Dedicated to the memory of Cabiria Andreian-Cazacu

AN EXACT PENALIZATION APPROACH OF SET-VALUED OPTIMIZATION PROBLEMS

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In this paper, we use the penalty method to study constrained minimization problems for set-valued maps and develop Clarke's exact penalty principle at both local and global cases. Also, by using the notion of merit functions, we generalize Clarke's exact principle for them.

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1. INTRODUCTION

Many problems in economics, engineering and applied sciences are modeled by nonlinear global minimization problems. Most attention in the search for global solutions is rather than local ones of nonlinear optimization problems. Already, most research efforts have been appropriated to globally solving either unconstrained problems or problems with simple constraints.
Recently, the more difficult case of global optimization problems with general constraints has been as well explored, and different approaches are explained (see [2, 12, 15]). Also, a specific attention is given to use of some augmented Lagrangian functions to agreement the general constraints (see [8, 16, 11]).
Penalty approach is an important and useful technique for solving constrained optimization. The idea of the penalty approach is not new; Eremin [7] and Zangwill [19] introduced the notion of exact penalization for nonlinear constrained optimization and the exact penalty results were demonstrated by Ioffe [10]. Zaslavski [19, 21] used the penalty approach to study a class of constrained minimization problems on complete metric spaces and created the generalized exact penalty property and obtained an estimation of the exact penalty. Also, by using the penalty approach, he studied three constrained nonconvex minimization problems with Lipschitzian functions (on bounded sets):

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1. an equality constrained problem in Banach space with a locally Lipschitzian constraint function,

2. an inequality constrained problem in a Banach space with a locally Lipschitzian constraint function,

3. a problem in a finite-dimensional space with mixed constraints and smooth constraint functions.

A penalty function is called to have the exact penalty property [6, 9] if there exists a penalty coefficient for which a solution of an unconstrained penalized problem is a solution of the corresponding constrained problem. Ye [17] studied Clarke's exact penalty principle at both global and local for vector valued functions. Also, Durea and Strugariu [4] employed a new method to penalize a constrained non solid vector optimization problem by means of a scalarization functional applied to the constrained system. In [5] they used a double penalization procedure in order a set-valued optimization problem with functional constrained to an unconstrained one. Recently, Strekalovsky [14] consider a nonconvex optimization problem with the cost function and the inequality constraints given by d.c functions (difference of two convex functions) and showed that the original problem is reduced to a problem without inequality constraints by means of the exact penalization techniques.
The main purpose of this paper is to extend results in [4, 17] from single-valued function to set-valued maps. In the following, we express Clarke's exact penalty principle [3] for real-valued functions.

Theorem 1.1 (Clarke's exact penalty principle). *Let S be a subset of a normed space X and $f : X \to \mathbb{R}$ be Lipschitz of rank L_f on S. Let x belong to a set $C \subset S$ and suppose that f attains a minimum over C at x. Then, for any $L \geq L_f$, the function $g(y) = f(y) + Ld_C(y)$ attains a minimum over S at x. Conversely, suppose that C is closed. Then, for any $L > L_f$, any other point minimizing g over S must also minimize the function f over C.*

In this paper by using the concept of K-minimizers for set-valued maps, we develop and prove Clarke's exact penalty principle at both local and global cases. The paper is organized as follows: Section 2 prepares briefly some preliminary notions and results used in sequel. In Section 3, we give some conditions under which the constrained set-valued optimization problem and the unconstrained exact penalized problem are exactly equivalent. Also, we express Clarke's exact principle for merit functions which is a generalization of penalty functions.

