreover we apply the identity

$$\Delta_h^N f(x) = \sum_{k=0}^{N} (-1)^{N-k} \binom{N}{k} f(x+kh). \tag{1}$$

Then we obtain (with the usual modifications in the case $q = \infty$ and/or $v = \infty$)

$$\begin{aligned}
&\Big(\int_0^1 t^{-sq-d\frac{q}{v}} \Big\| \Big(\int_{B(0,t)} |\Delta_h^N f(x)|^v dh\Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d)\Big\|^q \frac{dt}{t}\Big)^{\frac{1}{q}} \\
&\qquad \le C_1 \|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)} + C_1 \Big\| \Big(\int_{B(0,1)} |\Delta_h^N f(x)|^v dh\Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d)\Big\| \\
&\qquad \le C_2 \|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)} + C_2 \Big\| \Big(\int_{B(x,N)} |f(z)|^v dz\Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d)\Big\|.
\end{aligned}$$

Next we cover the ball $B(x,N)$ with $(2N+1)^d$ small balls with radius one and use the translation-invariance of the Morrey spaces. Then Proposition 1 leads to

$$\begin{aligned}
\Big\| \Big(\int_{B(x,N)} |f(z)|^v dz\Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d)\Big\| &\le C_3 \Big\| \Big(\int_{B(x,1)} |f(z)|^v dz\Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d)\Big\| \\
&\le C_4 \|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\spadesuit)}.
\end{aligned}$$

*Step* 2. Now we will deal with the case $a = \infty$.

*Substep* 2.1. Here at first we prove that there is a constant $C > 0$ independent of $f \in L_p^{loc}(\mathbb{R}^d)$ such that $\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(v,1)} \le C\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(v,\infty)}$. But of course this is obvious.

*Substep* 2.2. Next we will prove that for $f \in \mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ there is a constant $C > 0$ independent of $f$ such that $\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(v,\infty)} \le C\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(v,1)}$. At first because of the monotonicity of $\int_{B(0,t)} |\Delta_h^N f(x)|^v dh$ in $t$ and formula (1) we obtain (with modifications in the case $q = \infty$ and/or $v = \infty$)

$$\Big(\int_1^\infty t^{-sq-d\frac{q}{v}} \Big\| \Big(\int_{B(0,t)} |\Delta_h^N f(x)|^v dh\Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d)\Big\|^q \frac{dt}{t}\Big)^{\frac{1}{q}} \le C_1 \|f|\mathcal{M}^u_p(\mathbb{R}^d)\|$$

$$+ C_1\Big(\sum_{j=1}^{\infty} 2^{-jsq} 2^{-jd\frac{q}{v}} \Big\| \Big( \int_{B(x,N2^j)} |f(z)|^v dz \Big)^{\frac{1}{v}} \Big| \mathcal{M}_p^u(\mathbb{R}^d) \Big\|^q \Big)^{\frac{1}{q}}.$$

Now we want to cover the ball $B(x, N2^j)$ with $(2N \cdot 2^j + 1)^d$ small balls with radius one. Let $i \in \{1, 2, \ldots, (2N \cdot 2^j + 1)^d\}$ and $w_i$ appropriate displacement vectors such that $\bigcup_{i=1}^{(2N\cdot 2^j+1)^d} B(x + w_i, 1) \supset B(x, N2^j)$. Then we get

$$\Big(\sum_{j=1}^{\infty} 2^{-jsq} 2^{-jd\frac{q}{v}} \Big\| \Big( \int_{B(x,N2^j)} |f(z)|^v dz \Big)^{\frac{1}{v}} \Big| \mathcal{M}_p^u(\mathbb{R}^d) \Big\|^q \Big)^{\frac{1}{q}}$$
$$\leq \Big(\sum_{j=1}^{\infty} 2^{-jsq} 2^{-jd\frac{q}{v}} \Big\| \Big( \sum_{i=1}^{(2N\cdot 2^j+1)^d} \int_{B(x+w_i,1)} |f(z)|^v dz \Big)^{\frac{1}{v}} \Big| \mathcal{M}_p^u(\mathbb{R}^d) \Big\|^q \Big)^{\frac{1}{q}}.$$

We put $\mu = \min(p, v)$ and remember that we have $s > d \max(0, \frac{1}{p} - \frac{1}{v})$. Then the translation-invariance of the Morrey spaces in combination with Proposition 1 and step 1 of this proof leads to

$$\Big(\sum_{j=1}^{\infty} 2^{-jsq} 2^{-jd\frac{q}{v}} \Big\| \Big( \sum_{i=1}^{(2N\cdot 2^j+1)^d} \int_{B(x+w_i,1)} |f(z)|^v dz \Big)^{\frac{1}{v}} \Big| \mathcal{M}_p^u(\mathbb{R}^d) \Big\|^q \Big)^{\frac{1}{q}}$$
$$\leq C_2\Big(\sum_{j=1}^{\infty} 2^{-jsq} 2^{-jd\frac{q}{v}} 2^{jd\frac{q}{\mu}} \Big\| \Big( \int_{B(x,1)} |f(z)|^v dz \Big)^{\frac{1}{v}} \Big| \mathcal{M}_p^u(\mathbb{R}^d) \Big\|^q \Big)^{\frac{1}{q}}$$
$$\leq C_3 \Big\| \Big( \int_{B(x,1)} |f(z)|^v dz \Big)^{\frac{1}{v}} \Big| \mathcal{M}_p^u(\mathbb{R}^d) \Big\| \leq C_4 \|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(v,1)}.$$

*Step* 3. At last we look at the case $1 < a < \infty$. But here the proof is just a simple consequence of the things we did before. □

Now it is really easy to prove Theorem 1.

