Dedicated to the memory of Cabiria Andreian-Cazacu

EIGENVALUE PROBLEMS FOR $p(x)$-KIRCHHOFF-TYPE EQUATIONS WITH NEUMANN BOUNDARY CONDITIONS

SAID TAARABTI, ZAKARIA EL ALLALI, KHALIL BEN HADDOUCH, and LOUBNA MOUTAOUEKKIL

Communicated by Lucian Beznea

This paper is concerned with the existence of nontrivial weak solutions for a $p(x)$-Kirchhoff-type problem. By using the Mountain Pass theorem of Ambrosetti and Rabinowitz, Ekeland's variational principle and the theory of the variable exponent Sobolev spaces, we establish conditions for the existence of weak solutions.

AMS 2010 Subject Classification: 35J60, 35B05.

Key words: $p(x)$-Kirchhoff type equation, variational method, Mountain Pass theorem, Neumann boundary conditions.

1. INTRODUCTION

In this paper, we are interested with the following problem

$$\tag{1.1} \begin{cases} -M\left(\displaystyle\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\mathrm{d}x\right)\Delta^2_{p(x)}u = \lambda V(x)|u|^{q(x)-2}u & \text{in } \Omega,\\ \frac{\partial u}{\partial \nu} = \frac{\partial}{\partial \nu}(|\Delta u|^{p(x)-2}\Delta u) = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$ with smooth boundary $\partial\Omega$, $\Delta^2_{p(x)}u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$, is the $p(x)$-biharmonic operator, λ is a positive real number, p, q are continuous function on $\overline{\Omega}$, $V > 0$ is an indefinite weight function and $M(t)$ is a continuous real-valued function.
Problem (1.1) is called a nonlocal one because of the presence of the term M, which implies that the equation in (1.1) is no longer pointwise identities. This provokes some mathematical difficulties which make the study of such a problem particularly interesting. Nonlocal differential equations are also called Kirchhoff type equations. To be more precise, Kirchhoff in [25] has investigated the following equation

$$\tag{1.2} \rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

which is a generalization of the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration. A distinguishing feature of equation (1.2) is that the equation contains a nonlocal coefficient $\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x$ which depends on the average $\frac{1}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x$, and hence the equation is no longer a pointwise identity. Nonlocal effects also find various applications in biological systems. The parameters in (1.2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density and ρ_0 is the initial tension.

Lions [27] has proposed an abstract framework for the Kirchhoff type equations. After the work of Lions [27], various equations of Kirchhoff type have been studied extensively, see e.g. [3, 11]. The study of Kirchhoff type equations has already been extended to the case involving the $p-$Laplacian (for details, see [7, 8, 11]), $p(x)$-Laplacian (see [9, 10, 18, 28]) and $p(x)$-biharmonic (see [24, 14]).

We also mention that fourth order elliptic equations arise in a large variety of applications such as: Micro-Electro Mechanical systems, thin film theory, surface diffusion on solids, interface dynamics, flow in Hele-Shaw cells, and phase field models of multiphase systems (see [4, 22, 30] and the references therein). There is also another important class of physical problems leading to higher order partial differential equations. An example of this is Kuramoto-Sivashinsky equation which models pattern formation in different physical contexts, such as chemical reaction diffusion systems and a cellular gas flame in the presence of external stabilizing factors (see [34]). Recently contributions concerning a fourth order elliptic problems with $p(x)$-biharmonic operators can be found in [5, 23, 32].

Motivated by the above papers and the results in [6], [31] and [1], we consider (1.1) to study the existence of weak solutions.
Our paper is organized as follows. We first present some necessary preliminary results on variable exponent Sobolev spaces. Next, we give the main results and proofs about the existence of weak solutions.

2. PRELIMINARIES

In order to deal with $p(x)$-biharmonic operator problems, we need some results on the spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ and some properties of $p(x)$-

biharmonic operator, which we will use later (for details, see [20, 26]).
Define the generalized Lebesgue space by

$$L^{p(x)}(\Omega)=\left\{u:\ \Omega\longrightarrow\mathbb{R},\quad \text{measurable and}\quad \int_\Omega |u(x)|^{p(x)}\mathrm{d}x<\infty\right\},$$

where $p\in C_+(\overline{\Omega})$ and

$$C_+(\overline{\Omega})=\left\{h\in C(\overline{\Omega}):\ \ h(x)>1,\ \ \forall x\in\overline{\Omega}\right\}.$$

Denote

$$p^+=\max_{x\in\overline{\Omega}}p(x),\quad p^-=\min_{x\in\overline{\Omega}}p(x),$$

and for all $x\in\overline{\Omega}$ and $k\geq 1$

$$p^*(x)=\begin{cases}\frac{Np(x)}{N-p(x)} & \text{if } p(x)<N,\\ +\infty & \text{if } p(x)\geq N,\end{cases}$$

and

$$p_k^*(x)=\begin{cases}\frac{Np(x)}{N-kp(x)} & \text{if } kp(x)<N,\\ +\infty & \text{if } kp(x)\geq N.\end{cases}$$

The space $L^{p(x)}(\Omega)$ is endowed with the Luxemburg norm, which is defined by

$$|u|_{p(x)}=\inf\left\{\mu>0;\ \ \int_\Omega\left|\frac{u(x)}{\mu}\right|^{p(x)}\mathrm{d}x\leq 1\right\},$$

and the space $(L^{p(x)}(\Omega),|.|_{p(x)})$ is a Banach.

