Dedicated to the memory of Cabiria Andreian-Cazacu

MULTIPLE SOLUTIONS FOR A NEUMANN PROBLEM TYPE WITH INDEFINITE WEIGHT IN SOBOLEV SPACES WITH VARIABLE EXPONENTS

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In this paper we study the existence of multiple solutions for the $p(x)$-biharmonic operator under Neumann type boundary conditions. By using the variational approach and the Mountain Pass theorem, we obtain the existence of solutions of the problem

$$\begin{cases} \Delta\left(|\Delta u|^{p(x)-2}\Delta u\right) = \lambda V(x)|u|^{q(x)-2}u & \text{in} \quad \Omega \\ \frac{\partial}{\partial\nu}\left(|\Delta u|^{p(x)-2}\Delta u\right) = m(x)|u|^{p(x)-2}u & \text{on} \quad \partial\Omega, \\ \frac{\partial u}{\partial\nu} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

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1. INTRODUCTION

The aim of this paper is to study the existence of multiple solutions for the following problem involving the $p(x)$-biharmonic operator with Neumann type boundary condition

$$\begin{cases} \Delta\left(|\Delta u|^{p(x)-2}\Delta u\right) = \lambda V(x)|u|^{q(x)-2}u & \text{in} \quad \Omega \\ \frac{\partial}{\partial\nu}\left(|\Delta u|^{p(x)-2}\Delta u\right) = m(x)|u|^{p(x)-2}u & \text{on} \quad \partial\Omega, \\ \frac{\partial u}{\partial\nu} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N,\ N \geq 2$ is a bounded smooth domain, λ is a positive parameter, $m \in L^\infty(\partial\Omega)$ with $\inf_{x\in\partial\Omega} m(x) > 0$, $p(x) : \overline{\Omega} \to \mathbb{R}$ is a continuous function, and V is an indefinite weight function in a generalized Lebesgue space $L^{s(x)}(\Omega)$ such that $V > 0$ in an open set $\Omega_0 \subset \Omega$ where $|\Omega_0| > 0$.

The operator $\Delta^2_{p(x)}u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is called the $p(x)$-biharmonic operator of fourth order where p is a continuous non-constant function. This differential operator is a natural generalization of the p-biharmonic operator $\Delta_p^2 u := \Delta(|\Delta u|^{p-2}\Delta u)$, where $p > 1$ is a real constant. However, the $p(x)$-biharmonic operator is not homogeneous, this fact causes many problems, some classical theories and such as the Lagrange multiplier theorem and theory of Sobolev spaces are not applicable in many problems involving this operator.

As it is well known, the study of the problems involving the $p(x)$ growth arise in a large variety of applications as elastic mechanics, fluid dynamics, electrorheological fluids, image processing, flow in porous media, calculus of variations, non-linear elasticity theory, heterogeneous porous media models, see [22, 9, 28, 3] and the references therein for more information.

Before giving our main results, let us briefly recall literature concerning related non-linear equations involving $p(x)$-biharmonic operator. The existence and multiplicity of solutions of elliptic equations with variable exponents involving the $p(x)$-biharmonic operator have been extensively investigated using various methods, especially variational technics, and have received much attention. In this context, among others, we would like to mention [5, 6, 14, 15, 18, 19, 20, 26] and the references therein. Recently, in the case of $p(x)$-Laplacian we refer the reader to [1, 2, 11].

At this point, when $p(x) \equiv p$ is a constant, the problem (1.1) have been studied in [24, 8, 10] with their references. For $p = 2$, we refer the reader to [23] and references therein.

Let us impose the following hypotheses throughout this paper:

$\mathbf{H}_1(V)$: $V \in L^{s(x)}(\Omega)$ and there exists a measurable set $\Omega_0 \subset \Omega$ of a positive measure such that $V(x) > 0$, a. e. $x \in \Omega_0$.

$\mathbf{H}_2(V)$: Let $f(x,u) = V(x)|u|^{q(x)-2}u$. There exist $t_0 > 0$ and a ball B with $\overline{B} \subset \Omega$ such that

$$\int_B F(x,t_0)dx > 0, \text{ where } F(x,t) = \int_0^t f(x,s)ds.$$

$\mathbf{H}(m)$: $m \in L^\infty(\partial\Omega)$ with $m^- = \inf_{x\in\partial\Omega} m(x) > 0$.