*Proof of Theorem 1.* To prove Theorem 1 we just have to use Theorem 3 with $a = \infty$. Then we get exactly what we want. □

It is also possible to describe the Besov-Morrey spaces by a generalisation of a modulus of smoothness.

THEOREM 4. *Let* $0 < p \leq u < \infty$, $0 < q \leq \infty$ *and* $s > d \max(0, \frac{1}{p} - 1)$. *Let* $N \in \mathbb{N}$ *with* $N > s$. *Then a function* $f \in L^{loc}_{\max(1,p)}(\mathbb{R}^d)$ *belongs to* $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ *if and only if*

$$\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\omega)} := \|f|\mathcal{M}_p^u(\mathbb{R}^d)\|$$
$$+ \Big( \int_0^{\infty} t^{-sq} \Big[ \sup_{|h|\leq t} \|\Delta_h^N f|\mathcal{M}_p^u(\mathbb{R}^d)\| \Big]^q \frac{dt}{t} \Big)^{\frac{1}{q}} < \infty.$$

*The quasinorms* $\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|$ *and* $\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\omega)}$ *are equivalent for* $f \in L^{loc}_{\max(1,p)}(\mathbb{R}^d)$. *In the case* $q = \infty$ *the usual modifications have to be made.*

*Proof. Step* 1. At first we prove that for $f \in L^{loc}_{\max(1,p)}(\mathbb{R}^d)$ there is a constant $C > 0$ independent of $f$ such that $\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\| \le C\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\omega)}$. To prove this because of $f \in L^{loc}_{\max(1,p)}(\mathbb{R}^d) \subset L^{loc}_p(\mathbb{R}^d)$ and $s > d\max(0, \frac{1}{p} - 1)$ we can apply Theorem 3 with $a = \infty$ and $v = p$. Now we can use Fubini's theorem and obtain the desired result.

*Step* 2. Now we prove that for $f \in L^{loc}_{\max(1,p)}(\mathbb{R}^d) \cap \mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ there is a constant $C > 0$ independent of $f$ such that $\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(\omega)} \le C\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|$. At first we can apply Theorem 3 with $v = p$ and get

$$\|f|\mathcal{M}^u_p(\mathbb{R}^d)\| \le \|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(p,\infty)} \le C_1\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|.$$

To deal with the other term in the following we will use some ideas from Triebel, see chapter 2.5.11. in [20]. We transform the integral concerning $t$ into a sum. Then we obtain

$$\Big(\int_0^\infty t^{-sq}\Big[\sup_{|h|\le t}\|\Delta^N_h f|\mathcal{M}^u_p(\mathbb{R}^d)\|\Big]^q\frac{dt}{t}\Big)^{\frac{1}{q}} \le C_1\Big(\sum_{k=-\infty}^{\infty}2^{ksq}\Big[\sup_{|h|\le 2^{-k}}\|\Delta^N_h f|\mathcal{M}^u_p(\mathbb{R}^d)\|\Big]^q\Big)^{\frac{1}{q}}$$

with modifications in the case $q = \infty$. Now let $(\varphi_j)_{j\in\mathbb{N}_0}$ be a smooth dyadic decomposition of the unity. We put $\varphi_j = 0$ for $j < 0$. Then because of $s > d\max(0, \frac{1}{p} - 1)$ and $f \in \mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ for every $k \in \mathbb{Z}$ we have

$$f = \sum_{m=-\infty}^{\infty}\mathcal{F}^{-1}[\varphi_{k+m}\mathcal{F}f]$$

with convergence not only in $\mathcal{S}'(\mathbb{R}^d)$ but also in $\mathcal{M}^u_p(\mathbb{R}^d)$. Let $\tau = \min(1,p,q)$. We get

$$\Big(\sum_{k=-\infty}^{\infty}2^{ksq}\Big[\sup_{|h|\le 2^{-k}}\|\Delta^N_h f|\mathcal{M}^u_p(\mathbb{R}^d)\|\Big]^q\Big)^{\frac{\tau}{q}} \le \sum_{m=-\infty}^{\infty}\Big(\sum_{k=-\infty}^{\infty}2^{ksq}\Big[\sup_{|h|\le 2^{-k}}\Big\|\Delta^N_h\mathcal{F}^{-1}[\varphi_{k+m}\mathcal{F}f]\Big|\mathcal{M}^u_p(\mathbb{R}^d)\Big\|\Big]^q\Big)^{\frac{\tau}{q}}.$$

Next we split up the outer sum concerning $m$. We have $\sum_{m=-\infty}^{\infty} = \sum_{m=-\infty}^{-1} + \sum_{m=0}^{\infty}$. In what follows at first we will deal with the case $m < 0$.