2. PRELIMINARIES

Let X and Y be normed spaces. The closed unit ball of $x \in X$ and the distance of x from $C \subset X$ are denoted by B_X and $d_C(x) := \inf_{c \in C} \|x - c\|$, respectively. Recall that $K \subset Y$ is a cone if $\lambda y \in K$ for all $y \in K$ and $\lambda \geq 0$, a convex cone is one for which $\lambda_1 y_1 + \lambda_2 y_2 \in K$ for all $\lambda_1, \lambda_2 \geq 0$, and a cone is pointed if $K \cap (-K) = \{\circ\}$, where $\circ$ denotes the zero element in Y. Now, let Y be a normed space and K be a cone in Y. We consider the preference relation for two vectors $x, y \in Y$ induced by cone K given as follows:

$$x \preceq y \Longleftrightarrow y - x \in K,$$
$$x \prec y \Longleftrightarrow y - x \in K \backslash \{\circ\}.$$

In particular, if $Y = \mathbb{R}^N$ and $K = \mathbb{R}^{\mathbb{N}}_+ := \{z \in \mathbb{R}^{\mathbb{N}} : z$ has nonnegative components$\}$ then we have a preference in the Pareto sense, and if $Y = \mathbb{R}^{\mathbb{N}}$ and $K = \operatorname{int} \mathbb{R}^{\mathbb{N}}_+ \cup \{\circ\}$, where int$C$ denotes the interior of set C, then we have a preference in the weak Pareto sense.

The Aubin property is an important extension of Lipschitz continuity to multifunctions. It has been introduced by Aubin [1] and is also referred to as "Lipschitz-like" continuity. Now, we consider K-Lipschitz property near a point in the following sense that is not the same as the usual Lipschitz continuity.

Definition. Let S be a subset of X. Suppose that $F : X \to 2^Y$ and K is a cone of Y. We say that F is K-Lipschitz on S (of rank L_F) if there is a constant $L_F > 0$ and an element $e \in K$ with $\|e\| = 1$ such that

$$F(x_1) + L_F \|x_1 - x_2\| e \subseteq F(x_2) + K, \quad \forall x_1, x_2 \in S$$

Let $\bar{x} \in K$. We say that F is K-Lipschitz near $\bar{x}$ if there is U, a neighborhood of $\bar{x}$, such that F is K-Lipschitz on U.

Definition. A point $(\bar{x}, \bar{y}) \in GrF$ is called to be a global K-minimizer point for F on C if $\bar{x} \in C$ and

$$(F(C) - \bar{y}) \cap (-K \backslash \{\circ\}) = \emptyset.$$

If one replaces in the pervious formula the set C with $C \cap U$, where U is a neighborhood of $\bar{x}$, then local solutions are obtained.

The following lemma is an extension of Lemma 2.5 [13] in the case that C is not necessarily closed.

LEMMA 2.1 ([17]). *Let C be a nonempty set and let $\bar{x} \in C$. Then, for any $\varepsilon > \delta > 0$ and any $y \in \bar{B}(\bar{x}, \frac{\varepsilon - \delta}{2})$*

$$d_C(y) = d_{C \cap \bar{B}(\bar{x}, \varepsilon)}(y).$$

Moreover, if C is a closed subset of a finite-dimensional space, then δ can be chosen as $\circ$ in the above statement.

3. MAIN RESULTS

In this section, we present some results for global and local exact penalization for distance functions in set-valued optimization problems. We first give an extension of Clarke's exact penalty principle.

THEOREM 3.1 (Global Exact Penalization for Distance Function). *Let X and Y be normed spaces, let $S \subset X$, $C \subset S$, and let $K \subset Y$ be a convex and pointed cone. Suppose that $F : S \to 2^Y$ is K-Lipschitz on S of rank L_F and e is the element in K given by the K-Lipschitz continuity of F.*

(i) *Assume that $K\backslash\{\circ\}$ is an open set. Then any global K-minimizer of F on C is a global K-minimizer of the set-valued exact penalty function $F(x) + L_F d_C(x)e$ on S.*

(ii) *Assume that either C is closed or that $K\backslash\{\circ\}$ is an open set. Then for any $L > L_F$, $(\bar{x}, \bar{y})$ is a global K-minimizer of F on C if and only if it is a global K-minimizer of the set-valued exact penalty function $F(x) + Ld_C(x)e$ on S.*

Proof. By the K-Lipschitz continuity of F, there is a constant $L_F > 0$ and an element $e \in K$ with $\|e\| = 1$, such that

$$F(x) + L_F\|x - x^*\|e \subseteq F(x^*) + K \quad \forall x, x^* \in S \tag{1}$$