Proposition 2.1 ([?]). *The space $(L^{p(x)}(\Omega),|.|_{p(x)})$ is separable, uniformly convex, reflexive and its conjugate space is $L^{q(x)}(\Omega)$ where $q(x)$ is the conjugate function of $p(x)$, i.e.*

$$\frac{1}{p(x)}+\frac{1}{q(x)}=1,\quad \forall x\in\Omega.$$

For all $u\in L^{p(x)}(\Omega)$ and $v\in L^{q(x)}(\Omega)$ the Hölder's type inequality

$$\left|\int_\Omega u\,v\,\mathrm{d}x\right|\leq\left(\frac{1}{p^-}+\frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)} \tag{2.1}$$

holds true.
Moreover, if $p_1,p_2,p_3\in C_+(\overline{\Omega})$ and $\dfrac{1}{p_1(x)}+\dfrac{1}{p_2(x)}+\dfrac{1}{p_3(x)}=1$, then for any $u\in L^{p_1(x)}(\Omega)$, $v\in L^{p_2(x)}(\Omega)$ and $w\in L^{p_3(x)}(\Omega)$ the following inequality holds

$$\int_\Omega |u\,v\,w|\mathrm{d}x\leqslant\left(\frac{1}{p_1^-}+\frac{1}{p_2^-}+\frac{1}{p_3^-}\right)|u|_{p_1(x)}|v|_{p_2(x)}|w|_{p_3(x)}. \tag{2.2}$$

Furthermore, if we define the mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ by

$$\rho(u) = \int_\Omega |u|^{p(x)} \mathrm{d}x,$$

then the following relations hold

$$|u|_{p(x)} < 1 (= 1, > 1) \Leftrightarrow \rho(u) < 1 (= 1, > 1), \tag{2.3}$$

$$|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leqslant \rho(u) \leqslant |u|_{p(x)}^{p^+} \tag{2.4}$$

$$|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leqslant \rho(u) \leqslant |u|_{p(x)}^{p^-} \tag{2.5}$$

$$|u_n - u|_{p(x)} \longrightarrow 0 \Leftrightarrow \rho(u_n - u) \longrightarrow 0. \tag{2.6}$$

We recall also the following proposition, which will be needed later.

Proposition 2.2 ([12]). *Let p and q be measurable functions such that $p \in L^\infty(\Omega)$ and $1 < p(x)q(x) \leqslant \infty$, for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega), u \neq 0$. Then*

$$\begin{aligned} &|u|_{p(x)q(x)} \leqslant 1 \quad \Rightarrow |u|_{p(x)q(x)}^{p^+} \leqslant \left||u|^{p(x)}\right|_{q(x)} \leqslant |u|_{p(x)q(x)}^{p^-}, \\ &|u|_{p(x)q(x)} \geqslant 1 \quad \Rightarrow |u|_{p(x)q(x)}^{p^-} \leqslant \left||u|^{p(x)}\right|_{q(x)} \leqslant |u|_{p(x)q(x)}^{p^+} \end{aligned} \tag{2.7}$$

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined by

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : \ D^\alpha u \in L^{p(x)}(\Omega), \ |\alpha| \leq k\},$$

where

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_N^{\alpha_N}},$$

is the derivation in distribution sense, with $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^{i=N} \alpha_i$.

We know that the space $W^{k,p(x)}(\Omega)$ equipped with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha| \leq k} |D^\alpha u|_{p(x)},$$

is a Banach, separable and reflexive space. For more details, we refer to ([19], [21], [29], [**?**]).

Proposition 2.3 (Proposition 2.4 of [14]). *The norm $\|u\|_{2,p(x)}$ is equivalent to the norm $\|u\| = |\Delta u|_{p(x)}$ in the space X.*

In order to discuss problem (1.1), we need to choose the following subspace of $W^{2,p(x)}(\Omega)$

$$X = \{u \in W^{2,p(x)}(\Omega) : \ \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0\},$$

which is considered in ([16, 32]) and ([5]). They have proved that X is a nonempty, well defined and closed subspace of $W^{2,p(x)}(\Omega)$. For this they have showed the following boundary trace embedding theorem for variable exponent Sobolev spaces.

THEOREM 2.1 ([16]). *Let Ω be a bounded domain in $\mathbb{R}^N$ with C^2 boundary. If $2p(x) \geq N \geq 2$ for all $x \in \overline{\Omega}$, then for all $q \in C_+(\Omega)$ there is a continuous boundary trace embedding*

$$W^{2,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega), \tag{2.8}$$

and

$$W^{2,p(x)}(\Omega) \hookrightarrow W^{1,p(x)}(\partial\Omega). \tag{2.9}$$

PROPOSITION 2.4 ([16]). *If $2p(x) \geq N$ for all $x \in \overline{\Omega}$, then the set*

$$X = \{u \in W^{2,p(x)}(\Omega) : \ \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0\}$$

is a closed subspace of $W^{2,p(x)}(\Omega)$.

Remark 2.1. $(X; \|.\|)$ is a Banach, separable and reflexive space.

PROPOSITION 2.5 ([15]). *If we put*

$$J(u) = \int_\Omega |\Delta u|^{p(x)} \mathrm{d}x,$$

then for all $u, u_n \in X$ then the following relations hold true

(i) $\|u\| < 1 \ (= 1; > 1) \Longleftrightarrow J(u) < 1 \ (= 1; > 1)$,

(ii) $\|u\| \leq 1 \Longrightarrow \|u\|^{p^+} \leq J(u) \leq \|u\|^{p^-}$,

(iii) $\|u\| \geq 1 \Longrightarrow \|u\|^{p^-} \leq J(u) \leq \|u\|^{p^+}$,
for all $u_n \in X$, we have

(iv) $\|u_n\| \longrightarrow 0 \Longleftrightarrow J(u_n) \longrightarrow 0$,

(v) $\|u_n\| \longrightarrow \infty \Longleftrightarrow J(u_n) \longrightarrow \infty$.