Here we have to start with some preliminary considerations. For every $|h| \le 1$ and $x \in \mathbb{R}^d$ there is a constant $C_2 > 0$ independent of $f$ and $x$ such that

$$|(\Delta^N_{2^{-k}h}\mathcal{F}^{-1}[\varphi_{k+m}\mathcal{F}f])(x)| \le C_2 2^{-kN} \sup_{|x-y|\le N2^{-k}} \sum_{|\alpha|=N} |(D^\alpha \mathcal{F}^{-1}[\varphi_{k+m}\mathcal{F}f])(y)|,$$

see formula (6) in chapter 2.5.10. of [20]. Moreover for $j \in \mathbb{Z}$ and $a > 0$ we will define the function

$$(\varphi_j^* f)(x) = \sup_{y\in\mathbb{R}^d} \frac{|(\mathcal{F}^{-1}[\varphi_j \mathcal{F}f])(x-y)|}{1 + (2^{j+2}|y|)^a}.$$

Notice that for $j < 0$ because of $\varphi_j = 0$ we also have $\varphi_j^* f = 0$. Then for $|\alpha| = N$ and $y \in \mathbb{R}^d$ there is a constant $C_3 > 0$ independent of $f$ and $y$ such that

$$|(D^\alpha \mathcal{F}^{-1}[\varphi_{k+m}\mathcal{F}f])(y)| \le C_3 2^{(k+m)N} (\varphi^*_{k+m} f)(y),$$

see formula (7) in chapter 2.5.10. of [20]. When we use this estimates because of the properties of the function $\varphi_j^* f$ we find (with modifications in the case $q = \infty$)

$$\begin{aligned}
&\sum_{m=-\infty}^{-1} \Big( \sum_{k=-\infty}^{\infty} 2^{ksq} \Big[ \sup_{|h|\le 2^{-k}} \Big\| \Delta^N_h \mathcal{F}^{-1}[\varphi_{k+m}\mathcal{F}f] \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\| \Big]^q \Big)^{\frac{\tau}{q}} \\
&\quad \le C_4 \sum_{m=-\infty}^{-1} 2^{(N-s)m\tau} \Big( \sum_{k=-\infty}^{\infty} 2^{(k+m)sq} \Big\| (\varphi^*_{k+m} f)(x) \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\|^q \Big)^{\frac{\tau}{q}} \\
&\quad \le C_5 \Big( \sum_{j=0}^{\infty} 2^{jsq} \Big\| (\varphi^*_j f)(x) \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\|^q \Big)^{\frac{\tau}{q}}.
\end{aligned}$$

In the last step we used $k + m = j$ and $N > s$. Now let $a > \frac{d}{p}$. Then by Lemma 1.1.7. from [7] the boundedness of $\varphi_j^* f$ in $\mathcal{M}^u_p(\mathbb{R}^d)$ can be reduced to the boundedness of the Hardy-Littlewood maximal function, see Lemma 3 of this paper. We get

$$\sum_{m=-\infty}^{-1} \Big( \sum_{k=-\infty}^{\infty} 2^{ksq} \Big[ \sup_{|h|\le 2^{-k}} \Big\| \Delta^N_h \mathcal{F}^{-1}[\varphi_{k+m}\mathcal{F}f] \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\| \Big]^q \Big)^{\frac{\tau}{q}} \le C_6 \| f | \mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \|^\tau.$$

Next we will deal with the case $m \ge 0$. We use formula (1) from the proof of Theorem 3. Remember the translation-invariance of the Morrey spaces. Then we obtain

$$\sum_{m=0}^{\infty} \Big( \sum_{k=-\infty}^{\infty} 2^{ksq} \Big[ \sup_{|h|\le 2^{-k}} \Big\| \Delta^N_h \mathcal{F}^{-1}[\varphi_{k+m}\mathcal{F}f] \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\| \Big]^q \Big)^{\frac{\tau}{q}}$$

$$\leq C_7 \sum_{m=0}^{\infty} 2^{-ms\tau} \Big( \sum_{k=-\infty}^{\infty} 2^{ksq} 2^{msq} \|\mathcal{F}^{-1}[\varphi_{k+m}\mathcal{F}f](x)|\mathcal{M}_p^u(\mathbb{R}^d)\|^q \Big)^{\frac{\tau}{q}}$$
$$\leq C_8 \|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^\tau.$$

In the last step we used $s > 0$. So the proof is complete. □

Characterisations in terms of differences can be used to investigate whether some functions belong to a space $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ or not.