To prove (i), suppose to the contrary that $(\bar{x}, \bar{y})$ is a global K-minimizer of F on C but not a global K-minimizer for $F(x) + L_F d_C(x)e$ on S. Then there exists $(x, y) \in GrF$ with $x \in S$ such that

$$y + L_F d_C(x)e \prec \bar{y} \tag{2}$$

Since $K\backslash\{\circ\}$ is open, from (2), there exists a small enough $\varepsilon > 0$ such that

$$y + L_F d_C(x)e \prec \bar{y} - L_F\varepsilon e \tag{3}$$

By definition of the distance function, there exists $x^*_\varepsilon \in C$ such that $\|x - x^*_\varepsilon\| \leq d_C(x) + \varepsilon$. Hence, (1) implies that, there exists $y^*_\varepsilon \in F(x^*_\varepsilon)$ such that

$$\begin{aligned} y^*_\varepsilon &\preceq y + L_F\|x - x^*_\varepsilon\| && \text{by (1)} \\ &\preceq y + L_F(d_C(x) + \varepsilon)e \prec \bar{y} && \text{by (3),} \end{aligned}$$

which implies $y^*_\varepsilon \prec \bar{y}$. This contradicts the fact that $(\bar{x}, \bar{y})$ K-minimizes F on C, and therefore the conclusion of (i) holds.

We now prove (ii). Suppose to the contrary that $(\bar{x}, \bar{y})$ is a global K-minimizer of F on C but not a global K-minimizer for $F(x) + Ld_C(x)e$ on S and $L > L_F$. Then there exists $(x, y) \in GrF$ with $x \in S$ such that

$$y + Ld_C(x)e \prec \bar{y} \tag{4}$$

Observe that x cannot lie in the set C, since else $\bar{x}$ would not be a global K-minimizer of F on C. Assume that C is closed. Then $x \notin C$ implies that $d_C(x) > 0$. Since $\frac{L}{L_F} > 1$, one can choose $x^* \in C$ such that $\|x - x^*\| < \frac{L}{L_F} d_C(x)$. Hence, one has

$$(5) \qquad y + L_F\|x - x^*\|e \prec y + Ld_C(x)e \prec \bar{y}.$$

Now, by using the K-Lipschitz property of F (relation (1)), there exists $y^* \in F(x^*)$ such that

$$(6) \qquad y^* \preceq y + L_F\|x - x^*\|e.$$

Hence, by using relations (5) and (6), we have

$$y^* \prec \bar{y},$$

which contradicts the fact that $(\bar{x}, \bar{y})$ is a global K-minimizer of F on C, and hence the necessity (ii) under the assumption that C is closed is proved. When C is not closed but $K \backslash \{\circ\}$ is open, the rest of the proof is similar to the proof of (i). Therefore the necessity in (ii) holds.

To prove the sufficiency in (ii) by contradiction suppose that $(\bar{x}, \bar{y})$ is a minimizer of $F(x) + Ld_C(x)e$ on S with $\bar{x} \in C$ but not a minimizer of F on C and $L > L_F$. Then there is $(x, y) \in GrF$ with $x \in C$ such that $y \prec \bar{y}$. Since $d_C(x) = 0$, the above relationship implies that

$$y + Ld_C(x)e \prec \bar{y} + Ld_C(\bar{x})e.$$

contradicting the fact that $(\bar{x}, \bar{y})$ is a minimizer of $F(x) + Ld_C(x)e$ on S. Now, it remains to prove that it is not possible to have $\bar{x} \notin C$ and to have $(\bar{x}, \bar{y})$ be a minimizer of $F(x) + Ld_C(x)e$ on S. When C is closed, since $\bar{x} \notin C$ and $\frac{L}{L_F} > 1$, one can pick $x^* \in C$ such that $\|x - x^*\| < \frac{L}{L_F} d_C(\bar{x})$. Then

$$(7) \qquad \bar{y} + L_F\|x^* - \bar{x}\|e \prec \bar{y} + Ld_C(\bar{x})e.$$

Also, by using the K-Lipschitzian of F, there exists $y^* \in F(x^*)$ such that $y^* \prec \bar{y} + L_F\|x^* - \bar{x}\|e$. Hence by using relation (7) $y^* \prec \bar{y} + Ld_C(\bar{x})e$. Therefore, we obtain