PROPOSITION 2.6 ([5]). *Let $p \in C_+(\overline{\Omega})$ such that $2p(x) > N$ for all $x \in \overline{\Omega}$. Then*

(1) *there exists a continuous and compact embedding of* $W^{2,p(x)}(\Omega)$ *into* $L^{q(x)}(\Omega)$, *for all* $q \in C_+(\Omega)$.

(2) *there exists a continuous embedding of* $W^{2,p(x)}(\Omega)$ *into* $C(\overline{\Omega})$.

PROPOSITION 2.7 (see [17]). *Let* X *be a Banach space and*

$$\Lambda(u) = \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\mathrm{d}x.$$

The functional $\Lambda : X \longrightarrow \mathbb{R}$ *is convexe. The mapping* $\Lambda' : X \longrightarrow X'$ *is a strictly monotone, bounded homeomorphisme and of* (S_+), *namely*

$$u_n \rightharpoonup u \quad \text{and} \quad \overline{\lim_{n\to\infty}}\langle \Lambda'(u_n), u_n - u\rangle \leq 0 \quad \text{implies} \quad u_n \longrightarrow u.$$

Definition 2.1. We say that $u \in X$ is a weak solution of (1.1) if

$$M\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\mathrm{d}x\right)\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta\varphi \mathrm{d}x = \lambda\int_\Omega V(x)|u|^{q(x)-2}u\,\varphi\,\mathrm{d}x,$$

where $\varphi \in X$.

We associate to the problem (1.1) the energy functional, defined as $I : X \longrightarrow \mathbb{R}$,

$$I_\lambda(u) = \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\mathrm{d}x\right) - \lambda\int_\Omega \frac{1}{q(x)}V(x)|u|^{q(x)}\mathrm{d}x,$$

where $\widehat{M}(t) = \displaystyle\int_0^t M(s)\mathrm{d}s$. Then $I_\lambda \in C^1(X, \mathbb{R})$ and

$$\begin{aligned}\langle I_\lambda'(u), v\rangle = M&\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\mathrm{d}x\right)\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta v\mathrm{d}x\\ &- \lambda\int_\Omega V(x)|u|^{q(x)-2}uv\mathrm{d}x,\end{aligned}$$

for any $u, v \in X$. Hence, we can notice that critical points of functional I_λ are the weak solutions for problem (1.1).
Denote by $s'(x)$ the conjugate exponent of the function $s(x)$ and put $\alpha(x) = \frac{s(x)q(x)}{s(x)-q(x)}$. Thus, by the proposition 2.6 the embeddings $X \hookrightarrow L^{s'(x)q(x)}(\Omega)$ and $X \hookrightarrow L^{\alpha(x)}(\Omega)$ are compact and continuous.

Hereafter $M(t)$ and $V(x)$ are supposed to verify the following assumptions

(M1) There exists $m_2 \geq m_1 > 0$ and $\beta \geq \alpha > 1$ such that $m_1 t^{\alpha-1} \leq M(t) \leq m_2 t^{\beta-1}$.

(M2) For all $t \in \mathbb{R}^+, \widehat{M}(t) \geq M(t)t$.

(V1) $V(x) \in L^{s(x)}(\Omega)$ and there exists a measurable set $\Omega_0 \subset \Omega$ of positive measure such that $V(x) > 0$, a.e. $x \in \Omega_0$.

For simplicity, we use c_i, to denote the general nonnegative or positive constant (the exact value may change from line to line).

3. MAIN RESULTS

THEOREM 3.1. *Assume that* M *satisfies* $(M1)$, $(M2)$, $(V1)$, *and the function* $q \in C(\overline{\Omega})$ *satisfies*

$$\beta p^+ < q^- \leq q^+ < p^*(x). \tag{3.1}$$

Then for any $\lambda > 0$ *problem (1.1) possesses a nontrivial weak solution.*

LEMMA 3.1. *There exist* $\nu > 0$ *and* $a > 0$ *such that* $I_\lambda(u) \geqslant a > 0$ *for any* $u \in X$ *with* $\|u\| = \nu$.

Proof. Since the embedding $X \hookrightarrow L^{s'(x)q(x)}(\Omega)$ is continuous, we can find a constant $c_3 > 0$ such that

$$|u|_{s'(x)q(x)} \leq c_3\|u\|, \quad \forall u \in X. \tag{3.2}$$

According to the fact that

$$|u(x)|^{q(x)} \leq |u(x)|^{q^+} + |u(x)|^{q^-}, \quad \forall x \in \overline{\Omega}. \tag{3.3}$$

From the (3.3), we obtain

$$\begin{aligned}\int_\Omega V(x)|u|^{q(x)}\mathrm{d}x &\leq |V|_{s(x)} \left||u|^{q(x)}\right|_{s'(x)} \\ &\leq |V|_{s(x)} \left(|u|^{q^+}_{q(x)s'(x)} + |u|^{q^-}_{q(x)s'(x)}\right).\end{aligned} \tag{3.4}$$

Combining (3.2) and (3.4), we get

$$\int_\Omega V(x)|u|^{q(x)}\mathrm{d}x \leq |V|_{s(x)} \left(c_3^{q^+}\|u\|^{q^+} + c_3^{q^-}\|u\|^{q^-}\right). \tag{3.5}$$