Lemma 5. *Let $s > 0$, $1 \leq p \leq u < \infty$ and $0 < q \leq \infty$. Let $\alpha < 0$, $\delta \geq 0$ and $\vartheta > 0$ with $\vartheta$ very small . $\rho \in C_0^\infty(\mathbb{R}^d)$ is a smooth cut-off function with $\rho(x) = 1$ for $x \in B(0, \vartheta)$ and $\rho(x) = 0$ for $|x| > 2\vartheta$. We put*

$$f_{\alpha,\delta}(x) = \rho(x)|x|^\alpha(-\ln|x|)^{-\delta}.$$

(i) *Let $\delta = 0$. Then we have $f_{\alpha,0} \in \mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ if and only if we have either $s < \frac{d}{u} + \alpha$ or $s = \frac{d}{u} + \alpha$ and $q = \infty$.*

(ii) *Let $\delta > 0$. Then we have $f_{\alpha,\delta} \in \mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ if and only if we have either $s < \frac{d}{u} + \alpha$ or $s = \frac{d}{u} + \alpha$ with $\delta q > 1$.*

*Proof.* To prove sufficiency we use a version of Theorem 3 with $v = p$ and $a$ small. At first we transform the quasinorm $\|f_{\alpha,\delta}|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(p,a)}$ like it is described in step 1 of the proof from Theorem 4. After this we proceed like it is explained in the proof of Lemma 1 from chapter 2.3.1. in [15]. Also to prove necessity we can use the techniques that are described in the proof of this lemma. For that purpose in the case $0 < s < 1$ we have to use theorem 3 with $v = 1$, $N = 1$ and $a$ small. In the case $s \geq 1$ at first we have to apply Theorem 3.3. from [5]. Then we can proceed like in the case $0 < s < 1$ and obtain the desired result. □

So if $\alpha$ is increasing the regularity of $f_{\alpha,\delta}$ with respect to the parameter $s$ is increasing. If $\delta$ becomes larger the regularity is increasing only with respect to the fine index $q$.

*Remark* 6. It is also possible to prove Lemma 5 with $0 < p \leq u < \infty$ and $s > \sigma_p$. But then the proof of necessity becomes more technical in the case $s \geq 1$.

## 4. BESOV-MORREY SPACES AND DIFFERENCES : NECESSARY CONDITIONS

As you can see in Theorem 3 some conditions concerning the parameter $s$ do appear. In detail the restrictions $s > d\max(0, \frac{1}{p} - 1, \frac{1}{p} - \frac{1}{v})$ and $N > s$

can be found. In this chapter our main goal is to investigate whether these conditions are not only sufficient but also necessary. For this purpose we will define the following function spaces.

*Definition* 7. Let $s \in \mathbb{R}$, $0 < p \leq u < \infty$, $0 < q \leq \infty$, $0 < v \leq \infty$, $1 \leq a \leq \infty$ and $N \in \mathbb{N}$. Then $\mathbf{N}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$ is the collection of all $f \in L^{loc}_{\max(p,v)}(\mathbb{R}^d)$ such that $\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(v,a)}$ is finite.

In what follows we investigate in which cases we have $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{N}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$ in the sense of different sets.

## 4.1. The necessity of $s > 0$

Proposition 2. *Let $s \in \mathbb{R}$, $0 < p \leq u < \infty$, $0 < q \leq \infty$, $0 < v \leq \infty$, $1 \leq a \leq \infty$ and $N \in \mathbb{N}$ with $N > s$. Then we have $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{N}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$ if either $s < 0$ or $s = 0$ with $p \geq 2$ and $q > 2$ or $s = 0$ with $1 \leq p < 2$ and $q > p$. Moreover in the case $s = 0$ and $a = \infty$ we have $\mathcal{N}^0_{u,p,q}(\mathbb{R}^d) \neq \mathbf{N}^{0,N,\infty}_{u,p,q,v}(\mathbb{R}^d)$.*

*Proof. Step* 1. At first we have either $s < 0$ or $s = 0$ with $p \geq 2$ and $q > 2$ or $s = 0$ with $1 \leq p < 2$ and $q > p$. In each of these cases the spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ contain singular distributions, see Lemma 2. So a characterisation of $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ in terms of differences is not possible.

*Step* 2. Now we look at the case $s = 0$ and $a = \infty$. In the case $q = \infty$ the spaces $\mathcal{N}^0_{u,p,\infty}(\mathbb{R}^d)$ contain singular distributions, see Lemma 2. So in what follows we can assume $0 < q < \infty$. Let $f \in C^\infty_0(\mathbb{R}^d)$ with $f(x) = 1$ for $|x| \leq 1$ and $f(x) = 0$ for $|x| > 2$. Then because of Lemma 1 we have $f \in \mathcal{N}^0_{u,p,q}(\mathbb{R}^d)$. But we find $f \notin \mathbf{N}^{0,N,\infty}_{u,p,q,v}(\mathbb{R}^d)$. For this we refer to the proof of Proposition 5.1. in [8]. We omit the details. □

## 4.2. Is the condition $s > d\left(\frac{1}{p} - 1\right)$ necessary?

In this subsection we want to investigate whether the condition $s > d(\frac{1}{p} - 1)$ is necessary. To give a satisfactory answer to this question seems to be not so easy. But some results are already known.

