Dedicated to the memory of Cabiria Andreian-Cazacu

REPRODUCING PROPERTY FOR ITERATED PARABOLIC OPERATORS OF FRACTIONAL ORDER

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We consider a weighted version of Bergman type spaces with respect to iterated parabolic operators of fractional order on the upper half space. We discuss reproducing properties and the orthogonal projection.

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1. INTRODUCTION

The Bergman space was introduced as a space of all square integrable analytic functions on the unit disc in the complex plane. Since analytic functions are also harmonic functions, harmonic Bergman spaces have been investigated (see, for example, [10]). Moreover, parabolic Bergman spaces were introduced (see [6]), and discussed in relation to harmonic Bergman spaces on the upper half space in the $(n+1)$-dimensional Euclidean space. In [5], the authors discussed iterated parabolic operators, where some kind of mean value property plays an important role. Mean value properties for iteratael heat operator were discussed in [7] and [8].

H. Koo, K. Nam and H. Yi ([2]) investigated weighted harmonic Bergman kernels on half spaces, which were extended to parabolic Bergman spaces by Y. Hishikawa ([3]).

In this paper, we consider weighted spaces and give the explicit form of reproducing kernels for the Bergman type spaces of iterated parabolic operators, with systematic use of the orthogonal polynomials and fractional derivatives. We remark that the proof of reproducing property is accomplished elementarily without using approximating processes.

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Let $\boldsymbol{H}$ be the upper half space of the $(n+1)$-dimensional Euclidean space. We denote by $X = (x,t)$ a point in $\boldsymbol{H} = \boldsymbol{R}^n \times \boldsymbol{R}_+$, where $\boldsymbol{R}_+ := (0,\infty)$. For $1 \le p < \infty$, $\lambda > -1$, and an integer $m \ge 1$, we define a weighted polyharmonic Bergman space on $\boldsymbol{H}$ by

$$\boldsymbol{b}^{m,p,\lambda} := \{u \in C^{2m}(\boldsymbol{H}) | \Delta^m u = 0, \int_{\boldsymbol{H}} |u(x,t)|^p t^\lambda \mathrm{d}x\mathrm{d}t < \infty\},$$

where Δ denotes the Laplacian on $\boldsymbol{H}$,

$$\Delta := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial t^2} =: \Delta_x + \partial_t^2.$$

See [1] for general theory of polyharmonic functions.

Next, we consider a parabolic operator

$$L^{(\alpha)} := \partial_t + (-\Delta_x)^\alpha$$

for $0 < \alpha \le 1$, together with a fundamental solution

$$(1) \qquad W^{(\alpha)}(x,t) = \begin{cases} (2\pi)^{-n} \int_{\boldsymbol{R}^n} e^{-t|\xi|^{2\alpha}} e^{ix\cdot\xi} \mathrm{d}x & t > 0 \\ 0 & t \le 0. \end{cases}$$

Weighted polyparabolic Bergman spaces are defined by

$$\boldsymbol{b}_\alpha^{m,p,\lambda} := \{u \in C^\infty(\boldsymbol{H}) | \forall t > 0, \forall(\beta,j) \in \boldsymbol{N}_0^{n+1}, \sup_{x\in\boldsymbol{R}^n} |\partial_t^j \partial_x^\beta u(t,x)| < \infty,$$
$$(\delta_{\boldsymbol{H}})^k \big(L^{(\alpha)}\big)^k u \in L^p(\boldsymbol{H}, V^\lambda) \text{ for } \forall k \in \boldsymbol{N}_0, \text{ and } \big(L^{(\alpha)}\big)^m u = 0\},$$

where $\boldsymbol{N}_0 = \boldsymbol{N} \cup \{0\}$, $\mathrm{d}V^\lambda(x,t) = t^\lambda \mathrm{d}t\mathrm{d}x$, and $\delta_{\boldsymbol{H}}$ denotes the distance function from $\partial\boldsymbol{H}$, i.e., $\delta_{\boldsymbol{H}}(x,t) = t$. These spaces turn out to be closed in $L^p(\boldsymbol{H}, V^\lambda)$ in Section 4. We also remark that $\boldsymbol{b}_{1/2}^{m,p,\lambda} = \boldsymbol{b}^{m,p,\lambda}$ in Section 5.3.

When $m = 1$, we denote $\boldsymbol{b}^{1,p,\lambda}$, $\boldsymbol{b}_\alpha^{1,p,\lambda}$ by $\boldsymbol{b}^{p,\lambda}$, $\boldsymbol{b}_\alpha^{p,\lambda}$, and call them harmonic Bergman spaces, parabolic Bergman spaces, respectively.

In this paper, we give integral kernels explicitely by using the Laguerre polynomials and the fractional derivatives of the fundamental solution $W^{(\alpha)}$ of $L^{(\alpha)}$, and discuss their reproducing properties.

Let $(p_j^\lambda)_{j=0}^\infty$ be a sequence of orthonormal polynomials in $L^2(\boldsymbol{R}_+, e^{-t} t^\lambda \mathrm{d}t)$ with $\deg p_k^\lambda(t) = k$, i.e.,

$$\int_0^\infty p_j^\lambda(t) p_k^\lambda(t) e^{-t} t^\lambda \mathrm{d}t = \delta_{jk}.$$

The kernel

$$(2) \qquad k^{m,\lambda}(t,s) = \sum_{k,l=0}^{m-1} c_{k,l}^{m,\lambda} t^k s^l := \sum_{j=0}^{m-1} p_j^\lambda(t) p_j^\lambda(s)$$

has a reproducing property

$$\int_0^\infty k^{m,\lambda}(t,s)\rho(s)e^{-s}s^\lambda \mathrm{d}s = \rho(t) \tag{3}$$

for every polynomial ρ in t of degree less than m. Here, we define the polyparabolic Bergman kernel $\mathcal{K}_\alpha^{m,\lambda}$ by

$$\mathcal{K}_\alpha^{m,\lambda}(X,Y) = \sum_{k,l=0}^{m-1} 2^{k+l+\lambda+1} c_{k,l}^{m,\lambda} t^k s^l \mathcal{W}_\alpha^{k+l+\lambda+1}(X,Y),$$

where $c_{k,l}^{m,\lambda}$ are constants determined in (2) and

$$\mathcal{W}_\alpha^{k+l+\lambda+1}((x,t),(y,s)) := (-\partial_t)^{k+l+\lambda+1} W^{(\alpha)}(x-y,t+s). \tag{4}$$

Here, $(-\partial_t)^\nu$ stands for the fractional derivative in t for $\nu \in \boldsymbol{R}$ (see Section 2). We also define a kernel

$$\mathcal{W}_\alpha^{(\beta,\kappa)}((x,t),(y,s)) := (-\partial_t)^\kappa \partial_x^\beta W^{(\alpha)}(x-y,t+s) \tag{5}$$

for $\beta \in \boldsymbol{N}_0^n$, $\kappa > -(n+|\beta|)/(2\alpha)$.