Hence, from (3.5), we deduce that for any $u \in X$ with $\|u\| < 1$, we have

$$\begin{aligned}I_\lambda(u) &\geqslant \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^+} - \frac{\lambda}{q^-}|V|_{s(x)}\left(c_3^{q^-}\|u\|^{q^-} + c_3^{q^+}\|u\|^{q^+}\right) \\ &\geqslant \left(\frac{m_1}{\alpha(p^+)^\alpha} - \frac{\lambda}{q^-}|V|_{s(x)}\left(c_3^{q^-}\|u\|^{q^- - \alpha p^+} + c_3^{q^+}\|u\|^{q^+ - \alpha p^+}\right)\right)\|u\|^{\alpha p^+}.\end{aligned}$$

Since $\alpha p^+ \leq q^- \leq q^+$, the functional $g : [0,1] \longrightarrow \mathbb{R}$ defined by

$$g(t) = \frac{m_1}{\alpha (p^+)^\alpha} - \frac{\lambda}{q^-} C_1 \left(c_3^{q^-} t^{q^- - \alpha p^+} + c_3^{q^+} t^{q^+ - \alpha p^+} \right),$$

is positive on neighborhood of the origin. So the result of Lemma 3.1 follows. $\square$

Lemma 3.2. *There exists $e \in X$ with $\|e\| > \nu$ (where ν is given in Lemma 3.1) such that $I_\lambda(e) < 0$.*

Proof. Let $\psi \in C_0^\infty(\Omega), \psi \geqslant 0$ and $\psi \neq 0$ and $t > 1$. By (M1) we have

$$\begin{aligned} I_\lambda(t\psi) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)} |\Delta t\psi|^{p(x)} \mathrm{d}x \right) - \lambda \int_\Omega \frac{1}{q(x)} V(x) |t\psi|^{q(x)} \mathrm{d}x \\ &\leqslant \tfrac{m_2}{\beta} \left(\int_\Omega \frac{1}{p(x)} |\Delta t\psi|^{p(x)} \mathrm{d}x \right)^\beta - \lambda \tfrac{t^{q^-}}{q^+} \int_\Omega V(x) |\psi|^{q(x)} \mathrm{d}x \\ &\leqslant \tfrac{m_2}{\beta (p^-)^\beta} t^{\beta p^+} \left(\int_\Omega |\Delta \psi|^{p(x)} \mathrm{d}x \right)^\beta - \lambda \tfrac{t^{q^-}}{q^+} \int_\Omega V(x) |\psi|^{q(x)} \mathrm{d}x. \end{aligned}$$

Then, since $\beta p^+ < q^-$, we deduce that $\lim_{t\to\infty} I_\lambda(t\psi) = -\infty$. Then for $t > 1$ large enough, we can take $e = t\psi$ such that $\|e\| > \nu$ and $I_\lambda(e) < 0$. $\square$

Proof of Theorem 3.1. By Lemma 3.1 and Lemma 3.2 and the Mountain Pass Theorem of Ambrosetti and Rabinowitz [2], we deduce the existence of a sequence $(u_n) \subset X$ such that

$$I_\lambda(u_n) \to c_3 > 0, \quad I_\lambda'(u_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.6}$$

We prove that u_n is bounded in X.

By contradiction. Suppose that, $\|u_n\| \to \infty$ and $\|u_n\| > 1$ for all n. By (3.6) and (M1)-(M2), for n large enough, we have

$$\begin{aligned} 1 + c_3 + \|u_n\| \geqslant\; & I_\lambda(u_n) - \frac{1}{q^-} \langle I_\lambda'(u_n), u_n \rangle \\ \geqslant\; & M\left(\int_\Omega \frac{1}{p(x)} |\Delta u_n|^{p(x)} \mathrm{d}x \right) \int_\Omega \frac{1}{p(x)} |\Delta u_n|^{p(x)} \mathrm{d}x \\ & - \lambda \int_\Omega \frac{1}{q(x)} V(x) |u_n|^{q(x)} \mathrm{d}x \\ & - \frac{1}{q^-} M\left(\int_\Omega \frac{1}{p(x)} |\Delta u_n|^{p(x)} \mathrm{d}x \right) \int_\Omega \frac{1}{p(x)} |\Delta u_n|^{p(x)} \mathrm{d}x \\ & + \frac{\lambda}{q^-} \int_\Omega V(x) |u_n|^{q(x)} \mathrm{d}x \end{aligned}$$

$$\geqslant \frac{m_1}{\alpha(p^+)^{\alpha-1}}\left(\frac{1}{p^+}-\frac{1}{q^-}\right)\|u_n\|^{\alpha p^-}$$
$$+\lambda\left(\frac{1}{q^-}-\frac{1}{q(x)}\right)\int_\Omega V(x)|u_n|^{q(x)}\mathrm{d}x$$
$$\geqslant \frac{m_1}{\alpha(p^+)^{\alpha-1}}\left(\frac{1}{p^+}-\frac{1}{q^-}\right)\|u_n\|^{\alpha p^-}$$
$$+\lambda\left(\frac{1}{q^-}-\frac{1}{q(x)}\right)|V|_{s(x)}\left(c_3^{q^-}\|u_n\|^{q^-}+c_3^{q^+}\|u_n\|^{q^+}\right).$$

Taking into account (3.1) holds true and passing to the limit as $n\to\infty$, we obtain a contradiction. It follows that (u_n) is bounded in X. By the reflexivity of X, for a subsequence still denoted (u_n), we have

$$u_n \rightharpoonup u \quad \text{in } X.$$

Since $q^+<p^-$, it follows from proposition 2.6 that $u_n \rightharpoonup u$ in $L^{q(x)}(\Omega)$. We will show that

$$\lim_{n\to\infty}\int_\Omega V(x)|u_n|^{q(x)-2}u_n(u_n-u)\mathrm{d}x=0. \tag{3.7}$$