Proposition 3. *Let $0 < p \leq u < \infty$, $0 < q \leq \infty$, $s \geq 0$, $0 < v \leq \infty$ and $1 \leq a \leq \infty$. We have $N \in \mathbb{N}$ with $N > s$. Let $0 < p < 1$. Then we have $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{N}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$ if either $s < d\frac{p}{u}(\frac{1}{p} - 1)$ or $s = d\frac{p}{u}(\frac{1}{p} - 1)$ with $q > 1$.*

*Proof.* In both cases the spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ contain singular distributions, see Lemma 2. So a characterisation of $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ in terms of differences is not possible. □

In the case $0 < p < 1$ and $d\frac{p}{u}(\frac{1}{p} - 1) < s \leq d(\frac{1}{p} - 1)$ we do not know whether we have $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) = \mathbf{N}^{s,N,a}_{u,p,q,1}(\mathbb{R}^d)$ or $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{N}^{s,N,a}_{u,p,q,1}(\mathbb{R}^d)$. But there is the following result.

Proposition 4. *Let $s > 0$, $0 < p < u < \infty$, $0 < q \leq \infty$ and $N \in \mathbb{N}$ with $N > s$. Let $0 < p < 1$ and $d\frac{p}{u}(\frac{1}{p} - 1) < s \leq d(\frac{1}{p} - 1)$. Then there is the continuous embedding $\mathbf{N}^{s,N,\infty}_{u,p,q,1}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$.*

*Proof.* To prove this result we can use the techniques that are described in step 2 of the proof from Theorem 2.5.10. in [20]. The main tool for the proof is a classical construction from approximation theory that can be found in [11], see chapter 5.2.1. Here we apply the version from [17], see the proof of Lemma 10. Notice that $f \in \mathbf{N}^{s,N,\infty}_{u,p,q,1}(\mathbb{R}^d)$ implies $f \in L^{loc}_1(\mathbb{R}^d) \cap \mathcal{M}^u_p(\mathbb{R}^d)$. □

At the moment it is not clear whether we have $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathbf{N}^{s,N,\infty}_{u,p,q,1}(\mathbb{R}^d)$ for $d\frac{p}{u}(\frac{1}{p} - 1) < s \leq d(\frac{1}{p} - 1)$ and $0 < p < 1$.

### 4.3. Is the condition $s > d\left(\frac{1}{p} - \frac{1}{v}\right)$ necessary?

The following result is not optimal but tells us that the condition $s > d(\frac{1}{p} - \frac{1}{v})$ is at least partly necessary.

Proposition 5. *Let $s \geq 0$, $0 < p \leq u < \infty$ and $0 < q \leq \infty$. Let $0 < p < v < \infty$, $1 \leq a \leq \infty$ and $N \in \mathbb{N}$ with $N > s$. We have $s < d\frac{p}{u}(\frac{1}{p} - \frac{1}{v})$. Then we have $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{N}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$.*

*Proof. Step* 1. At first we look at the case $0 < v \leq 1$. Here $s < d\frac{p}{u}(\frac{1}{p} - \frac{1}{v})$ implies $s < d\frac{p}{u}(\frac{1}{p} - 1)$. But from Proposition 3 we know that in this case it is not possible to describe the spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ in terms of differences.

*Step* 2. Now we work with the case $1 < v < \infty$. We will use some ideas from [9], see Proposition 5.2. Let $s < d\frac{p}{u}(\frac{1}{p} - \frac{1}{v})$. We will argue by contradiction. Our first assumption is $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) = \mathbf{N}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$ as sets. Then $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ can not contain singular distributions and $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \subset L^{loc}_1(\mathbb{R}^d)$ follows. Our second assumption is a sharpening of the first one. We assume that the identity $Id : \mathbf{N}^{s,N,a}_{u,p,q,v}(B(0, \frac{1}{8N})) \rightarrow \mathcal{N}^s_{u,p,q}(B(0, \frac{1}{8N}))$ is a continuous operator. Here $\mathbf{N}^{s,N,a}_{u,p,q,v}(B(0, \frac{1}{8N}))$ is defined to be the set of all $f \in \mathbf{N}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$

satisfying $\operatorname{supp} f \subset B(0, \frac{1}{8N})$. We will first disprove assumption two, afterwards assumption one.

*Substep* 2.1. We use $f \in \mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ with $\operatorname{supp} f \subset B(0, \frac{1}{4N})$. We will prove that there is a $C > 0$ independent of $f$ such that $\| f | L_v(\mathbb{R}^d)\| \leq C \| f | \mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|$. Because of our assumption we can start with (modifications in the case $q = \infty$)

$$\| f | \mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\| \geq C_1 \Big( \int_0^a t^{-sq-d\frac{q}{v}} \Big\| \Big( \int_{B(0,t)} |\Delta^N_h f(x)|^v dh \Big)^{\frac{1}{v}} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\|^q \frac{dt}{t} \Big)^{\frac{1}{q}}.$$

Next instead of the supremum we choose the ball $B(0, \frac{N+1}{4})$. Then we obtain

$$\| f | \mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\| \geq C_2 \Big( \int_0^1 t^{-sq-d\frac{q}{v}} \Big( \int_{B(0,\frac{N+1}{4})} \Big( \int_{B(0,t)} |\Delta^N_h f(x)|^v dh \Big)^{\frac{p}{v}} dx \Big)^{\frac{q}{p}} \frac{dt}{t} \Big)^{\frac{1}{q}}.$$

Now we are exactly in the same situation as it is described in step 1 of the proof from Proposition 5.2 in [9]. So we can proceed like there and get the desired result.