$\mathbf{H}(p,q,s)$: $q^+ < p^- < \frac{N}{2} < s(x), \forall x \in \overline{\Omega}$, where $s(x) \in C_+(\overline{\Omega})$.

Inspired by the above results, in the present paper, we are interested in discussing the existence of at least two weak solutions for the problem (1.1). Precisely, we give below the statement of results that will be proved.

THEOREM 1.1. *Suppose that $H(m)$, $H_1(V)$, $H_2(V)$ and $H(p,q,s)$, there exists $\lambda_\star > 0$ such that for any $\lambda > \lambda_\star$ the problem (1.1) has at least one nontrivial solution.*

Next, we consider the problem (1.1) in the case $V : \Omega \longrightarrow \mathbb{R}$ is a non-negative function. Let us assume that the function $V \in L^{\infty}(\Omega)$ and the function $q \in C_+(\Omega)$ satisfy the following conditions

(V_1) There exist an $x_0 \in \Omega$ and two positive constants r and R with $0 < r < R$ such that $\overline{B_R(x_0)} \subset \Omega$ and $V(x) = 0$ for $x \in \overline{B_R(x_0) \setminus B_r(x_0)}$ while $V(x) > 0$ for $x \in \Omega \setminus \overline{B_R(x_0) \setminus B_r(x_0)}$;

(Q_1) $1 < q(x) < p^{\star}(x) = \frac{Np(x)}{N-p(x)}$ for all $x \in \overline{\Omega}$;

(Q_2) Either

$(Q_2^{'})$ $\max\limits_{x \in \overline{B_r(x_0)}} q(x) < p^- \leqslant p^+ < \min\limits_{x \in \overline{\Omega \setminus B_R(x_0)}} q(x).$

or

$(Q_2^{''})$ $\max\limits_{x \in \overline{\Omega \setminus B_R(x_0)}} q(x) < p^- \leqslant p^+ < \min\limits_{x \in \overline{B_r(x_0)}} q(x).$

THEOREM 1.2. *Assume that the conditions* (V_1) *and* $(Q_1) - (Q_2)$ *are satisfied. Then there exists* $\lambda^{\star\star} > 0$ *such that any* $\lambda \in (0, \lambda^{\star\star})$ *is an eigenvalue of problem (1.1). Moreover, for any* $\lambda \in (0, \lambda^{\star\star})$ *problem (1.1) has at least two nonnegative non-trivial weak solutions.*

This paper is organized as follows. In section 2, we present some preliminaries and basic facts about the variable exponent Lebesgue and Sobolev spaces that will be used in Section 3. In the last section, we give the proofs of our main results.

2. PRELIMINARIES

To study $p(x)$-biharmonic operator problems, we need some results on the spaces $L^{p(x)}(\Omega)$, $W^{k,p(x)}(\Omega)$ and some properties of $p(x)$-biharmonic operator, which will be needed later (for details, see [16, 21]). Define the generalized Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R}, \quad \text{measurable and} \quad \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\},$$

where $p \in C_+(\overline{\Omega})$ and

$$C_+(\overline{\Omega}) = \left\{ h \in C(\overline{\Omega}) : \quad h(x) > 1, \ \ \forall x \in \overline{\Omega} \right\}.$$

Denote

$$p^+ = \max_{x \in \overline{\Omega}} p(x), \quad p^- = \min_{x \in \overline{\Omega}} p(x),$$

and for all $x \in \overline{\Omega}$ and $k \geq 1$

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases}$$

and

$$p_k^*(x) = \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if } kp(x) < N, \\ +\infty & \text{if } kp(x) \geq N. \end{cases}$$

One introduces in $L^{p(x)}(\Omega)$ the following norm

$$|u|_{p(x)} = \inf\left\{\mu > 0; \quad \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\},$$

and the space $(L^{p(x)}(\Omega), |.|_{p(x)})$ is a Banach.