$$y^* + Ld_C(x^*)e \prec \bar{y} + Ld_C(\bar{x})e.$$

This contradicts the fact that $(\bar{x}, \bar{y})$ is a K-minimizer of $F(x) + Ld_C(x)e$ on S. So, $\bar{x}$ must lie in C. Now suppose that C is not closed but $K \backslash \{\circ\}$ is open. Let $\varepsilon > 0$ and $x_\varepsilon^* \in C$ be such that $\|\bar{x} - x_\varepsilon^*\| \leq d_C(\bar{x}) + \epsilon$. Then by using K-Lipschitzian property of F there exists $y_\varepsilon^* \in F(x_\varepsilon^*)$ such that

$$y_\varepsilon^* \preceq \bar{y} + L_F(d_C(\bar{x}) + \varepsilon)e \prec \bar{y} + L(d_C(\bar{x}) + \varepsilon)e.$$

Hence, we have

$$y_\varepsilon^* + Ld_C(x_\varepsilon^*)e \prec \bar{y} + Ld_C(\bar{x})e + L\varepsilon e.$$

Since $\varepsilon > 0$ is arbitrary and $K\backslash\{\circ\}$ is open, this contradicts the fact that $(\bar{x}, \bar{y})$ is a global K-minimizer of $F(x) + Ld_C(x)e$ on S. Therefore $\bar{x}$ must lie in C. $\square$

Example 3.2. Suppose that $X = Y = \mathbb{R}$, $S = [-1, 5], C = [-1, 1], K = [0, +\infty[, e = 1$ and $F : X \to 2^Y$ given by

$$F(x) = \begin{cases} |x| & -1 \leq x \leq 1; \\ |x-3|-1 & 1 \leq x \leq 5. \end{cases}$$

By some computation we can see that F is K-Lipschitz on S of rank $L_F = 1$. Also, C is closed, $K\backslash\{\circ\}$ is an open set and $(0, 0)$ is a global K-minimizer of F on C. Therefore Theorem 3.1 implies that it is a global K minimizer of the set-valued exact penalty function $F(x) + Ld_C(x)e$ on S for any $L \geq L_F$.

THEOREM 3.3 (Local Exact Penalization for Distance Functions). *Let X and Y be normed spaces, $C \subset S \subset X$, and let $K \subset Y$ be a convex and pointed cone and also $\bar{x} \in S$. Suppose that there exists a positive constant ε such that $F : S \to 2^Y$ is K-Lipschitz on $\bar{B}(\bar{x}, \varepsilon)$ of rank L_F. Let e be an element in K given by the Lipschitz continuity of F.*

(i) *Assume that $K\backslash\{\circ\}$ is an open set. Let $(\bar{x}, \bar{y}) \in GrF$ be a local K-minimizer of F on C. Then, for any $L \geq L_F$, it is a local K-minimizer of the exact penalty function $F(x) + Ld_C(x)e$ on S. Assume that C is closed. Then, for any $L > L_F$, if $(\bar{x}, \bar{y}) \in GrF$ is a local K-minimizer of F on C then it is a local K-minimizer of the exact penalty function $F(x) + Ld_C(x)e$ on S.*

(ii) *Assume that either C is closed or $K\backslash\{\circ\}$ is an open set, and $L > L_F$. Suppose that $(\bar{x}, \bar{y}) \in GrF$ is a K-minimizer of the exact penalty function $F(x) + Ld_C(x)e$ on $S \cap \bar{B}(\bar{x}, \varepsilon)$ and $C \cap \bar{B}(\bar{x}, \varepsilon) \neq \phi$. Then $(\bar{x}, \bar{y}) \in GrF$ is a K-minimizer of F on $C \cap \bar{B}(\bar{x}, \varepsilon)$.*