*Substep* 2.2. In this substep we will work with function spaces on smooth and bounded domains. They have been introduced in Definition 3. As domain we choose the ball $B(0, \frac{1}{8N})$. We want to prove that we have the continuous embedding $\mathcal{N}^s_{u,p,q}(B(0, \frac{1}{8N})) \hookrightarrow L_v(B(0, \frac{1}{8N}))$. To prove this we use the methods that are described in step 2 of the proof from Proposition 5.2 in [9]. Our main tool for the proof is the result from substep 2.1. We omit the details.

*Substep* 2.3. Next we want to use Lemma 4. From the substep before we know $\mathcal{N}^s_{u,p,q}(B(0, \frac{1}{8N})) \hookrightarrow L_v(B(0, \frac{1}{8N}))$. Then because of $p < v$ Lemma 4 tells us that we have $s \geq d\frac{p}{u}(\frac{1}{p} - \frac{1}{v})$. But we assumed $s < d\frac{p}{u}(\frac{1}{p} - \frac{1}{v})$. This is a contradiction. So our assumption on the continuity of the identity must have been wrong.

Let $(f_j)_j$ be a convergent sequence in $\mathbf{N}^{s,N,a}_{u,p,q,v}(B(0, \frac{1}{8N}))$ with limit $f \in \mathbf{N}^{s,N,a}_{u,p,q,v}(B(0, \frac{1}{8N}))$. In addition we assume $\lim_{j\to\infty} f_j = g$ in $\mathcal{N}^s_{u,p,q}(B(0, \frac{1}{8N}))$. The first fact implies convergence in $L_p(\mathbb{R}^d)$, see Theorem 3 and Definition 7. This yields convergence almost everywhere for an appropriate subsequence $(f_{j_\ell})_\ell$. The second assumption applied to this subsequence yields the existence of extensions $h_{j_\ell}$ of $f_{j_\ell} - g$ such that $\lim_{\ell\to\infty} \| h_{j_\ell} | \mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\| = 0$. We can assume $\operatorname{supp} h_{j_\ell} \subset [-1, 1]^d$ and find $\lim_{\ell\to\infty} \| f_{j_\ell} - g | L_1(B(0, \frac{1}{8N}))\| = 0$. By switching to a further subsequence we conclude $f = g$ almost everywhere. So we have proved that the identity $Id : \mathbf{N}^{s,N,a}_{u,p,q,v}(B(0, \frac{1}{8N})) \to \mathcal{N}^s_{u,p,q}(B(0, \frac{1}{8N}))$ is a closed linear operator. The Closed Graph Theorem yields that $Id$ must be continuous. But this contradicts our previous conclusion. Therefore also our assumption concerning the equality of the sets must be wrong. This proves $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{N}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$ as claimed. $\square$

### 4.4. **The necessity of $N>s$**

Proposition 6. *Let $0 < p \le u < \infty$, $s \ge 0$ and $0 < q \le \infty$. Let $0 < v \le \infty$, $1 \le a \le \infty$ and $N \in \mathbb{N}$ with $N \le s$. Then we have $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \neq \mathbf{N}^{s,N,a}_{u,p,q,v}(\mathbb{R}^d)$ if either $N < s$ or $N = s$ with $q \neq \infty$ or $N = s$ with $q = \infty$, $u = p$ and $v \ge 1$.*

*Proof. Step* 1. At first we have either $N < s$ and $0 < q \le \infty$ or $N = s$ and $0 < q < \infty$. We work with a function $f \in C_0^\infty(\mathbb{R}^d)$ that has a support in $B(0, 3N+3)$. In the ball $B(0, 2N+2)$ this function looks like $f(x_1, x_2, \ldots, x_d) = e^{x_1+x_2+x_3+\ldots+x_d}$. Then because of Lemma 1 we have $f \in \mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$. But it is $\|f|\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\|^{(v,a)} = \infty$. For that we refer to Proposition 5.5 in [9], see also Proposition 5.10 in [8]. We omit the details.