PROPOSITION 2.1 ([21, 17]). *The space $(L^{p(x)}(\Omega), |.|_{p(x)})$ is separable, uniformly convex, reflexive and its conjugate space is $L^{q(x)}(\Omega)$ where $q(x)$ is the conjugate function of $p(x)$ i.e*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1, \quad \forall x \in \Omega.$$

For all $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ the Hölder's type inequality

$$\left|\int_\Omega uv dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right) |u|_{p(x)}|v|_{q(x)} \tag{2.1}$$

holds true.

Moreover, if $p_1, p_2, p_3 \in C_+(\overline{\Omega})$ and $\frac{1}{p_1(x)} + \frac{1}{p_2(x)} + \frac{1}{p_3(x)} = 1$, then for any $u \in L^{p_1(x)}(\Omega)$, $v \in L^{p_2(x)}(\Omega)$ and $w \in L^{p_3(x)}(\Omega)$ the following inequality holds

$$\int_\Omega |uvw| dx \leqslant \left(\frac{1}{p_1^-} + \frac{1}{p_2^-} + \frac{1}{p_3^-}\right) |u|_{p_1(x)}|v|_{p_2(x)}|w|_{p_3(x)}. \tag{2.2}$$

Furthermore, if we define the mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ by $\rho(u) = \int_\Omega |u(x)|^{p(x)} dx$, then the following relations hold.

PROPOSITION 2.2 ([21, 16]).

(i) $|u|_{p(x)} < 1 (= 1, > 1) \Leftrightarrow \rho(u) < 1 (= 1, > 1)$.

(ii) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leqslant \rho(u) \leqslant |u|_{p(x)}^{p^+}$

(iii) $|u_n - u|_{p(x)} \longrightarrow 0 \Leftrightarrow \rho(u_n - u) \longrightarrow 0$.

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined by

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : \quad D^\alpha u \in L^{p(x)}(\Omega), \ |\alpha| \leq k\},$$

where

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_N^{\alpha_N}},$$

is the derivation in distribution sense, with $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^{N} \alpha_i$. The space $W^{k,p(x)}(\Omega)$, equipped with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha| \leq k} |D^\alpha u|_{p(x)},$$

also becomes a Banach, separable and reflexive space. For more details, we refer to [17, ?, 27].

PROPOSITION 2.3 (Fan and Zhao [17]). *Let $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$. Then there is a continuous embedding*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

If we replace $\leq$ with $<$, the embedding is compact.

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

In order to discuss problem (1.1), we need to choose the following subspace of $W^{2,p(x)}(\Omega)$

$$X = \left\{ u \in W^{2,p(x)}(\Omega) / \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{in} \quad \partial\Omega \right\}.$$

Let us choose on $W^{2,p(x)}(\Omega)$ the norm defined by

$$\|u\| = \inf \left\{ \mu > 0 : \int_\Omega \left| \frac{\Delta u}{\mu} \right|^{p(x)} dx + \int_\Omega \left| \frac{u}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$

Note that $(X, \|.\|)$ is a separable and reflexive Banach space.

Let $\|u\|_m = \inf \left\{ \mu > 0 : \int_\Omega \left| \frac{\Delta u}{\mu} \right|^{p(x)} dx + \int_{\partial\Omega} m(x) \left| \frac{u}{\mu} \right|^{p(x)} dx \leq 1 \right\}$ for $u \in X$. Then, $\|u\|_m$ is equivalent to $\|u\|$, moreover, if we define the so called modular which is defined by

$$J(u) = \int_\Omega |\Delta u|^{p(x)} dx + \int_{\partial\Omega} m(x) |u|^{p(x)} dx,$$

we have the following proposition (similar to proposition ([13])).

PROPOSITION 2.4.