In fact, from the Hölder type inequality, we have

$$\begin{aligned}\int_\Omega V(x)|u_n|^{q(x)-2}u_n(u_n-u)\mathrm{d}x &\le |V(x)|_{s(x)}\left||u_n|^{q(x)-2}u_n(u_n-u)\right|_{s'(x)}\\ &\le |V(x)|_{s(x)}\left||u_n|^{q(x)-2}u_n\right|_{\frac{q(x)}{q(x)-1}}|u_n-u|_{\alpha(x)}\\ &\le |V|_{s(x)}\left(1+|u_n|_{q(x)}^{q^+-1}\right)|u_n-u|_{\alpha(x)}.\end{aligned}$$

Since X is continuously embedded in $L^{q(x)}(\Omega)$ and (u_n) is bounded in X, it follows that (u_n) is bounded in $L^{q(x)}(\Omega)$. On the other hand, since the embedding $X\hookrightarrow L^{\alpha(x)}(\Omega)$ is compact where $\alpha(x)=\frac{s(x)q(x)}{s(x)-q(x)}$, we deduce that $|u_n-u|_{\alpha(x)}\longrightarrow 0$ as $n\longrightarrow+\infty$.
So, we obtain

$$\lim_{n\to\infty}\int_\Omega V(x)|u_n|^{q(x)-2}u_n(u_n-u)\mathrm{d}x=0. \tag{3.8}$$

Using (3.6), we infer that

$$\lim_{n\to\infty}\langle I'_\lambda(u_n),u_n-u\rangle=0. \tag{3.9}$$

Since (u_n) is bounded in X. passing to a subsequence, if necessary, we may assume that

$$\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\mathrm{d}x\to t_0\ge 0 \quad \text{as} \quad n\to\infty.$$

If $t_0 = 0$ then (u_n) converges strongly to $u = 0$ in X and the proof is complete. If $t_0 > 0$ then since the function M is continuous, we obtain

$$M\left(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\mathrm{d}x\right) \to M(t_0) \geq 0 \quad \text{as} \quad n \to \infty.$$

Thus, by (M1), for suffciently large n, we have

$$0 < c_4 \leq M\left(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}\mathrm{d}x\right) \leq c_5. \tag{3.10}$$

From (3.8)-(3.10), we deduce that

$$\lim_{n\to\infty} \int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n(\Delta u_n - \Delta u)\mathrm{d}x = 0. \tag{3.11}$$

Using proposition 2.7, we conclude that actually (u_n) converges strongly to u in X. Then by relation (3.6), we have

$$I_\lambda(u) = c_3 > 0, \quad I_\lambda^{'}(u) = 0.$$

That is, u is a nontrivial weak solution of (1.1). □

THEOREM 3.2. *If we assume that* $(M1)$, $(M2)$, $(V1)$ *hold and* $q \in C^+(\overline{\Omega})$ *satisfies*

$$1 < q^- \leq q^+ < \alpha p^-, \tag{3.12}$$

then there exists $\lambda^* > 0$ *such that for any* $\lambda > \lambda^*$, *the problem (1.1) possesses a nontrivial weak solution.*

Under the theorem's conditions, we want to construct a global minimizer of the functional. We start with the following auxiliary result.

LEMMA 3.3. *The functional* I_λ *is coercive on* X.

Proof. By Theorem 3.1 and Proposition 2.5, we deduce that for all $u \in X$,

$$I_\lambda(u) \geqslant \frac{m_1}{\alpha(p^+)^\alpha}\left(\int_\Omega |\Delta u|^{p(x)}\mathrm{d}x\right)^\alpha - \frac{\lambda}{q^-}|V|_{s(x)}\left(c_3^{q^-}\|u\|^{q^-} + c_3^{q^+}\|u\|^{q^+}\right).$$

Now we set $\|u\| > 1$, then

$$I_\lambda(u) \geqslant \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^-} - \frac{\lambda}{q^-}|V|_{s(x)}\left(c_3^{q^-}\|u\|^{q^-} + c_3^{q^+}\|u\|^{q^+}\right).$$

As $q^- \leq q^+ < \alpha p^-$, I_λ is coercive, that is $I_\lambda(u) \to \infty$ as $\|u\| \to \infty$. □

Proof of Theorem 3.2. $I_\lambda(u)$ is a coercive functional and weakly lower semi-continuous on X. These two facts enable us to apply [[31], Theorem 1.2] in order to find that there exists $u_\lambda \in X$ a global minimizer of I_λ and thus a weak solution of problem (1.1).
We show u_λ is not trivial for λ large enough. Letting $t_0 > 1$ be a constant and Ω_1 be an open subset of Ω with $|\Omega_1| > 0$, we assume that $v_0 \in C_0^\infty(\overline{\Omega})$ is such that $v_0(x) = t_0$ for any $x \in \overline{\Omega_1}$ and $0 \le v_0(x) \le t_0$ in $\Omega \setminus \Omega_1$. We have

$$\begin{aligned} I_\lambda(v_0) &= \widehat{M}\left(\int_\Omega |\Delta v_0|^{p(x)}\mathrm{d}x\right) - \lambda \int_\Omega \frac{1}{q(x)} V(x)|v_0|^{q(x)}\mathrm{d}x \\ &\le c_6 - \frac{\lambda}{q^+}\int_\Omega V(x)|v_0|^{q(x)}\mathrm{d}x \\ &\le c_6 - \frac{\lambda}{q^+} t_0^{q^-}\int_{\Omega_1} V(x)\mathrm{d}x. \end{aligned} \tag{3.13}$$

So there exists $\lambda^* > 0$ such that $I_\lambda(v_0) < 0$ for any $\lambda \in [\lambda^*; +\infty)$. It follows that for any $\lambda \ge \lambda^*$, u_λ is a nontrivial weak solution of problem (1.1) for λ large enough. □

THEOREM 3.3. *If $(M1)$, $(M2)$, $(V1)$ hold and and $q \in C_+(\overline{\Omega})$ satisfies*

$$1 < q(x) < p(x) < p^*(x), \tag{3.14}$$

then there exists $\lambda^{**} > 0$ such that for any $\lambda \in (0; \lambda^{**})$, problem (1.1) possesses a nontrivial weak solution.