*Step* 2. Now we will deal with the case $s = N$, $q = \infty$, $p = u$ and $v \ge 1$. Here we will use some ideas from Oswald, see [13]. Although we assume $u = p$ we will keep both numbers different in notation to point out why the assumption $u = p$ is needed. We fix $r \in \mathbb{N}$ with $r > 4$ such that $2^{r+1} \ge N + 4$. Take $\phi \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\phi \subset B(0,1) \cap [0,1)^d$ such that $\phi(\cdot - 32^{r-2}(1,1,\ldots,1)^T)$ fulfills moment conditions up to order $L \in \mathbb{N}_0 \cup \{-1\}$ with $L \ge \max(-1, \sigma_p - N)$. There is a set $D \subset \operatorname{supp}\phi$ with $|D| > \frac{|\operatorname{supp}\phi|}{2}$ such that for all $x \in D$ and $|\gamma| \le N$ we have $|D^\gamma \phi(x)| > C > 0$. There is a set $\tilde{D} \subset D$ such that for all $x \in \partial\tilde{D}$ we have $2^{-10} > \operatorname{dist}(x, \partial D) > 2^{-20}$. For $k \in \mathbb{N}$ we put $n_k = r(k-1) + 2$ and $a_k = 2^{n_k(\frac{d}{u} - N)}$. Put $x_r = 32^{r-2}(1,1,\ldots,1)^T$. Now let $f(x) = \sum_{k=1}^\infty a_k \phi(2^{n_k}x - x_r)$. For $k \in \mathbb{N}$ we have $\operatorname{supp}\phi(2^{n_k}\cdot - x_r) \subset B(2^{-n_k}x_r, 2^{-n_k}) \cap Q_{n_k,x_r}$ and $\operatorname{supp} f \subset B(0, 4\sqrt{d}\cdot 32^{r-2} + 4)$. For $k, t \in \mathbb{N}$ with $k \neq t$ we can find $\operatorname{supp}\ \phi(2^{n_k}\ \cdot\ - x_r) \cap \operatorname{supp}\ \phi(2^{n_t}\ \cdot\ - x_r) = \emptyset$. Moreover if we fix a large $l \in \mathbb{N}$ and $h \in \mathbb{R}^d$ with $|h| \le 2^{-n_{l+1}}2^{-rl}$ for $k, t \in \mathbb{N}$ with $0 < k < t < l - 4$ we have $\operatorname{supp}\phi(2^{n_k}\cdot + N2^{n_k}h - x_r) \cap \operatorname{supp}\phi(2^{n_t}\cdot + N2^{n_t}h - x_r) = \emptyset$. Now we want to prove $\|f|\mathcal{N}^N_{u,p,\infty}(\mathbb{R}^d)\| < \infty$. For that purpose we use a characterisation of the Besov-Morrey spaces by means of atomic decompositions. Details concerning that topic can be found in [14], see Theorems 2.30 and 2.36, and in [16], see Corollary 4.10 and Theorem 4.12. A short summary of the main ideas also can be found in [3]. The function $f$ is constructed in a way such that Proposition 2.12. from [3] can be applied and we obtain

$$\|f|\mathcal{N}^N_{u,p,\infty}(\mathbb{R}^d)\| \le C_1 \sup_{k\in\mathbb{N}} 2^{n_k(N-\frac{d}{u})} \left\| |a_k| \chi^{(u)}_{n_k,x_r}(x) \middle| \mathcal{M}^u_p(\mathbb{R}^d) \right\| \le C_1 < \infty.$$

In the last step we used $\|\chi^{(u)}_{n_k,x_r}(x)|\mathcal{M}^u_p(\mathbb{R}^d)\| = 1$, see the remark after Definition 2.9 in [3]. Next we will prove that we have $\|f|\mathcal{N}^N_{u,p,\infty}(\mathbb{R}^d)\|^{(v,a)} = \infty$.

To prove this at first we fix a large number $l \in \mathbb{N}$ with $l > 10$. Then for $|h| \leq 2^{-n_{l+1}} 2^{-rl}$ because of the disjoint supports of the involved functions we obtain

$$\Big\| \Big( \int_{B(0,t)} |\Delta_h^N f(x)|^v dh \Big)^{\frac{1}{v}} \Big| \mathcal{M}_p^u(\mathbb{R}^d) \Big\|^p$$
$$\geq \Big\| \Big( \int_{|h| \leq \min(t, 2^{-n_{l+1}} 2^{-rl})} \Big( \sum_{k=1}^{l-6} a_k |\Delta_h^N (\phi(2^{n_k} \cdot - x_r))(x)| \Big)^v dh \Big)^{\frac{1}{v}} \Big| \mathcal{M}_p^u(\mathbb{R}^d) \Big\|^p .$$