Proof. (i) Assume that $K\backslash\{\circ\}$ is an open set and $L \geq L_F$. Let $(\bar{x}, \bar{y})$ be a local K-minimizer of F on C but not a local K-minimizer of the exact penalty function $F(x) + Ld_C(x)e$ on S. Hence, there exists a closed ball $\bar{B}(\bar{x}, \varepsilon)$ such that $(\bar{x}, \bar{y})$ is a global K-minimizer of F over $\bar{B}(\bar{x}, \varepsilon) \cap S$ and F is K-Lipschitz of rank L_F on $\bar{B}(\bar{x}, \varepsilon)$. It follows from Theorem 3.1 (i) that $(\bar{x}, \bar{y})$ is a global K-minimizer of $F(x) + Ld_{C \cap \bar{B}(\bar{x}, \varepsilon)}(x)e$ on $\bar{B}(\bar{x}, \varepsilon) \cap S$. Now by using Lemma (2.1), $(\bar{x}, \bar{y})$ is a global K-minimizer of $F(x) + Ld_C(x)e$ on $\bar{B}(\bar{x}, \frac{\varepsilon}{3}) \cap S$. So $(\bar{x}, \bar{y})$ is a local K-minimizer of $F(x) + Ld_C(x)e$ on S.

By using Theorem 3.1 (ii) in place of Theorem 3.1 (i) in the above proof, one can prove (i) under the assumption that C is closed and $L > L_F$.

(ii) Suppose that $(\bar{x}, \bar{y})$ is a K-minimizer of $F(x) + Ld_C(x)e$ on $S \cap \bar{B}(\bar{x}, \varepsilon)$ with $\bar{x} \in C$ but not a local K-minimizer of F on $C \cap \bar{B}(\bar{x}, \varepsilon)$. Therefore, there is $(x,y) \in GrF$ with $x \in C \cap \bar{B}(\bar{x},\varepsilon)$ such that $y \prec \bar{y}$, which implies that

$$y + Ld_C(x)e \prec \bar{y} + Ld_C(\bar{x})e,$$

which contradicts with $(\bar{x},\bar{y})$ is a K-minimizer of $F(x)+Ld_C(x)e$ on $S \cap \bar{B}(\bar{x},\varepsilon)$. Now, we show that it is not possible to have $\bar{x} \notin C$, $C \cap \bar{B}(\bar{x},\varepsilon) \neq \phi$ for some $\varepsilon > 0$ and $(\bar{x},\bar{y})$ being a K-minimizer of $F(x) + Ld_C(x)e$ on $S \cap \bar{B}(\bar{x},\varepsilon)$. By assumption, F is K-Lipschitz on $\bar{B}(\bar{x},\varepsilon)$ of rank L_F and $C \cap \bar{B}(\bar{x},\varepsilon) \neq \phi$. When C is closed, since $\bar{x} \notin C$ and $\frac{L}{L_F} > 1$, one can put $(x^*,y^*) \in GrF$ with $x^* \in C \cap \bar{B}(\bar{x},\varepsilon)$ to be such that $\|\bar{x}-x^*\| < \frac{L}{L_F} d_C(\bar{x})$ and $y^* \preceq \bar{y} + L_F\|x^*-\bar{x}\|e$. Hence, we deduce

$$y^* \prec \bar{y} + Ld_C(x)e.$$

Now, the above implies that $y^* + Ld_C(x^*)e \prec \bar{y} + Ld_C(\bar{x})e$. This contradicts the fact that $(\bar{x},\bar{y})$ is a K-minimizer of $F(x) + Ld_C(x)e$ on $S \cap \bar{B}(\bar{x},\varepsilon)$. Therefore $\bar{x}$ must lie in C. Consider the case when $K\backslash\{\circ\}$ is open. Let $\delta > 0$ be a small enough with $x^*_\delta \in C$ be such that $\|\bar{x} - x^*_\delta\| \leq d_C(\bar{x}) + \delta$ and x^*_δ is included in the ball $\bar{B}(\bar{x},\varepsilon)$. Then by K-Lipschitzation of F, there exists $y^*_\delta \in F(x^*_\delta)$ such that

$$y^*_\delta \preceq \bar{y} + L_F\|x^*_\delta - \bar{x}\|e \preceq \bar{y} + L_F(d_C(\bar{x}) + \delta)e \prec \bar{y} + L(d_C(\bar{x}) + \delta)e.$$