We plan to apply Ekeland variational principle [13] to get a nontrivial solution to problem (1.1). We start with two auxiliary results.

LEMMA 3.4. *There exists $\lambda^{**} > 0$ such that for any $\lambda \in (0; \lambda^{**})$ there are $\rho, a > 0$ such that $I_\lambda(u) \ge a > 0$ for any $u \in X$ with $\|u\| = \rho$.*

Proof. Since the embedding $X \hookrightarrow L^{s'(x)q(x)}(\Omega)$ is continuous, we can find a constant $c_3 > 0$ such that

$$|u|_{s'(x)q(x)} \le c_3\|u\|, \forall u \in X. \tag{3.15}$$

Let us fix $\rho \in (0,1)$ such that $\rho < \frac{1}{c_3}$. Then the relation (3.15) implies $|u|_{s'(x)q(x)} < 1$, for all $u \in X$ with $\|u\| = \rho$. Thus,

$$\int_\Omega V(x)|u|^{q(x)}\mathrm{d}x \le |V|_{s(x)} \left||u|^{q(x)}\right|_{s'(x)} \le |V|_{s(x)}|u|^{q^-}_{q(x)s'(x)}, \tag{3.16}$$

for all $u \in X$ with $\|u\| = \rho$.
Combining (3.15) and (3.16), we obtain

$$\int_\Omega V(x)|u|^{q(x)}\mathrm{d}x \le c_3^{q^-}|V|_{s(x)}\|u\|^{q^-}. \tag{3.17}$$

Using the hypotheses (M1) and (3.17), we deduce that for any $u \in X$ with $\|u\| = \rho$, the following holds

$$\begin{aligned} I_\lambda(u) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\mathrm{d}x\right) - \lambda \int_\Omega \frac{1}{q(x)} V(x)|u|^{q(x)}\mathrm{d}x \\ &\geq \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^+} - \frac{\lambda}{q^-} c_3^{q^-}|V|_{s(x)}\|u\|^{q^-} \\ &= \rho^{q^-}\left(\frac{m_1}{\alpha(p^+)^\alpha}\rho^{\alpha p^+ - q^-} - \frac{\lambda}{q^-}c_3^{q^-}|V|_{s(x)}\right). \end{aligned}$$

Putting

$$\lambda^{**} = \frac{m_1 q^-}{2\alpha(p^+)^\alpha c_3^{q^-}|V|_{s(x)}}\rho^{\alpha p^+ - q^-}, \tag{3.18}$$

then for any $\lambda \in (0, \lambda^{**})$ and $u \in X$ with $\|u\| = \rho$, there exists $a = \frac{\rho^{\alpha p^+}}{2\alpha(p^+)^\alpha}$ such that $I_\lambda(u) \geq a > 0$. □

Lemma 3.5. *For any $\lambda \in (0, \lambda^{**})$ given by (3.18), there exists $\varphi \in X$ such that $\varphi \geq 0, \varphi \neq 0$ and $I_\lambda(t\varphi) < 0$ for all $t > 0$ small enough.*

Proof. Assumption (3.14) implies that $q(x) < \beta p(x)$. There exist $\epsilon_0 > 0$ such that $q^- + \epsilon_0 < \beta p^-$.

Since $q \in C(\overline{\Omega_0})$, there exists an open set $\Omega_1 \subset \Omega_0$ such that $|q(x) - q^-| < \epsilon_0$ for all $x \in \Omega_1 \cap \Omega_0$. Thus, we deduce $q(x) < q^- + \epsilon_0 < \beta p^-$ for all $x \in \Omega_1 \cap \Omega_0$.

Take $\varphi \in C_0^\infty(\Omega)$ such that $supp(\varphi) \supset \overline{\Omega_1}$, $\varphi(x) = 1$ for all $x \in \overline{\Omega_1}$ and $0 \leq \varphi \leq 1$ in Ω_0.

Then for any $t \in (0,1)$, we have

$$\begin{aligned} I_\lambda(t\varphi) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\Delta t\varphi|^{p(x)}\mathrm{d}x\right) - \lambda\int_\Omega \frac{1}{q(x)}V(x)|t\varphi|^{q(x)}\mathrm{d}x \\ &\leq \frac{m_2}{\beta}\left(\int_\Omega \frac{1}{p(x)}|\Delta t\varphi|^{p(x)}\mathrm{d}x\right)^\beta - \lambda\int_{\Omega_1}\frac{1}{q(x)}t^{q(x)}V(x)|\varphi|^{q(x)}\mathrm{d}x \\ &\leq \frac{m_2}{\beta(p^-)^\beta}t^{\beta p^-}\left(\int_\Omega |\Delta\varphi|^{p(x)}\mathrm{d}x\right)^\beta - \frac{\lambda}{q^+}t^{q^- + \epsilon_0}\int_{\Omega_1} V(x)|\varphi|^{q(x)}\mathrm{d}x < 0, \end{aligned}$$