We define the integral operator $K_\alpha^{m,\lambda}$ by

$$K_\alpha^{m,\lambda} f(X) := \int_{\boldsymbol{H}} \mathcal{K}_\alpha^{m,\lambda}(X,Y) f(Y) \mathrm{d}V^\lambda(Y). \tag{6}$$

The results of this paper are the following theorems.

THEOREM 1. *Let* $0 < \alpha \le 1$, $m \in \boldsymbol{N}$, $1 \le p < \infty$, $-1 < \lambda, \eta < \infty$, $u \in \boldsymbol{b}_\alpha^{m,p,\lambda}$ *and* $X \in \boldsymbol{H}$. *Then, we have*

$$u(X) = K_\alpha^{m,\eta} u(X) \quad \left(= \int_{\boldsymbol{H}} \mathcal{K}_\alpha^{m,\eta}(X,Y) u(Y) \mathrm{d}V^\eta(Y) \right),$$

when $\eta \ge \lambda$.

THEOREM 2. *The above integral operator* $K_\alpha^{m,\eta}$ *is bounded on* $L^p(\boldsymbol{H}, V^\lambda)$, *when* $(\lambda+1)/p < \eta + 1$, *i.e., when*

$$\begin{cases} \eta \ge \lambda, & 1 < p \le \infty \\ \eta > \lambda, & p = 1. \end{cases}$$

THEOREM 3. *The integral operator* $K_\alpha^{m,\lambda}$ *is the orthogonal projection from* $L^2(\boldsymbol{H}, V^\lambda)$ *onto* $\boldsymbol{b}_\alpha^{m,2,\lambda}$.

The paper organized as follows. In Section 2, we summarize necessary results on parabolic Bergman spaces. We first establish reproducing properties for polyparabolic functions in Section 3. Next, in Section 4, as an application, we investigate some properties of polyparabolic Bergman spaces, for example, completeness, boundedness of projections. Finally, in Section 5, we discuss dualities for $1 < p < \infty$. As for the case of $p = 1$, we need further discussions, and we shall treat this problem elsewhere. We also comment on boundedness of point evaluations, norm inequalities, and the relation with polyharmonic Bergman spaces.

Constants: In the rest of the paper we use the same letter C to denote various positive constants which may change at each occurrence.

2. PRELIMINARIES

2.1. Fractional Laplacian and fundamental solution

Fractional Laplacians are given by the Fourier multipliers $(-\Delta_x)^\alpha = \mathcal{F}_x^{-1}|\xi|^{2\alpha}\mathcal{F}_x$, where

$$\mathcal{F}_x f(\xi) := \int_{\boldsymbol{R}^n} f(x)e^{-ix\cdot\xi}\mathrm{d}x \quad \text{and} \quad \mathcal{F}_x^{-1} f(x) := \frac{1}{(2\pi)^n}\int_{\boldsymbol{R}^n} f(\xi)e^{ix\cdot\xi}\mathrm{d}\xi.$$

Note that when $0 < \alpha < 1$, for $\varphi \in C_c^\infty(\boldsymbol{R}^n)$,

$$(-\Delta_x)^\alpha\varphi(x) = -c_{n,\alpha}\lim_{\delta\downarrow 0}\int_{|y|>\delta}\big(\varphi(x-y)-\varphi(x)\big)|y|^{-n-2\alpha}\,\mathrm{d}y,$$

where $C_c^\infty(\boldsymbol{R}^n)$ stands for the totality of C^∞-functions with compact support and

$$c_{n,\alpha} = -4^\alpha\pi^{-n/2}\Gamma\big((n+2\alpha)/2\big)/\Gamma(-\alpha),$$

i.e., $(-\Delta_x)^\alpha$ is a convolution operator defined by $-c_{n,\alpha}\mathrm{p.f.}|x|^{-n-2\alpha}$. Let $C_b^2(\boldsymbol{R}^n)$ be the space of all C^2-functions with bounded derivatives. We also remark that for $f \in C_b^2(\boldsymbol{R}^n)$, putting

$$F(x) := \lim_{\delta\downarrow 0}\int_{|y|>\delta}\big(f(x-y)-f(x)\big)|y|^{-n-2\alpha}\,\mathrm{d}y,$$

we have

$$\int_{\boldsymbol{R}^n} F(x)\varphi(x)dx = \int_{\boldsymbol{R}^n} f(x)(-\Delta_x)^\alpha\varphi(x)\mathrm{d}x \tag{7}$$

for every $\varphi \in C_c^\infty(\boldsymbol{R}^n)$. In general, for locally integrable functions f and F satisfying (7), we write $(-\Delta_x)^\alpha f = F$ (weakly). In [3], Hishikawa introduced weighted parabolic Bergman spaces, defined by

$$\boldsymbol{b}_\alpha^p(\lambda) := \{h \in C(\boldsymbol{H}) | L^{(\alpha)}h = 0 \text{ (weakly)}, h \in L^p(\boldsymbol{H}, V^\lambda)\}$$

for $1 \le p < \infty$ and $\lambda > -1$. We remark that these are the same space as $\boldsymbol{b}_\alpha^{p,\lambda}$ stated in the introduction.

Note that $W^{(\alpha)}(x,t) = \mathcal{F}_x^{-1}(e^{-t|\xi|^{2\alpha}})(x)$ for $(x,t) \in \boldsymbol{H}$. Then we have

$$W^{(1)}(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} \quad \text{and} \quad W^{(1/2)}(x,t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{t}{\left(t^2 + |x|^2\right)^{\frac{n+1}{2}}}$$

for $(x,t) \in \boldsymbol{H}$. In general, we have shown in [6] that

$$0 \le W^{(\alpha)}(x,t) \le Ct(t + |x|^{2\alpha})^{-\frac{n}{2\alpha}-1} \le C(t + |x|^{2\alpha})^{-\frac{n}{2\alpha}}.$$

2.2. **Fractional derivatives**

We recall here some estimates of the fundamental solution and parabolic Bergman functions according to [3].