Hence the above implies that

$$y^*_\delta + Ld_C(x^*_\delta)e \prec \bar{y} + Ld_C(\bar{x})e + L\delta e.$$

Since $\delta > 0$ is arbitrary and $K\backslash\{\circ\}$ is an open set, this contradicts the fact $(\bar{x},\bar{y})$ is a K-minimizer of $F(x) + Ld_C(x)e$ on $S \cap \bar{B}(\bar{x},\varepsilon)$. Therefore $\bar{x}$ must lie in C. □

In Theorems 3.1 and 3.3, we showed that under suitable conditions the distance function is an exact penalty function in set-valued optimization problems. In 1997, Ye et al. [18] introduced the concept of merit functions that by using them we can characterize exact penalty functions.

Definition ([18]). Let X be a normed space and $C \subset S \subset X$. We call a function $\psi : S \to \mathbb{R}$ a merit function if

1. $\psi(y) \geq 0, \quad \forall y \in S,$
2. $\psi(y) = 0 \quad$ if and only if $\; y \in C.$

It is easy to see that if C is a closed set, then the distance function is a merit function. Also, we can obtain some merit functions that are more appropriate than the distance function.

Definition ([17]). We say that a merit function $\psi : S \to \mathbb{R}$ is a global error bound function if $\psi(x) \geq d_C(x)$ for every $x \in S$.

Because an error bound function is restricted from below by the distance function, we can obtain similar result for merit function.
In next theorems we present the global and local exact penalty results for merit functions. Since, the proof are similar to the proof of Theorems 3.1 and 3.3, respectively, we omit them.

THEOREM 3.4 (Global Exact Penalty for Merit Function). *Let X and Y be normed spaces, let $C \subset S \subset X$, and let $K \subset Y$ be a convex and pointed cone. Asseme that $F : S \to 2^Y$ is K-Lipschitz on S and that e is the element in K given by the K-Lipschitz continuity of F. Suppose that $\psi : S \to \mathbb{R}$ is a global error bound function.*

(i) *Assume that $K \setminus \{\circ\}$ is an open set. Then any global K-minimizer of F on C is a global K-minimizer of the exact penalty function $F(x) + L_F\psi(x)e$ on S.*

(ii) *Assume that either C is closed or that $K \setminus \{\circ\}$ is an open set. Then $(\bar{x}, \bar{y})$ is a global K-minimizer of F on C if and only if it is a global K-minimizer of the exact penalty function $F(x) + L\psi(x)e$ on S for any $L > L_F$.*

THEOREM 3.5 (Local Exact Penalty For Merit Function). *Let X and Y be normed spaces, let $C \subset S \subset X$, and let $K \subset Y$ be a convex and pointed cone. Let $\bar{x} \in S$. Suppose that one can find a positive constant $\varepsilon > 0$ such that $F : S \to 2^Y$ is K-Lipschitz on $\bar{B}(\bar{x}, \varepsilon)$ of rank L_F and $\psi : S \to \mathbb{R}$ is an error bound function on $\bar{B}(\bar{x}, \varepsilon)$. Let e be an element in K given by the Lipschitz continuity of F. Then the following statements hold.*

(i) *Assume that $K \setminus \{\circ\}$ is an open set. For any $L \geq L_F$, if $(\bar{x}, \bar{y})$ is a local K-minimizer of F on $C \subset S$, then it is a local K-minimizer of the exact penalty function $F(x) + L\psi(x)e$ on S. Conversely, assume that C is closed. Then, for any $L > L_F$, if $(\bar{x}, \bar{y})$ is a local K-minimizer of F on $C \subset S$ then it is a local K-minimizer of the exact penalty function $F(x) + L\psi(x)e$ on S.*

(ii) *Assume that either C is closed or that $K \setminus \{\circ\}$ is an open set, and that $L > L_F$. Suppose that $(\bar{x}, \bar{y})$ is a K-minimizer of the exact penalty function $F(x) + L\psi(x)e$ on $S \cap \bar{B}(\bar{x}, \varepsilon)$, and that $C \cap \bar{B}(\bar{x}, \varepsilon) \neq \phi$. Then $(\bar{x}, \bar{y})$ is a K-minimizer of the function F on $C \cap \bar{B}(\bar{x}, \varepsilon)$.*

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