for all $t < \delta^{\frac{1}{\beta p^- - q^- - \epsilon_0}}$ with

$$0 < \delta < \min\left\{1, \frac{\lambda\beta(p^-)^\beta}{m_2 q^+}\frac{\int_{\Omega_1} V(x)|\varphi|^{q(x)}\mathrm{d}x}{\left(\int_\Omega |\Delta\varphi|^{p(x)}\mathrm{d}x\right)^\beta}\right\},$$

we conclude that $I_\lambda(t\varphi) < 0$, the proof is complete. $\square$

Proof of Theorem 3.3. Let λ^{**} be defined as in (3.18) and $\lambda \in (0, \lambda^{**})$. By Lemma 3.4, it follows that on the boundary of the ball centered at the origin and of radius $\rho \in X$, we have

$$\inf_{\partial B_\rho(0)} I_\lambda(u) > 0.$$

On the other hand, from Lemma 3.5, there exists $\varphi \in X$ such that

$$I_\lambda(t\varphi) < 0 \quad \text{for} \quad t > 0 \quad \text{small enough.}$$

Moreover, for $u \in B_\rho(0)$,

$$I_\lambda(u) \geq \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^+} - \frac{\lambda}{q^-} c_3^{q^-} |V|_{s(x)} \|u\|^{q^-} \quad \text{for} \quad u \in B_\rho(0).$$

It follows that

$$-\infty < c_8 = \inf_{\overline{B_\rho(0)}} I_\lambda(u) < 0.$$

Let $0 < \epsilon < \inf_{\partial B_\rho(0)} I_\lambda - \inf_{B_\rho(0)} I_\lambda$. Then by applying Ekeland variational principle [13] to the functional $I_\lambda : \overline{B_\rho(0)} \longrightarrow \mathbb{R}$, there exist $u_\epsilon \in \overline{B_\rho(0)}$ such that

$$I_\lambda(u_\epsilon) < \inf_{\overline{B_\rho(0)}} I_\lambda + \epsilon$$

$$I_\lambda(u_\epsilon) < I_\lambda + \epsilon\|u - u_\epsilon\|, \quad u \neq u_\epsilon.$$

Since

$$I_\lambda(u_\epsilon) \leq \inf_{\overline{B_\rho(0)}} I_\lambda + \epsilon \leq \inf_{B_\rho(0)} I_\lambda + \epsilon < \inf_{\partial B_\rho(0)} I_\lambda,$$

we deduce that $u_\epsilon \in B_\rho(0)$. Now, we define $K_\lambda : \overline{B_\rho(0)} \longrightarrow \mathbb{R}$ by $K_\lambda(u) = I_\lambda(u) + \epsilon\|u - u_\epsilon\|$. It is clear that u_ϵ is a minimum point of K_λ and thus

$$\frac{K_\lambda(u_\epsilon + tv) - K_\lambda(u_\epsilon)}{t} \geq 0,$$

for small $t > 0$ and $v \in B_\rho(0)$. The above relation yields

$$\frac{I_\lambda(u_\epsilon + tv) - I_\lambda(u_\epsilon)}{t} + \epsilon\|v\| \geq 0.$$

Letting $t \to 0$ it follows that $\langle I_\lambda'(u_\epsilon), v\rangle + \epsilon\|v\| > 0$ and we infer that $\|I_\lambda'(u_\epsilon)\| \leq \epsilon$. We deduce that there exists a sequence $(v_n) \subset B_1(0)$ such that

$$I_\lambda(v_n) \longrightarrow c_8, \quad I_\lambda'(v_n) \longrightarrow 0. \tag{3.19}$$

It is clear that (v_n) is bounded in X. Thus, there exists $u_2 \in X$ such that, up to a subsequence, (v_n) converges weakly to u_2 in X. Actually, with similar arguments as those used in the proof of theorem 3.1, we can show that $v_n \to u_2$ in X. Thus, by relation (3.19),

$$I_\lambda(u_2) = c_8, \quad I'_\lambda(u_2) = 0.$$

i.e., u_2 is a nontrivial weak solution for problem (1.1). □

REFERENCES

[1] G. A. Afrouzi and M. Mirzapou, *Problems for $p(x)$-Kirchhoff type equations.* Electron. J. Differential Equations **2013** (2013), 253.

[2] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical points theory and applications.* J. Funct. Anal. **14** (1973), 349 – 381.

[3] A. Arosio and S. Panizzi, *On the well-posedness of the Kirchhoff string.* Trans. Amer. Math. Soc. **348** (1996), 305 – 330.

[4] S. Baraket and V. Rădulescu, *Combined effects of concave-convex nonlinearities in a fourth-order problem with variable exponent.* Adv. Nonlinear Stud. **16** (2016), 409 – 419.

[5] K. Ben Haddouch, Z. El Allali, A. Ayoujil, and N. Tsouli, *Continuous spectrum of a fourth order eigenvalue problem with variable exponent under Neumann boundary conditions.* Annals of the University of Craiova, Mathematics and Computer Science Series **42** (2015), *1*, 42 – 55.

[6] N. T. Chung, *Multiplicity results for a class of $p(x)$-Kirchoff type equations with combined nonlinearities.* Electron. J. Qual. Theory Differ. Equ. **42** (2012), 1 – 13.

[7] F. J. S. A. Corrêa and G. M. Figueiredo, *On an elliptic equation of p-Kirchhoff type via variational methods.* Bull. Aust. Math. Soc. **74** (2006), 263 – 277.

[8] F. J. S. A. Corrêa and G. M. Figueiredo, *On a p-Kirchhoff equation via Krasnoselskii's genus.* Appl. Math. Letters **22** (2009), 819 – 822.