For negative real number κ and continuous function $\rho \in C(\boldsymbol{R}_+)$ satisfying $|\rho(t)| \le Ct^{\kappa'}$ $(t \in \boldsymbol{R}_+)$ with some $C > 0$ and $\kappa' < \kappa$, we define the fractional derivative of ρ with order κ by

$$(-\partial_t)^{\kappa} \rho(t) = \frac{1}{\Gamma(-\kappa)} \int_0^{\infty} \tau^{-\kappa-1} \rho(\tau + t)\, \mathrm{d}\tau, \quad t \in \boldsymbol{R}_+.$$

For $\kappa = 0$, we define $(-\partial_t)^0 \rho = \rho$. For positive real number κ, $\lceil \kappa \rceil$ denotes the smallest integer which is greater than or equal to κ. For $\rho \in C^{\lceil \kappa \rceil}(\boldsymbol{R}_+)$ satisfying $|(-\partial_t)^{\lceil \kappa \rceil} \rho(t)| \le Ct^{\kappa'}$ $(t \in \boldsymbol{R}_+)$ with some $C > 0$ and $\kappa' < \kappa - \lceil \kappa \rceil$, we define the fractional derivative of ρ with order κ by

$$(-\partial_t)^{\kappa} \rho(t) = (-\partial_t)^{\kappa - \lceil \kappa \rceil} (-\partial_t)^{\lceil \kappa \rceil} \rho(t), \quad t \in \boldsymbol{R}_+.$$

Example 1. ([3, Example 2.2]) Let $a > 0$ and $\nu \in \boldsymbol{R}$. Then we have

(i) $(-\partial_t)^{\nu} e^{-at} = a^{\nu} e^{-at}$,

(ii) if $\nu > -a$, $(-\partial_t)^{\nu} t^{-a} = t^{-a-\nu} \Gamma(a+\nu)/\Gamma(a)$.

We summarize necessary results from [3].

LEMMA 1 ([3, Theorem 3.1]). *Let $0 < \alpha \le 1$ and $\beta \in \boldsymbol{N}_0^n$. If $\kappa > -n/(2\alpha)$, then we have*

(i) *the derivatives $(-\partial_t)^{\kappa} \partial_x^{\beta} W^{(\alpha)} = \partial_x^{\beta} (-\partial_t)^{\kappa} W^{(\alpha)}$ are well-defined and there exists a positive constant C such that*

$$|(-\partial_t)^{\kappa} \partial_x^{\beta} W^{(\alpha)}(x,t)| \le C(t + |x|^{2\alpha})^{-\frac{n+|\beta|}{2\alpha}-\kappa} \tag{8}$$

for all $(x,t) \in \boldsymbol{H}$,

(ii) *if $\nu > -n/(2\alpha) - \kappa$, then $(-\partial_t)^{\nu} \partial_x^{\beta} (-\partial_t)^{\kappa} W^{(\alpha)} = \partial_x^{\beta} (-\partial_t)^{\kappa+\nu} W^{(\alpha)}$,*

(iii) $L^{(\alpha)}(\partial_x^\beta(-\partial_t)^\kappa W^{(\alpha)}) = 0$ *(weakly) on* $\boldsymbol{H}$, *and*

(iv) $\partial_x^\beta(-\partial_t)^\kappa W^{(\alpha)}(x,t) = t^{-\frac{n+|\beta|}{2\alpha}-\kappa}\partial_x^\beta(-\partial_t)^\kappa W^{(\alpha)}(t^{-\frac{1}{2\alpha}}x,1),\ \forall(x,t)\in\boldsymbol{H}$.

LEMMA 2 ([3, Proposition 5.1]). *If* $0<\alpha\le 1$, $1\le p<\infty$, $\lambda>-1$, *and* $\nu>-(\lambda+1)/p$, *then there exists a positive constant* C *such that*

$$\|t^\nu(-\partial_t)^\nu h\|_{L^p(\boldsymbol{H},V^\lambda)} \le C\|h\|_{L^p(\boldsymbol{H},V^\lambda)}$$

for all $h\in\boldsymbol{b}_\alpha^{p,\lambda}$. *Therefore,* $(-\partial_t)^\nu h\in\boldsymbol{b}_\alpha^{p,\lambda+\nu p}$ *if* $h\in\boldsymbol{b}_\alpha^{p,\lambda}$.

LEMMA 3 ([3, Proposition 4.1]). *Let* $0<\alpha\le 1$, $1\le p<\infty$, $\lambda>-1$. *If* $\nu>-(n/(2\alpha)+\lambda+1)/p$ *and* $\kappa+\nu>-(n/(2\alpha)+\lambda+1)/p$, *then*

$$(-\partial_t)^\kappa(-\partial_t)^\nu h(X) = (-\partial_t)^{\kappa+\nu}h(X)$$

for all $h\in\boldsymbol{b}_\alpha^{p,\lambda}$ *and* $X\in\boldsymbol{H}$.

LEMMA 4 ([3, Theorem 5.2]). *Let* $0<\alpha\le 1, 1\le p<\infty$ *and* $\lambda>-1$. *If* $\nu>-(\lambda+1)/p$ *and* $\kappa>(\lambda+1)/p$, *then*

$$h(X) = \frac{2^{\nu+\kappa}}{\Gamma(\nu+\kappa)}\int_{\boldsymbol{H}}\mathcal{W}_\alpha^\kappa(X,Y)(-\partial_t)^\nu h(Y)\,\mathrm{d}V^{\kappa+\nu-1}(Y)$$

for all $h\in\boldsymbol{b}_\alpha^{p,\lambda}$. *In the case of* $p=1$, *the equality also holds for* $\kappa=\lambda+1$.

2.3. Parabolic mean operators and their L^p-boundedness

We recall α-parabolic similarities. For $t>0$, $\tau_t^{(\alpha)}(y,s) := (t^{1/(2\alpha)}y, ts)$ is called an α-parabolic dilation. For $X=(x,t)\in\boldsymbol{H}$, the mapping $\Phi_X := T_x\circ\tau_t^{(\alpha)}$ on $\boldsymbol{H}$ is called a parabolic similarity, where $T_x(y,s)=(x+y,s)$ is a translation. Note that $\Phi_X(y,s) = (t^{1/(2\alpha)}y+x, ts)$ and from α-parabolic homogeneity follows

$$\mathcal{W}_\alpha^{(\beta,\kappa)}(\Phi_X(Y),\Phi_X(Z)) = t^{-\frac{n+|\beta|}{2\alpha}-\kappa}\mathcal{W}_\alpha^{(\beta,\kappa)}(Y,Z) \tag{9}$$

for (β,κ) with $\kappa>-(n+|\beta|)/(2\alpha)$.

