Dedicated to the memory of Cabiria Andreian-Cazacu

ON THE SPACE OF HOMOGENEOUS MODIFIED HARMONIC POLYNOMIALS IN FOUR DIMENSIONS

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The functions on $\mathbb{R}^3 \times (0,\infty)$ that are annihilated by the Laplace-Beltrami operator corresponding to the line-element $\mathrm{d}l^2 = s^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}t^2 + \mathrm{d}s^2)$ are called *modified harmonic*. In this note we prove a conjecture of Heinz Leutwiler concerning the space of homogeneous modified harmonic polynomials of a fixed degree.

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1. INTRODUCTION

The "upper half space" $\{(x,y,t,s) \in \mathbb{R}^4 \mid s > 0\}$ of $\mathbb{R}^4$ equipped with the line-element $\mathrm{d}l^2 = s^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}t^2 + \mathrm{d}s^2)$ becomes a Riemannian manifold, whose Laplace-Beltrami operator is $\frac{1}{s^2}\left(\Delta + \frac{2}{s} \cdot \frac{\partial}{\partial s}\right)$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial s^2}$. The functions u that are annihilated by this operator, or, more generally, the solutions of

$$s \cdot \Delta u + 2 \cdot \frac{\partial u}{\partial s} = 0$$

(waiving the restriction $s > 0$) are called *modified harmonic functions.* It is straightforward to see that this property passes from u to $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, and $\frac{\partial u}{\partial t}$.

In [1], Heinz Leutwiler introduces the space $\mathcal{H}_n(\mathbb{R}^4)$ of all homogeneous modified harmonic polynomials of degree n on $\mathbb{R}^4$ and shows that its dimension equals $\binom{n+2}{2}$. He further proves that if u is a modified harmonic function, then so is its *modified Kelvin transform*,

$$K[u](x,y,t,s) := \frac{1}{r^4} u\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{t}{r^2}, \frac{s}{r^2}\right),$$

where $r = \sqrt{x^2 + y^2 + t^2 + s^2}$.

Now, since $u(x,y,t,s) := \dfrac{1}{r^4}$ is a modified harmonic function (being the modified Kelvin transform of 1), so are its partial derivatives

$$u_{\alpha\beta\gamma} := \frac{\partial^n r^{-4}}{\partial x^\alpha \partial y^\beta \partial t^\gamma}$$

for $\alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}$, $\alpha+\beta+\gamma = n$, as well as their modified Kelvin transforms

$$\begin{aligned} v_{\alpha\beta\gamma}(x,y,t,s) &:= K[u_{\alpha\beta\gamma}](x,y,t,s) \\ &= \frac{1}{r^4} u_{\alpha\beta\gamma}\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{t}{r^2}, \frac{s}{r^2}\right) = r^{2n+4} \cdot \frac{\partial^n r^{-4}}{\partial x^\alpha \partial y^\beta \partial t^\gamma}, \end{aligned} \tag{1.1}$$

since $u_{\alpha\beta\gamma}$ is homogeneous of degree $-4-n$ (in fact, r^{-4} is homogeneous of degree -4, and every partial derivative reduces the degree of homogeneity by 1). Setting $R := r^2$, it follows by induction that $u_{\alpha\beta\gamma}$ has the form $R^{-2-n} \cdot P$, where P is a polynomial, whence $v_{\alpha\beta\gamma}$ is a polynomial too. Altogether, $v_{\alpha\beta\gamma} \in \mathcal{H}_n(\mathbb{R}^4)$.

In [1], Leutwiler conjectured that the $\binom{n+2}{2}$ polynomials $v_{\alpha\beta\gamma} \in \mathcal{H}_n(\mathbb{R}^4)$ are linearly independent (and therefore form a basis of $\mathcal{H}_n(\mathbb{R}^4)$). The purpose of this article is to prove this conjecture.