Now we use that for fixed $h$ with $|h| \leq \min(t, 2^{-n_{l+1}} 2^{-rl}) = t(r,l)$ and $k \in \mathbb{N}$ we have $\operatorname{supp} \Delta_h^N (\phi(2^{n_k} \cdot - x_r)) \subset B(0, \sqrt{d} 32^{r-2}(4+N) + 4)$. Because of this instead of the supremum we can choose the ball $B(0, \sqrt{d} 32^{r-2}(4+N)+4)$. For $v \geq 1$ we get

$$\Big\| \Big( \int_{B(0,t)} |\Delta_h^N f(x)|^v dh \Big)^{\frac{1}{v}} \Big| \mathcal{M}_p^u(\mathbb{R}^d) \Big\|^p$$
$$\geq C_1 \Big\| \Big( \int_{|h| \leq t(r,l)} \Big( \sum_{k=1}^{l-6} a_k |\Delta_h^N (\phi(2^{n_k} \cdot - x_r))(x)| \Big)^v dh \Big)^{\frac{1}{v}} \Big| L_p(\mathbb{R}^d) \Big\|^p$$
$$\geq C_2 t(r,l)^{dp(\frac{1}{v}-1)} \sum_{k=1}^{l-6} a_k^p 2^{-n_k d} \Big\| \int_{|h| \leq t(r,l)} |\Delta_{2^{n_k} h}^N (\phi(\cdot))(x)| dh \Big| L_p(\mathbb{R}^d) \Big\|^p .$$

Let $\eta = \frac{h}{|h|}$ and $\theta \in (0,1)$. Because of the properties of the sets $D$ and $\tilde{D}$ we obtain

$$\Big\| \Big( \int_{B(0,t)} |\Delta_h^N f(x)|^v dh \Big)^{\frac{1}{v}} \Big| \mathcal{M}_p^u(\mathbb{R}^d) \Big\|^p \geq C_2 t(r,l)^{dp(\frac{1}{v}-1)}$$
$$\times \sum_{k=1}^{l-6} a_k^p 2^{-n_k d} 2^{n_k N p} \Big\| \int_{|h| \leq \frac{t(r,l)}{N}} \Big| \frac{\partial^N \phi}{\partial \eta^N}(x + \theta N 2^{n_k} h) \Big| |h|^N dh \Big| L_p(\tilde{D}) \Big\|^p$$
$$\geq C_3 t(r,l)^{dp(\frac{1}{v}-1)} \sum_{k=1}^{l-6} a_k^p 2^{-n_k d} 2^{n_k N p} \Big\| \int_{|h| \leq \frac{t(r,l)}{N}} |h|^N dh \Big| L_p(\tilde{D}) \Big\|^p .$$

When we use $a_k = 2^{n_k(\frac{d}{u} - N)}$ we get

$$\Big\| \Big( \int_{B(0,t)} |\Delta_h^N f(x)|^v dh \Big)^{\frac{1}{v}} \Big| \mathcal{M}_p^u(\mathbb{R}^d) \Big\|^p \geq C_4 \sum_{k=1}^{l-6} 2^{n_k p \frac{d}{u}} 2^{-n_k d} t(r,l)^{Np + dp + \frac{dp}{v} - dp} .$$

Now because of $r > 4$ and $l > 10$ we have $2^{-n_{l+1}} 2^{-rl} < 1$. For $p = u$ this leads to

$$\Big(\|f|\mathcal{N}^N_{u,p,\infty}(\mathbb{R}^d)\|^{(v,a)}\Big)^p$$
$$\geq C_5 \sum_{k=1}^{l-6} 2^{n_k p \frac{d}{u}} 2^{-n_k d} \sup_{t\in[0,1]} t^{-Np-\frac{dp}{v}} \min(t, 2^{-n_{l+1}} 2^{-rl})^{Np+\frac{dp}{v}}$$
$$\geq C_5 \sum_{k=1}^{l-6} 2^{n_k p \frac{d}{u}} 2^{-n_k d} = C_5 \sum_{k=1}^{l-6} 1 = C_5 (l-6).$$

So if $l$ tends to infinity also $\|f|\mathcal{N}^N_{u,p,\infty}(\mathbb{R}^d)\|^{(v,a)}$ tends to infinity and the proof is complete. □

*Proof of Theorem 2.* To prove Theorem 2 we just have to combine the results from the Propositions 2, 3, 5 and 6. □

*Remark* 8. Let $s \in \mathbb{R}$, $0 \leq \tau < \infty$ and $0 < p, q \leq \infty$. Then the Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ are closely related to the Besov-Morrey spaces. See chapter 1.3 in [21] for a definition. The spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ can be characterised in terms of differences as well, see Theorem 1.4 in [9]. Whereas the sufficient conditions concerning the parameter $s$ that can be found there are almost the same as in Theorem 1 from this paper, the results concerning necessity look slightly different, see Theorem 1.5 in [9]. Here sometimes the parameter $\tau$ appears.

*Remark* 9. Let $s \in \mathbb{R}$, $0 < p \leq u < \infty$ and $0 < q \leq \infty$. Then the so-called Triebel-Lizorkin-Morrey spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ are another kind of function spaces that are related to the Besov-Morrey spaces. See [19] or chapter 1.3 in [21] for a definition. It is known that also the spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ can be described in terms of differences, see Theorem 2 and Theorem 6 in [8]. This characterisation looks similar to that we have found in Theorem 1 for $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$. But in the case of the Triebel-Lizorkin-Morrey spaces some additional conditions concerning the parameter $s$ that also depend on $q$ do appear. Notice that in [8] there also is a result similar to Theorem 2 where the necessity of the conditions concerning $s$ is investigated, see Theorem 4 and Theorem 7 in [8].

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