For a Radon measure ρ on $\boldsymbol{H}$, an α-parabolic mean operator with symbol ρ is defined by

$$\mathcal{I}_\rho f(X) := \int_{\boldsymbol{H}} f(\Phi_X(Y))\,\mathrm{d}\rho(Y).$$

The boundedness of a mean operator $\mathcal{I}_\rho$ is given by the following

LEMMA 5 ([9, Proposition 2]). *For* $1\le p\le\infty$, $\lambda>-1$ *and a Radon measure* ρ *on* $\boldsymbol{H}$, $\mathcal{I}_\rho$ *is bounded on* $L^p(\boldsymbol{H},V^\lambda)$ *if*

$$\int s^{-\frac{\lambda+1}{p}}\,\mathrm{d}|\rho|(y,s)<\infty.$$

Proof. By the Minkowski inequality, for a measurable function f on $\boldsymbol{H}$, we have

$$\begin{aligned}\|\mathcal{I}_\rho f\|_{L^p(\boldsymbol{H},V^\lambda)} &= \Big(\int_{\boldsymbol{H}}\Big|\int_{\boldsymbol{H}} f(t^{\frac{1}{2\alpha}}y+x,ts)\,\mathrm{d}\rho(y,s)\Big|^p \mathrm{d}V^\lambda(x,t)\Big)^{1/p}\\ &\le \int_{\boldsymbol{H}}\Big(\int_{\boldsymbol{H}} |f(t^{\frac{1}{2\alpha}}y+x,ts)|^p t^\lambda\,\mathrm{d}V(x,t)\Big)^{1/p}\mathrm{d}|\rho|(y,s)\\ &= \|f\|_{L^p(\boldsymbol{H},V^\lambda)}\int_{\boldsymbol{H}} s^{-\frac{\lambda+1}{p}}\,\mathrm{d}|\rho|(y,s),\end{aligned}$$

which proves the lemma. □

Some related operators are expressed as mean operators.

Example 2. Let $\eta > -1$, $\kappa > -n/(2\alpha) - 1 - \eta$ and $\lambda > -1$. Then,

$$\begin{aligned}\mathcal{I}_\alpha^{\kappa+\eta+1,\eta} f(X) &:= \int_{\boldsymbol{H}} \mathcal{W}_\alpha^{\kappa+\eta+1}(X_0,Y) f(\Phi_X(Y))\mathrm{d}V^\eta(Y)\\ &= t^\kappa \int_{\boldsymbol{H}} \mathcal{W}_\alpha^{\kappa+\eta+1}(X,Y) f(Y)\mathrm{d}V^\eta(Y)\end{aligned}$$

is an α-parabolic mean operator, where $X_0 := (0,1)$.

LEMMA 6. *Let $0 < \alpha \le 1$, $1 \le p < \infty$, $\lambda, \eta > -1$ and $\kappa > -n/(2\alpha) - 1 - \eta$. Then the mean operator $\mathcal{I}_\alpha^{\kappa+\eta+1,\eta}$ is bounded on $L^p(\boldsymbol{H}, V^\lambda)$ if $(\lambda+1)/p < \eta + 1$.*

Proof. The lemma follows from (8), Lemma 5, and the following integrability. □

LEMMA 7. *If $-1 < \lambda < \delta - n/(2\alpha) - 1$, then there exists a positive constant C such that*

$$\int_{\boldsymbol{H}} (t+s+|x-y|^{2\alpha})^{-\delta}\,\mathrm{d}V^\lambda(y,s) = Ct^{\lambda-\delta+\frac{n}{2\alpha}+1} < \infty \tag{10}$$

for all $(x,t) \in \boldsymbol{H}$.

LEMMA 8. *If $1 \le q \le \infty$ and $(\frac{n}{2\alpha} + 1 + \lambda)\frac{1}{q} - (\frac{n+|\beta|}{2\alpha} + \kappa) < 0$, then there exists a positive constant C such that*

$$\|\mathcal{W}_\alpha^{(\beta,\kappa)}(X,\cdot)\|_{L^q(\boldsymbol{H},V^\lambda)} \le Ct^{(\frac{n}{2\alpha}+1+\lambda)\frac{1}{q}-(\frac{n+|\beta|}{2\alpha}+\kappa)}. \tag{11}$$

Proof. The inequality follows from (8) and (10). □

3. REPRODUCING PROPERTIES

In this section, we shall show a generalized version of Theorem 1. We begin with the following Almansi type decomposition.

LEMMA 9. *For each* $u \in \boldsymbol{b}_\alpha^{m,p,\lambda}$, *there exist unique* $h_j \in \boldsymbol{b}_\alpha^{p,\lambda+jp}$ $(j = 0, \ldots, m-1)$ *such that*

$$u = h_0 + th_1 + \cdots + t^{m-1}h_{m-1}.$$

Proof. We shall show the lemma by induction. If $m = 1$, then the lemma is trivial. Next, we assume the lemma for $m-1$, and take any $u \in \boldsymbol{b}_\alpha^{m,p,\lambda}$. We put $h_{m-1} := (L^{(\alpha)})^{m-1}u/(m-1)!$ and $v := u - t^{m-1}h_{m-1}$. Then we can easily see that $t^{m-1}h_{m-1} \in L^p(\boldsymbol{H}, V^\lambda)$, $(L^{(\alpha)})^{m-1}v = 0$, and $t^k(L^{(\alpha)})^k v = t^k(L^{(\alpha)})^k u - Ct^{m-1}h_{m-1} \in L^p(\boldsymbol{H}, V^\lambda)$ for nonnegative integer k, where C is a constant depending on m and k. Thus we have $v = h_0 + \cdots + t^{m-2}h_{m-2}$ with suitable $h_0, \ldots, h_{m-2}$ by assumption, which implies the lemma for m. □

The opposite assertion of Lemma 9 is trivial, and hence, we have

$$\boldsymbol{b}_\alpha^{m,p,\lambda} = \{h_0 + th_1 + \cdots + t^{m-1}h_{m-1} \mid h_j \in \boldsymbol{b}_\alpha^{p,\lambda+jp}\ (j = 0, \ldots, m-1)\}. \tag{12}$$

We denote the correspondence of the above Almansi type decomposition by

$$H_j^m u := h_j, \tag{13}$$

and introduce an operator on $\boldsymbol{b}_\alpha^{m,p,\lambda}$ by

$$\mathcal{E}_t^\nu u := \sum_{j=0}^{m-1} t^j(-\partial_t)^\nu H_j^m u \tag{14}$$

for $u \in \boldsymbol{b}_\alpha^{m,p,\lambda}$ and $\nu > -(\lambda+1)/p$. A main theorem of this section is the following

THEOREM 4. *Let* $-1 < \lambda \leq \eta$ *and* $\nu > -(\lambda+1)/p$. *Then we have*

$$u(X) = \int_{\boldsymbol{H}} \mathcal{E}_t^{-\nu}\mathcal{K}_\alpha^{m,\eta+\nu}(X,Y)\mathcal{E}_t^\nu u(Y)\mathrm{d}V^{\eta+\nu}(Y)$$

for $u \in \boldsymbol{b}_\alpha^{m,p,\lambda}$, *where we use the following notation:*

$$\mathcal{E}_t^{-\nu}\mathcal{K}_\alpha^{m,\eta+\nu}((x,t),(y,s)) := \sum_{k,l=0}^{m-1} 2^{k+l+\eta+\nu+1}c_{k,l}^{m,\eta+\nu}t^k s^l \mathcal{W}_\alpha^{k+l+\eta+1}((x,t),(y,s)).$$