We close this introduction by listing the polynomials $v_{\alpha\beta\gamma}$ for $0 \le n \le 3$:
$v_{000} = 1$;
$v_{100} = -4x$, $v_{010} = -4y$, $v_{001} = -4t$;
$v_{200} = 4(5x^2 - y^2 - t^2 - s^2)$, $v_{110} = 24xy$, $v_{101} = 24xt$,
$v_{020} = 4(-x^2 + 5y^2 - t^2 - s^2)$, $v_{011} = 24yt$, $v_{002} = 4(-x^2 - y^2 + 5t^2 - s^2)$;
$v_{300} = 24x(-5x^2 + 3y^2 + 3t^2 + 3s^2)$, $v_{210} = 24y(-7x^2 + y^2 + t^2 + s^2)$,
$v_{201} = 24t(-7x^2 + y^2 + t^2 + s^2)$, $v_{102} = 24x(x^2 + y^2 - 7t^2 + s^2)$, $v_{111} = -192xyt$,
$v_{120} = 24x(x^2 - 7y^2 + t^2 + s^2)$, $v_{012} = 24y(x^2 + y^2 - 7t^2 + s^2)$,
$v_{021} = 24t(x^2 - 7y^2 + t^2 + s^2)$, $v_{030} = 24y(3x^2 - 5y^2 + 3t^2 + 3s^2)$,
$v_{003} = 24t(3x^2 + 3y^2 - 5t^2 + 3s^2)$.

2. PROOF OF LEUTWILER'S CONJECTURE

We introduce the new variables $X := x^2$, $Y := y^2$, $T := t^2$, $S := s^2$. Then, $R = r^2 = X + Y + T + S$. Furthermore, we relate every function f of the variables X, Y, T, S to the function

$$g(x,y,t,s) := f\,(X,Y,T,S)\big|_{\substack{X=x^2, Y=y^2, \\ T=t^2, S=s^2}},$$

where we assume $x, y, t, s \ge 0$. There follow relations among the partial derivatives of f and g:

$$\frac{\partial g}{\partial x}(x,y,t,s) = \frac{\partial f}{\partial X}(X,Y,T,S)\bigg|_{\substack{X=x^2, Y=y^2, \\ T=t^2, S=s^2}} \cdot 2x$$

$$= \left[\frac{\partial f}{\partial X}(X,Y,T,S) \cdot 2\sqrt{X} \right] \Bigg|_{\substack{X=x^2,Y=y^2,\\ T=t^2,S=s^2}} ,$$

which we shall express in the shorter form

$$\frac{\partial g}{\partial x} = 2\sqrt{X} \cdot \frac{\partial f}{\partial X} .$$

Under this convention, which we shall always use in the sequel, it further holds:

$$\frac{\partial^2 g}{\partial x^2} = 2 \cdot \frac{\partial f}{\partial X} + 4X \cdot \frac{\partial^2 f}{\partial X^2} , \qquad \frac{\partial^2 g}{\partial x \partial y} = 4\sqrt{XY} \cdot \frac{\partial^2 f}{\partial X \partial Y} , \quad \dots ,$$

$$\frac{\partial^3 g}{\partial x^3} = 12\sqrt{X} \cdot \frac{\partial^2 f}{\partial X^2} + 8X\sqrt{X} \cdot \frac{\partial^3 f}{\partial X^3} \quad \text{etc.}$$

For the proof of the conjecture we need the next three lemmas.

Lemma 1. *Let the notation be as above.*

1. *For $\alpha \in 2\mathbb{N} \cup \{0\}$ it holds:*

$$\frac{\partial^\alpha g}{\partial x^\alpha} = \sum_{i=0}^{\frac{\alpha}{2}} c_{\alpha,i} X^i \cdot \frac{\partial^{\frac{\alpha}{2}+i} f}{\partial X^{\frac{\alpha}{2}+i}}$$

with certain $c_{\alpha,i} \in \mathbb{N}$.

2. *For $\alpha \in 2(\mathbb{N} \cup \{0\}) + 1$ it holds:*

$$\frac{\partial^\alpha g}{\partial x^\alpha} = \sum_{i=0}^{\frac{\alpha-1}{2}} c_{\alpha,i+\frac{1}{2}} X^{i+\frac{1}{2}} \cdot \frac{\partial^{\frac{\alpha+1}{2}+i} f}{\partial X^{\frac{\alpha+1}{2}+i}}$$

with certain $c_{\alpha,i+\frac{1}{2}} \in \mathbb{N}$.