[9] G. Dai and R. Hao, *Existence of solutions for a $p(x)$-Kirchhoff-type equation.* J. Math. Anal. Appl. **359** (2009), 275 – 284.

[10] G. Dai and R. Ma, *Solutions for a $p(x)$-Kirchhoff-type equation with Neumann boundary data.* Nonlinear Anal. **12** (2011), 2666 – 2680.

[11] M. Dreher, *The wave equation for the p-Laplacian.* Hokkaido Math. J. **36** (2007), 21 – 52.

[12] D. Edmunds and J. Rákosník, *Sobolev embeddings with variable exponent.* Studia Math. **143** (2000), 267 – 293.

[13] I. Ekeland, *On the variational principle.* J. Math. Anal. Appl. **47** (1974), 324 – 353.

[14] Z. El Allali and S. Taarabti, *Existence and multiplicity of the solution for the $p(x)$-Kirchhoff equation via genus theory.* Communications in Applied Analysis, **23** (2019), *1*, 79 – 95.

[15] A. R. El Amrouss and A. Ourraoui. *Existence of solutions for a boundary problem involving $p(x)$-biharmonic operator.* Bol. Soc. Parana. Mat. (3) **31** (2013), *1*, 179 – 192.

[16] A. R. El Amrouss, F. Moradi, and M. Moussaoui, *Existence and multiplicity of solutions for a $p(x)$-biharmonic problem with Neumann boundary condition.* Accepted for publication in Bol. Soc. Parana. Mat.

[17] A. R. El Amrouss, F. Moradi, and M. Moussaoui, *Existence of solutions for fourth-order PDEs with variable exponents.* Electron. J. Differential Equations **153** (2009), 1 –13.

[18] X. L. Fan, *On nonlocal $p(x)$-Laplacian Dirichlet problems.* Nonlinear Anal. **72** (2010), 3314 – 3323.

[19] X. L. Fan and X. Fan, *A Knobloch-type result for $p(x)$-Laplacian systems.* J. Math. App. **282** (2003), 453 – 464.

[20] X. L. Fan, J. S. Shen, and D. Zhao, *Sobolev embedding theorems for spaces $W^{k,p(x)(\Omega)}$.* J. Math. Anal. Appl. **262** (2001), 749 – 760.

[21] X.L. Fan and D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$.* J. Math. Anal. Appl. **263** (2001), 424 – 446.

[22] A. Ferrero and G. Warnault, *On solutions of second and fourth order elliptic equations with power-type nonlinearities.* Nonlinear Anal. **70** (2009), 2889 – 2902.

[23] K. Kefi and V. Rădulescu, *On a $p(x)$-biharmonic problem with singular weights.* Z. Angew. Math. Phys. **68** (2017), *4*, 80.

[24] K. Kefi and V. Rădulescu, *Small perturbations of nonlocal biharmonic problems with variable exponent and competing nonlinearities.* Rendiconti Lincei Matematica E Applicazioni, **29** (2018), *3*, 439 – 463.

[25] G. Kirchhoff, *Mechanik.* Teubner, Leipzig, 1883.

[26] O. Kováčik and J. Rákosník, *On space $L^{p(x)}$ and $W^{k,p(x)}$.* Czechoslovak Math. J. **41(116)** (1991), 592 – 618.

[27] J.-L. Lions, *On some questions in boundary value problems of mathematical physics.* In: G. M. de la Penha and L. A. J. Medeiros eds., Proceedings of International Symposium on Continum Mechanics and Partial Differential Equations, Rio de Janeiro 1977. Math. Stud., North Holland vol. 30, 1978, pp. 284 – 346.

[28] M. Mihăilescu and G. Moroşanu, *Existence and multiplicity of solutions for an anisotropic elliptic problem involving variable exponent growth conditions.* Applicable Anal. **89** (2010), 257 – 271.

[29] M. Mihăilescu, *Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplacian operator.* Nonlinear Anal. T.M.A. **67** (2007), 1419 – 1425.

[30] T. G. Myers, *Thin films with high surface tension.* SIAM Rev. **40** (1998), 441 – 462.

[31] M. Struwe, *Variational methods: Applications to Nonlinear Partial Differential Equatios and Hamiltonian systems.* Springer-Verlag, Berlin, 1996.

[32] S. Taarabti, Z. El Allali, and K. Ben Haddouch, *Eigenvalues of the $p(x)$-biharmonic operator with indefinite weight under Neumann boundary conditions.* Bol. Soc. Parana. Mat. **36** (2018), *1*, 195 – 213.

[33] J. H. Yao, *Solution for Neumann boundary problems involving the $p(x)$-Laplacian operators*, Nonlinear Anal. T.M.A. **68** (2008), 1271 – 1283.

[34] W. Wang and E. Canessa, *Biharmonic pattern selection.* Physical Review E **47** (1993), 1243 – 1248.

National School of Applied Sciences of Agadir
Laboratory of Systems Engineering and Information Technologies (LISTI)
Ibn Zohr University, Morocco
`s.taarabti@uiz.ac.com`

Multidisciplinary Faculty of Nador
Team of Modeling and Scientific Computing
Department of Mathematics
Mohammed First University, Oujda, Morocco
`z.elallali@ump.ma`

National School of Applied Sciences of Fez
Sidi Mohamed Ben Abdellah University, Morocco
`khalil.benhaddouch@usmba.ac.ma`

Multidisciplinary Faculty of Nador
Team of Modeling and Scientific Computing
Department of Mathematics
Mohammed First University, Oujda, Morocco
`anits@yahoo.fr`