Proof. Let $u = h_0 + th_1 + \cdots + t^{m-1}h_{m-1} \in \boldsymbol{b}_\alpha^{m,p,\lambda}$, where $h_j \in \boldsymbol{b}_\alpha^{p,\lambda+jp}$ for $j = 0, \ldots, m-1$. First, we shall show that, for $j, k, l = 0, 1, \ldots, m-1$,

$$\int_{\boldsymbol{H}} t^k s^l \mathcal{W}_\alpha^{k+l+\eta+1}(X,Y) s^j \big((-\partial_t)^\nu h_j\big)(Y) \mathrm{d}V^{\eta+\nu}(Y) = Ct^k(-\partial_t)^{k-j}h_j(X) \tag{15}$$

with some constant C. Since $(-\partial_t)^\nu h_j = (-\partial_t)^{\nu+j-k}(-\partial_t)^{k-j}h_j$ by Lemma 3 and since $(-\partial_t)^{k-j}h_j \in \boldsymbol{b}_\alpha^{p,\lambda+kp}$ by Lemma 2, it follows from Lemma 4 that

$$(-\partial_t)^{k-j}h_j(X) = \frac{2^{j+l+\eta+\nu+1}}{\Gamma(j+l+\eta+\nu+1)} \int_{\boldsymbol{H}} \mathcal{W}_\alpha^{k+l+\eta+1}(X,Y)(-\partial_t)^\nu h_j(Y) \mathrm{d}V^{l+j+\eta+\nu}(Y),$$

whose integrability condition $k+l+\eta+1 > (\lambda+kp+1)/p$ for $1 < p < \infty$ or $k+l+\eta+1 \geq (\lambda+kp+1)/p$ for $p = 1$ is easily verified from the assumption.

Next, from (15), for each $j = 0, 1, \ldots, m-1$, we can write

$$\int_{\boldsymbol{H}} \mathcal{E}_t^{-\nu}\mathcal{K}_\alpha^{m,\eta+\nu}(X,Y)(-\partial_t)^\nu h_j(Y) \mathrm{d}V^{j+\eta+\nu}(Y) = \sum_{k=0}^{m-1} c'_{k,j} t^k (-\partial_t)^{k-j} h_j(X), \tag{16}$$

with some constants $c'_{k,j}$. We shall show $c'_{k,j} = \delta_{k,j}$. Taking a sufficiently large κ, such that $h_j(x,t) := (-2\partial_t)^\kappa W^{(\alpha)}(x, t+\tau) \in \boldsymbol{b}_\alpha^{2,\lambda+2j}$ for all $\tau > 0$, we shall show that

$$\int_{\boldsymbol{H}} \mathcal{E}_t^{-\nu}\mathcal{K}_\alpha^{m,\eta+\nu}(X,Y)(-\partial_t)^\nu h_j(Y) \mathrm{d}V^{j+\eta+\nu}(Y) = h_j(X). \tag{17}$$

The left hand side is equal to

$$\int_0^\infty \Big(\int_{\boldsymbol{R}^n} \sum_{k,l=0}^{m-1} c_{k,l}^{m,\eta+\nu} t^k s^l (-2\partial_t)^{k+l+\eta+1} W^{(\alpha)}(x-y, t+s) \times s^j (-2\partial_t)^{\nu+\kappa} W^{(\alpha)}(y, s+\tau) dy \Big) s^{\eta+\nu} \mathrm{d}s$$

$$= \int_0^\infty \mathcal{F}_x^{-1} \Big(\sum_{k,l=0}^{m-1} c_{k,l}^{m,\eta+\nu} t^k s^l (2|\xi|^{2\alpha})^{k+l+\eta+1} e^{-(t+s)|\xi|^{2\alpha}} \times s^j (2|\xi|^{2\alpha})^{\nu+\kappa} e^{-(s+\tau)|\xi|^{2\alpha}} \Big)(x) s^{\eta+\nu} \mathrm{d}s$$

$$= \mathcal{F}_x^{-1} \Big(\int_0^\infty \sum_{k,l=0}^{m-1} c_{k,l}^{m,\eta+\nu} (2t|\xi|^{2\alpha})^k (2s|\xi|^{2\alpha})^l e^{-(t+\tau+2s)|\xi|^{2\alpha}}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad \times s^j (2|\xi|^{2\alpha})^\kappa (2s|\xi|^{2\alpha})^{\eta+\nu+1} \mathrm{d}s\Big)(x) \\
&= \mathcal{F}_x^{-1}\Big(e^{-(t+\tau)|\xi|^{2\alpha}} (2|\xi|^{2\alpha})^{\kappa-j} \int_0^\infty k^{m,\eta+\nu}(2t|\xi|^{2\alpha}, 2s|\xi|^{2\alpha}) \\
&\qquad\qquad\qquad\qquad \times e^{-2s|\xi|^{2\alpha}} (2s|\xi|^{2\alpha})^{j+\eta+\nu+1} \mathrm{d}s\Big)(x) \\
&= \mathcal{F}_x^{-1}\Big(e^{-(t+\tau)|\xi|^{2\alpha}} (2|\xi|^{2\alpha})^{\kappa-j} \int_0^\infty k^{m,\eta+\nu}(2t|\xi|^{2\alpha}, s) s^j e^{-s}\, s^{\eta+\nu+1} \mathrm{d}s\Big)(x) \\
&= \mathcal{F}_x^{-1}\Big(e^{-(t+\tau)|\xi|^{2\alpha}} (2|\xi|^{2\alpha})^{\kappa-j} (2t|\xi|^{2\alpha})^j\Big)(x) \\
&= t^j \mathcal{F}_x^{-1}\Big((2|\xi|^{2\alpha})^\kappa e^{-(t+\tau)|\xi|^{2\alpha}}\Big)(x) \\
&= t^j(-2\partial_t)^\kappa W^{(\alpha)}(x, t+\tau),
\end{aligned}
$$

which shows (17), where we use the reproducing property (3).

Finally, comparing this with (16), we have

$$
t^j(-2\partial_t)^\kappa W^{(\alpha)}(x, t+\tau) = \sum_{k=0}^{m-1} c'_{k,j} t^k (-\partial_t)^{k-j} (-2\partial_t)^\kappa W^{(\alpha)}(x, t+\tau).
$$

Putting $(x,t) = (0,1)$, by Lemma 1 (iv), we obtain

$$
(-\partial_t)^\kappa W^{(\alpha)}(0,1) = \sum_{k=0}^{m-1} (1+\tau)^{-k+j} c'_{k,j} (-\partial_t)^{k-j+\kappa} W^{(\alpha)}(0,1)
$$

for all $\tau > 0$. Then we have $c'_{k,j} = 0$ for $k \neq j$ and $c'_{j,j} = 1$. This completes the proof. □

Remark 1. In the above proof, using Example 1 (ii) and computing constants exactly, we have an $m \times m$-matrix equality

$$
\Big(2^{k+l} c_{k,l}^{m,\lambda}\Big)_{k,l} = \Big(\Gamma(k+l+\lambda+1)\Big)_{k,l}^{-1}.
$$

4. POLYPARABOLIC BERGMAN SPACES

In this section, we consider the polyparabolic Bergman spaces, and discuss some properties. First, we consider the polyparabolic Bergman kernels.