Proof. 1. We only have to verify the inductive step from α to $\alpha+2$. Let A denote the right side of the assertion for an even α. Then,

$$\frac{\partial A}{\partial x} = \sum_{i=0}^{\frac{\alpha}{2}} 2c_{\alpha,i} \left(iX^{i-\frac{1}{2}} \cdot \frac{\partial^{\frac{\alpha}{2}+i} f}{\partial X^{\frac{\alpha}{2}+i}} + X^{i+\frac{1}{2}} \cdot \frac{\partial^{\frac{\alpha}{2}+i+1} f}{\partial X^{\frac{\alpha}{2}+i+1}} \right) ,$$

$$\frac{\partial^2 A}{\partial x^2} = \sum_{i=0}^{\frac{\alpha}{2}} 4c_{\alpha,i} \left[i(i-\frac{1}{2})X^{i-1} \cdot \frac{\partial^{\frac{\alpha}{2}+i} f}{\partial X^{\frac{\alpha}{2}+i}} \right.$$
$$\left. + (2i+\frac{1}{2})X^i \cdot \frac{\partial^{\frac{\alpha}{2}+i+1} f}{\partial X^{\frac{\alpha}{2}+i+1}} + X^{i+1} \cdot \frac{\partial^{\frac{\alpha}{2}+i+2} f}{\partial X^{\frac{\alpha}{2}+i+2}} \right]$$
$$= \sum_{i=0}^{\frac{\alpha+2}{2}} \left[2(4i+1)c_{\alpha,i} + 4c_{\alpha,i-1}\right] X^i \cdot \frac{\partial^{\frac{\alpha+2}{2}+i} f}{\partial X^{\frac{\alpha+2}{2}+i}}$$
$$+ \sum_{i=0}^{\frac{\alpha}{2}-1} 4(i+\frac{1}{2})(i+1)c_{\alpha,i+1} \cdot X^i \cdot \frac{\partial^{\frac{\alpha+2}{2}+i} f}{\partial X^{\frac{\alpha+2}{2}+i}}$$
$$= \sum_{i=0}^{\frac{\alpha+2}{2}} \left[4c_{\alpha,i-1} + 2(4i+1)c_{\alpha,i} + 2(i+1)(2i+1)c_{\alpha,i+1}\right] X^i \cdot \frac{\partial^{\frac{\alpha+2}{2}+i} f}{\partial X^{\frac{\alpha+2}{2}+i}},$$

where we have set $c_{\alpha,-1} = c_{\alpha,\frac{\alpha+2}{2}} = c_{\alpha,\frac{\alpha+4}{2}} = 0$. This completes the proof for even α.

2. If α is odd, then $\alpha - 1$ is even, so by what has just been proven,

$$\frac{\partial^\alpha g}{\partial x^\alpha} = \sum_{i=0}^{\frac{\alpha-1}{2}} 2c_{\alpha-1,i} \left(iX^{i-\frac{1}{2}} \cdot \frac{\partial^{\frac{\alpha-1}{2}+i} f}{\partial X^{\frac{\alpha-1}{2}+i}} + X^{i+\frac{1}{2}} \cdot \frac{\partial^{\frac{\alpha-1}{2}+i+1} f}{\partial X^{\frac{\alpha-1}{2}+i+1}} \right)$$
$$= \sum_{i=0}^{\frac{\alpha-1}{2}} \left[2c_{\alpha-1,i} + 2(i+1)c_{\alpha-1,i+1}\right] X^{i+\frac{1}{2}} \cdot \frac{\partial^{\frac{\alpha+1}{2}+i} f}{\partial X^{\frac{\alpha+1}{2}+i}}. \qquad \square$$

LEMMA 2. *The functions of X, Y, T of the form $X^i Y^j T^k$, where $i, j, k \in \frac{1}{2}\mathbb{N} \cup \{0\}$, are linearly independent.*

Proof. By the substitution $X = x^2$, $Y = y^2$, $T = t^2$, these functions become the monomials $x^{2i} y^{2j} t^{2k}$, which obviously are linearly independent.

$\square$

LEMMA 3. *For a function h of the form $h(X,Y,T,S) = \frac{1}{(X+Y+T+S)^k}$ it holds for every $l \in \mathbb{N} \cup \{0\}$:*

$$\frac{\partial^l h}{\partial X^l} = \frac{\partial^l h}{\partial Y^l} = \frac{\partial^l h}{\partial T^l} = \frac{\partial^l h}{\partial S^l} = \frac{(-1)^l \cdot (k)_l}{(X+Y+T+S)^{k+l}},$$

where $(k)_l := \prod_{i=0}^{l-1}(k+i)$ is the Pochhammer symbol.