LEMMA 10. *Let* $(k,\beta) \in \boldsymbol{N}_0 \times \boldsymbol{N}_0^n$. *Then for* $1 < q \leq \infty$ *and* $X \in \boldsymbol{H}$, *we have*

$$
\|\partial_t^k \partial_x^\beta \mathcal{K}_\alpha^{m,\lambda}(X,\,\cdot\,)\|_{L^q(\boldsymbol{H}, V^\lambda)} \leq C t^{-(\frac{|\beta|}{2\alpha}+k)-(\frac{n}{2\alpha}+1+\lambda)(1-\frac{1}{q})}. \tag{18}
$$

Proof. Recalling the form of $\mathcal{K}_\alpha^{m,\lambda}$, we see that $\partial_t^k \partial_x^\beta \mathcal{K}_\alpha^{m,\lambda}$ is a linear combination of

$$t^j s^l \mathcal{W}_\alpha^{(\beta,\, k+j+l+\lambda+1)}(X,Y),$$

which is bounded by

$$C(t+s+|x-y|^{2\alpha})^{-\frac{n+|\beta|}{2\alpha}-k-\lambda-1}$$

from Lemma 1 (i). Then applying (10), we have Lemma 10. □

Next, we shall show the L^p-boundedness of projections, which implies Theorem 2.

THEOREM 5. *Let* $1 \le p < \infty$. *If* $\eta + 1 > (\lambda+1)/p > 0$, *then the integral operator* $K_\alpha^{m,\eta}$ *is bounded on* $L^p(\boldsymbol{H}, V^\lambda)$ *and the image of* $K_\alpha^{m,\eta}$ *is equal to* $\boldsymbol{b}_\alpha^{m,p,\lambda}$. *Moreover,* $\boldsymbol{b}_\alpha^{m,p,\lambda}$ *is a Banach space.*

Proof. We first remark that $K_\alpha^{m,\eta}$ is a linear combination

$$K_\alpha^{m,\eta} = \sum_{k,l=0}^{m-1} 2^{k+l+\eta+1} c_{k,l}^{m,\eta} \mathcal{I}_\alpha^{k+l+\eta+1,l+\eta},$$

where $\mathcal{I}_\alpha^{k+l+\eta+1,l+\eta}$ is the operator defined in Example 2. Then, assuming $\eta+1 > (\lambda+1)/p$, we obtain the L^p-boundedness by Lemma 6, because $p > (\lambda+1)/(\eta+1) \ge (\lambda+1)/(l+\eta+1)$.

Next, we shall determine the image of $K_\alpha^{m,\eta}$. For $f \in L^p(\boldsymbol{H}, V^\lambda)$, remarking that

$$L^{(\alpha)}\Big(\int_{\boldsymbol{H}} \mathcal{W}_\alpha^{k+l+\eta+1}(\,\cdot\,,Y) f(Y)\,\mathrm{d}V^{l+\eta}(Y)\Big) = 0, \quad \text{(weakly)}$$

by [4, Theorem 3.1 (1)], we may regard

$$K_\alpha^{m,\eta} f = \sum_{k=0}^{m-1}\Big(\sum_{l=0}^{m-1} 2^{k+l+\eta+1} c_{k,l}^{m,\eta} \mathcal{I}_\alpha^{k+l+\eta+1,l+\eta} f\Big)$$

as an Almansi type decomposition, which shows $K_\alpha^{m,\eta} f \in \boldsymbol{b}_\alpha^{m,p,\lambda}$ by (12).

Finally, to show that $\boldsymbol{b}_\alpha^{m,p,\lambda}$ is a closed subspace of $L^p(\boldsymbol{H}, V^\lambda)$, we take any sequence (u_j) in $\boldsymbol{b}_\alpha^{m,p,\lambda}$ which converges to some $f \in L^p(\boldsymbol{H}, V^\lambda)$. Then we have

$$f = \lim_{j\to\infty} u_j = \lim_{j\to\infty} K_\alpha^{m,\eta} u_j = K_\alpha^{m,\eta} f \in \boldsymbol{b}_\alpha^{m,p,\lambda}.$$

This completes the proof. □

Finally, as a corollary, we have Theorem 3. In fact $K_\alpha^{m,\lambda}$ is self-adjoint on $L^2(\boldsymbol{H}, V^\lambda)$, because the kernel $\mathcal{K}_\alpha^{m,\lambda}$ is symmetric. Idempotent property $(K_\alpha^{m,\lambda})^2 = K_\alpha^{m,\lambda}$ follows from reproducing property. Then $K_\alpha^{m,\lambda}$ is an orthogonal projection from $L^2(\boldsymbol{H}, V^\lambda)$ onto $\boldsymbol{b}_\alpha^{m,2,\lambda}$.

5. APPLICATIONS

5.1. Dual spaces

First application of the reproducing property is to characterize dual spaces.

THEOREM 6. *Let $1 < p < \infty$, $\lambda > -1$, and $m \in \boldsymbol{N}$. Then the dual space of $\boldsymbol{b}_\alpha^{m,p,\lambda}$ is identified with $\boldsymbol{b}_\alpha^{m,p',\lambda}$ by*

$$\langle v, u\rangle_\lambda = \int_{\boldsymbol{H}} vu \,\mathrm{d}V^\lambda,$$

for $v \in \boldsymbol{b}_\alpha^{m,p',\lambda}, u \in \boldsymbol{b}_\alpha^{m,p,\lambda}$, where p' is the exponent conjugate to p.

Proof. For $v \in \boldsymbol{b}_\alpha^{m,p',\lambda}$, we put $T_v u := \langle v, u\rangle_\lambda$. Then $T_v \in (\boldsymbol{b}_\alpha^{m,p,\lambda})^*$. We shall show that the correspondence $\iota : v \to T_v$ is bijective, since it is easily follows from the Hölder inequality that ι is bounded linear.

To show that ι is injective, let $v \in \boldsymbol{b}_\alpha^{m,p',\lambda}$ and assume $T_v = 0$. Taking $X \in \boldsymbol{H}$ and considerling $u := \mathcal{K}_\alpha^{m,\lambda}(X, \cdot)$, we have $u \in \boldsymbol{b}_\alpha^{m,p,\lambda}$ and

$$0 = T_v u = \int_{\boldsymbol{H}} \mathcal{K}_\alpha^{m,\lambda}(X,Y)v(Y)\,\mathrm{d}V^\lambda(Y) = v(X),$$

by Theorem 1.

Next, we show the surjectivity. Take $T \in (\boldsymbol{b}_\alpha^{m,p,\lambda})^*$. By the Hahn-Banach theorem, there exists $f \in L^{p'}(\boldsymbol{H}, V^\lambda)$ such that $\|f\|_{L^{p'}(\boldsymbol{H},V^\lambda)} \le \|T\|$ and

$$Tu = \int_{\boldsymbol{H}} uf \,\mathrm{d}V^\lambda$$

for all $u \in \boldsymbol{b}_\alpha^{m,p,\lambda}$. Since

$$u(x) = \int_{\boldsymbol{H}} \mathcal{K}_\alpha^{m,\lambda}(X,Y)u(Y)\,\mathrm{d}V^\lambda(Y),$$

by the Fubini theorem, we have

$$\begin{aligned} Tu &= \int_{\boldsymbol{H}} u(X)f(X)\,\mathrm{d}V^\lambda(X) = \int_{\boldsymbol{H}}\int_{\boldsymbol{H}} u(Y)\mathcal{K}_\alpha^{m,\lambda}(X,Y)f(X)\,\mathrm{d}V^\lambda(Y)\mathrm{d}V^\lambda(X) \\ &= \int_{\boldsymbol{H}} u(Y)\int_{\boldsymbol{H}} \mathcal{K}_\alpha^{m,\lambda}(X,Y)f(X)\,\mathrm{d}V^\lambda(X)\,\mathrm{d}V^\lambda(Y). \end{aligned}$$

Theorem 2 shows that $v := \int_{\boldsymbol{H}} \mathcal{K}_\alpha^{m,\lambda}(X, \cdot)f(X)\,\mathrm{d}V^\lambda(X) \in \boldsymbol{b}_\alpha^{m,p',\lambda}$ and we have $T = T_v$.

Finally, using the open mapping theorem, we find that $\iota : \boldsymbol{b}_\alpha^{m,p',\lambda} \to (\boldsymbol{b}_\alpha^{m,p,\lambda})^*$ gives a norm-equivalent isomorphism. □

5.2. **Norm estimates**

Second application of the reproducing property is the boundedness of point evaluation and norm estimates.

THEOREM 7. *Let* $1 \le p < \infty$, $(k,\beta) \in \boldsymbol{N}_0 \times \boldsymbol{N}_0^n$, $\lambda > -1$, *and* $0 < \alpha \le 1$. *Then there exists a constant* C *such that*

$$|\partial_t^k \partial_x^\beta u(x,t)| \le C t^{-(\frac{n}{2\alpha}+\lambda+1)\frac{1}{p}-(\frac{|\beta|}{2\alpha}+k)} \|u\|_{L^p(\boldsymbol{H},V^\lambda)} \tag{19}$$

for $u \in \boldsymbol{b}_\alpha^{m,p,\lambda}$ *and* $(x,t) \in \boldsymbol{H}$.

Proof. Since

$$\partial_t^k \partial_x^\beta u(X) = \int \partial_t^k \partial_x^\beta \mathcal{K}_\alpha^{m,\lambda}(X,Y) u(Y)\, \mathrm{d}V^\lambda(Y),$$

(19) follows from the Hölder inequality and Lemma 10.
□

THEOREM 8. *Let* $1 \le p < \infty$, $(k,\beta) \in \boldsymbol{N}_0 \times \boldsymbol{N}_0^n$, $\lambda > -1$, *and* $0 < \alpha \le 1$. *Then we have*

$$\|t^{\frac{|\beta|}{2\alpha}+k} \partial_t^k \partial_x^\beta u\|_{L^p(\boldsymbol{H},V^\lambda)} \le C \|u\|_{L^p(\boldsymbol{H},V^\lambda)} \tag{20}$$

$$\|t^k (L^{(\alpha)})^k u\|_{L^p(\boldsymbol{H},V^\lambda)} \le C \|u\|_{L^p(\boldsymbol{H},V^\lambda)} \tag{21}$$

for $u \in \boldsymbol{b}_\alpha^{m,p,\lambda}$.

Proof. Let $\eta > (\lambda+1)/p - 1$. Since the kernel $\partial_t^k \partial_x^\beta \mathcal{K}_\alpha^{m,\eta}$ is a linear combination of

$$t^j s^l \mathcal{W}_\alpha^{(\beta,\, k+j+l+\eta+1)}(X,Y),$$

we have the homogeneity

$$\partial_t^k \partial_x^\beta \mathcal{K}_\alpha^{m,\eta}(\Phi_X(Y), \Phi_X(Z)) = t^{-\frac{n+|\beta|}{2\alpha}-(k+\eta+1)} \partial_t^k \partial_x^\beta \mathcal{K}_\alpha^{m,\eta}(Y,Z).$$

Therefore in a similar way as in Example 2, the integral operator $\mathcal{I}$ defined using the kernel

$$t^{\frac{|\beta|}{2\alpha}+k} \partial_t^k \partial_x^\beta \mathcal{K}_\alpha^{m,\eta}(X,Y)$$

is a mean operator whose symbol measure $\partial_t^k \partial_x^\beta \mathcal{K}_\alpha^{m,\eta}(X_0,Y)\mathrm{d}V^\eta(Y)$ is bounded by

$$C(1+s+|y|^{2\alpha})^{-\frac{n}{2\alpha}-(\eta+1)}\, \mathrm{d}V^\eta(Y)$$

by Lemma 1 (i) where $X_0 = (0,1)$. Then the $L^p(\boldsymbol{H},V^\lambda)$-boundedness of $\mathcal{I}$ follows from Lemma 5. Hence, we have

$$\|t^{\frac{|\beta|}{2\alpha}+k} \partial_t^k \partial_x^\beta u\|_{L^p(\boldsymbol{H},V^\lambda)} \le \|\mathcal{I}u\|_{L^p(\boldsymbol{H},V^\lambda)} \le C \|u\|_{L^p(\boldsymbol{H},V^\lambda)}$$

for $u \in \boldsymbol{b}_\alpha^{m,p,\lambda}$, which shows the norm inequality (20). To show (21), it suffices to show

$$\|t^{l+k}\partial_t^k\big((-\Delta_x)^\alpha\big)^l u\|_{L^p(\boldsymbol{H},V^\lambda)} \le C\|u\|_{L^p(\boldsymbol{H},V^\lambda)} \tag{22}$$

for $l \in \boldsymbol{N}_0$ and $u \in \boldsymbol{b}_\alpha^{m,p,\lambda}$. Since $u = K_\alpha^{m,\eta}u$ is a linear combination of

$$t^j \int_{\boldsymbol{H}} \mathcal{W}_\alpha^{j+l+\eta+1}(\,\cdot\,,Y)\,\mathrm{d}V^{l+\eta}(Y) =: t^j h_j$$

and $L^{(\alpha)}h_j = 0$ (weakly),

$$(-\Delta_x)^\alpha h_j = -\partial_t h_j \quad \text{(weakly)}.$$

Thus we have

$$\big((-\Delta_x)^\alpha\big)^l u = \mathcal{E}_t^l K_\alpha^{m,\eta}u,$$

where $\mathcal{E}_t$ was defined in (14), and hence (21) is obtained similarly as above. □