Proof. The claim follows easily by induction. □

We now start with the actual proof of the conjecture.

For the function f defined by $f(X,Y,T,S) = \frac{1}{(X+Y+T+S)^2}$, the last lemma gives for $k,l,m \in \mathbb{N}\cup\{0\}$:

$$\frac{\partial^k f(X,Y,T,S)}{\partial X^k} = \frac{(-1)^k \cdot (k+1)!}{(X+Y+T+S)^{k+2}},$$

$$\frac{\partial^{k+l} f(X,Y,T,S)}{\partial Y^l \partial X^k} = \frac{(-1)^{k+l} \cdot (k+1)! \cdot (k+2)_l}{(X+Y+T+S)^{k+2+l}} = \frac{(-1)^{k+l} \cdot (k+l+1)!}{(X+Y+T+S)^{k+l+2}},$$

$$\frac{\partial^{k+l+m} f(X,Y,T,S)}{\partial T^m \partial Y^l \partial X^k} = \frac{(-1)^{k+l+m} \cdot (k+l+m+1)!}{(X+Y+T+S)^{k+l+m+2}}. \tag{2.1}$$

The conjecture is proven if we show that the functions

$$\frac{\partial^n}{\partial x^\alpha \partial y^\beta \partial t^\gamma}\left[f(X,Y,T,S)\big|_{\substack{X=x^2,Y=y^2\\ T=t^2,S=s^2}}\right] = \frac{v_{\alpha\beta\gamma}(x,y,t,s)}{r^{2n+4}}$$

(see (1.1)) for $\alpha,\beta,\gamma \in \mathbb{N}\cup\{0\}$, $\alpha+\beta+\gamma=n$, are linearly independent. By reductio ad absurdum we assume that there exists a linear combination

$$\sum_{\substack{\alpha+\beta+\gamma=n\\ \alpha,\beta,\gamma\geq 0}} C_{\alpha,\beta,\gamma} \cdot \frac{\partial^n}{\partial x^\alpha \partial y^\beta \partial t^\gamma}\left[f(X,Y,T,S)\big|_{\substack{X=x^2,Y=y^2\\ T=t^2,S=s^2}}\right] = 0, \tag{2.2}$$

where not all $C_{\alpha,\beta,\gamma}$ vanish.

Let $\hat\alpha$ be the biggest value of α such that $C_{\alpha,\beta,\gamma} \neq 0$ for certain β,γ. Let then $\hat\beta$ be the biggest value of β for which $C_{\hat\alpha,\beta,\gamma} \neq 0$ for certain γ. Obviously, there is only one such value of γ, namely $\hat\gamma := n - \hat\alpha - \hat\beta$.

According to Lemma 1, the term with the biggest exponents of X, Y, and T in $\frac{\partial^n}{\partial x^\alpha \partial y^\beta \partial t^\gamma}\left[f(X,Y,T,S)\big|_{\substack{X=x^2,Y=y^2\\ T=t^2,S=s^2}}\right]$ is

$$c_{\alpha,\frac{\alpha}{2}} \cdot c_{\beta,\frac{\beta}{2}} \cdot c_{\gamma,\frac{\gamma}{2}} \cdot X^{\frac{\alpha}{2}} Y^{\frac{\beta}{2}} T^{\frac{\gamma}{2}} \cdot \frac{\partial^{\alpha+\beta+\gamma} f(X,Y,T,S)}{\partial T^\gamma \partial Y^\beta \partial X^\alpha}.$$

Therefore, after setting $S = 1-X-Y-T$ (restricting x,y,t to $\left[0,\frac{1}{\sqrt{3}}\right]$, that is, X,Y,T to $\left[0,\frac{1}{3}\right]$, which does not affect Lemma 2) and observing (2.1), the product $X^{\frac{\hat\alpha}{2}} Y^{\frac{\hat\beta}{2}} T^{\frac{\hat\gamma}{2}}$ appears only once in (2.2), and its coefficient is

$$C_{\hat\alpha,\hat\beta,\hat\gamma} \cdot c_{\hat\alpha,\frac{\hat\alpha}{2}} \cdot c_{\hat\beta,\frac{\hat\beta}{2}} \cdot c_{\hat\gamma,\frac{\hat\gamma}{2}} \cdot (-1)^n \cdot (n+1)! \neq 0,$$

which contradicts Lemma 2. At this point, the proof is completed.

REFERENCES

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