COROLLARY 1. *Let* $1 \le p < \infty$, $k = 0, 1, \ldots, m-1$, $\lambda > -1$, *and* $0 < \alpha \le 1$. *Then*

$$\|t^k H_k^m u\|_{L^p(\boldsymbol{H},V^\lambda)} \le C\|u\|_{L^p(\boldsymbol{H},V^\lambda)} \tag{23}$$

for $u \in \boldsymbol{b}_\alpha^{m,p,\lambda}$, *where* H_j^m *was defined in* (13).

5.3. **Polyharmonic Bergman spaces**

We already know the relation $\boldsymbol{b}^{p,\lambda} = \boldsymbol{b}_{1/2}^{p,\lambda}$. Now we give a relation with polyharmonic Bergman spaces.

First, we prove a lemma.

LEMMA 11. *If* $h_0, h_1, \ldots, h_{k-1}$ *are all harmonic in* $\boldsymbol{H}$, *then*

$$u = h_0 + th_1 + \ldots t^{k-1}h_{k-1}$$

is polyharmonic of degree k *in* $\boldsymbol{H}$.

Proof. We shall prove the lemma by induciton on k. The case $k = 1$ is trivial. Let $k > 1$.

$$\begin{aligned}\Delta u =& \sum_{j=1}^{k-1}\big(2(\nabla t^j)\cdot(\nabla h_j) + (\Delta t^j)h_j\big) + \sum_{j=1}^{k-1}\big(2jt^{j-1}\partial_t h_j + j(j-1)t^{j-2}h_j\big)\\ =& 2\partial_t h_1 + \sum_{j=1}^{k-3} t^j(j+1)\big(2\partial_t h_{j+1} + (j+1)h_{j+2}\big) + t^{k-2}2(k-1)(k-2)\partial_t h_{k-1}\end{aligned}$$

because h_j and $\partial_t h_j$ ($j = 0, 1, \ldots, k-1$) are harmonic. Since Δu is polyharmonic of degree $k-1$ by the induction assumption, we have the assertion for k. □

By the mean value properties for polyharmonic function [5, Lemma 5], we obtain the weighted version of [5, Theorem 6] and [5, Lemma 6].

PROPOSITION 1. *Let* $1 \le p < \infty$, $(k, \beta) \in \boldsymbol{N}_0 \times \boldsymbol{N}_0^n$, *and* $\lambda > -1$. *Then we have the following estimates*

(i) $|\partial_t^k \partial_x^\beta u(x,t)| \le C t^{-(k+|\beta|)-(n+\lambda+1)\frac{1}{p}}$

(ii) $|\partial_t^k \partial_x^\beta (-\Delta_x)^{1/2} u(x,t)| \le C t^{-(k+|\beta|+1)-(n+\lambda+1)\frac{1}{p}}$

(iii) $\|t^{k+|\beta|} \partial_t^k \partial_x^\beta u\|_{L^p(\boldsymbol{H},V^\lambda)} \le C \|u\|_{L^p(\boldsymbol{H},V^\lambda)}$

(iv) $\|t^{k+|\beta|+1} \partial_t^k \partial_x^\beta (-\Delta_x)^{1/2} u\|_{L^p(\boldsymbol{H},V^\lambda)} \le C \|u\|_{L^p(\boldsymbol{H},V^\lambda)}$

for $(x,t) \in \boldsymbol{H}$ *and* $u \in \boldsymbol{b}^{m,p,\lambda}$.

The proof proceeds in the same way as in [5]. Using this proposition, we can show the coincidence.

THEOREM 9. $\boldsymbol{b}_{1/2}^{m,p,\lambda} = \boldsymbol{b}^{m,p,\lambda}$.

Proof. If $u \in \boldsymbol{b}_{1/2}^{m,p,\lambda}$, then by Almansi type decomposition Lemma 9, there exist unique $h_j \in \boldsymbol{b}_{1/2}^{p,\lambda+jp}$ $(j = 0, \ldots, m-1)$ such that

$$u = h_0 + t h_1 + \cdots + t^{m-1} h_{m-1}.$$

Since we already know $\boldsymbol{b}_{1/2}^{p,\lambda+jp} = \boldsymbol{b}^{p,\lambda+jp}$, all h_j are harmonic in $\boldsymbol{H}$. Then $u \in \boldsymbol{b}^{m,p,\lambda}$, for the above lemma implies u is polyharmonic of degree m.

Let $u \in \boldsymbol{b}^{m,p,\lambda}$. By Proposition 1 (i), $\partial_t^j \partial_x^\beta u(t, \cdot)$ is bounded on $\boldsymbol{R}^n$ for $\forall t > 0$, $\forall (\beta, j) \in \boldsymbol{N}_0^{n+1}$, and $t^k \{(-\Delta_x)^{1/2}\}^k u$, $t^k \partial_t^k u \in L^p(\boldsymbol{H}, V^\lambda)$, we have

$$t^k \left(L^{(1/2)}\right)^k u \in L^p(\boldsymbol{H}, V^\lambda). \tag{24}$$

Now it suffices to show $\left(L^{(1/2)}\right)^m u = 0$. The case $m = 1$ is trivial. Let $m > 1$. Since $\Delta^{m-1} u$ is harmonic, $\Delta^{m-1} u \in \boldsymbol{b}^{p,\lambda+2(m-1)p} = \boldsymbol{b}_{1/2}^{p,\lambda+2(m-1)p}$, and hence $\Delta^{m-1}(L^{(1/2)} u) = L^{(1/2)}(\Delta^{m-1} u) = 0$. By (24), $\left(L^{(1/2)}\right) u \in \boldsymbol{b}^{m-1,p,\lambda+p}$, and hence we have $\left(L^{(1/2)}\right)^k u \in \boldsymbol{b}^{m-k,p,\lambda+kp}$ inductively. In particular,

$$\left(L^{(1/2)}\right)^{m-1} u = \boldsymbol{b}^{p,\lambda+(m-1)p} = \boldsymbol{b}_{1/2}^{p,\lambda+(m-1)p}$$

which shows $\left(L^{(1/2)}\right)^m u = 0$. $\square$

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