(i) $\|u\|_m < 1 \ (= 1; > 1) \Longleftrightarrow J(u) < 1 \ (= 1; > 1)$,

(ii) $\|u\|_m > 1 \Longrightarrow \|u\|_m^{p^-} \leq J(u) \leq \|u\|_m^{p^+}$, *for all $u_n \in X$, we have*

(iii) $\|u_n\|_m \longrightarrow 0 \Longleftrightarrow J(u_n) \longrightarrow 0,$

(iv) $\|u_n\|_m \longrightarrow \infty \Longleftrightarrow J(u_n) \longrightarrow \infty$

THEOREM 2.1 ([15]). *Let Ω be a bounded domain in $\mathbb{R}^N$ with C^2 boundary. If $2p(x) \geq N \geq 2$ for all $x \in \overline{\Omega}$, then for all $q \in C_+(\Omega)$ there is a continuous boundary trace embedding*

$$W^{2,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega), \tag{2.3}$$

and

$$W^{2,p(x)}(\Omega) \hookrightarrow W^{1,p(x)}(\partial\Omega). \tag{2.4}$$

PROPOSITION 2.5 ([15]). *If $2p(x) \geq N$ for all $x \in \overline{\Omega}$, then the set*

$$X = \{u \in W^{2,p(x)}(\Omega) : \ \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0\}$$

is a closed subspace of $W^{2,p(x)}(\Omega)$.

Definition 2.1. We say that u is a weak solution of the problem (1.1) if

$$\begin{aligned} &\int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v dx + \int_{\partial\Omega} m(x)|u|^{p(x)-2}uv d\sigma \\ &\qquad = \lambda \int_\Omega V(x)|u|^{q(x)}uv dx \quad \forall v \in X. \end{aligned} \tag{2.5}$$

PROPOSITION 2.6.

(1) *Let $\Phi_1(u) = \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)} + \int_{\partial\Omega} \frac{1}{p(x)} m(x)|u|^{p(x)}dx$. Then the functional $\Phi_1 : X \longrightarrow \mathbb{R}$ is sequentially weakly lower semicontinuous and $\Phi_1 \in C^1(X, \mathbb{R})$.*

(2) *The mapping $\Phi_1^{'} : X \longrightarrow X^{'}$ is a strictly monotone, bounded homeomorphism and is of type S_+.*

Proof. (1) Since $\int_{\partial\Omega} \frac{1}{p(x)}|\Delta u|^{p(x)}dx \in C^1(X, \mathbb{R})$ then $\Phi_1 \in C^1(X, \mathbb{R})$. By the continuity and convexity of Φ_1, we deduce that Φ_1 is sequentially weakly lower semi continuous.

(2) Since $\Phi_1^{'}$ is Fréchet derivative of Φ_1 then $\Phi_1^{'}$ is continuous and bounded. We set

$$U_p = \{x \in \Omega : p(x) \geq 2\}, \qquad U_p^{'} = \{x \in \partial\Omega : p(x) \geq 2\},$$

$$V_p = \{x \in \Omega : 1 < p(x) < 2\}, \qquad V_p^{'} = \{x \in \partial\Omega : 1 < p(x) < 2\}.$$

By using the similar argument as in the Proposition 3.3 of [7], we conclude that $\Phi_1^{'}$ is homeomorphism and is of type S_+. $\square$

3. MAIN RESULTS

We consider the energy function $\Phi : X \longrightarrow \mathbb{R}$ for the problem (1.1), defined by

$$\Phi(u) = \Phi_1(u) - \lambda\Phi_2(u),$$

where

$$\Phi_1(u) = \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx + \int_{\partial\Omega} \frac{m(x)}{p(x)}|u|^{p(x)}dx,$$

and

$$\Phi_2(u) = \int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}dx \quad \forall u \in X.$$

Proof of Theorem 1.1

Step 1. Φ is coercive. It is clear that Φ is even and $\Phi(0) = 0$. Since the embedding $X \hookrightarrow L^{s'(x)q(x)}(\Omega)$ is continuous, we can find a constant $c_3 > 0$ such that

$$|u|_{s'(x)q(x)} \leq c_3\|u\|_m, \quad \forall u \in X. \tag{3.1}$$

According to the fact that

$$|u(x)|^{q(x)} \leq |u(x)|^{q^+} + |u(x)|^{q^-}, \quad \forall x \in \overline{\Omega}. \tag{3.2}$$

From (3.2), we obtain

$$\int_\Omega V(x)|u|^{q(x)}dx \leq |V|_{s(x)}\left||u|^{q(x)}\right|_{s'(x)} \leq |V|_{s(x)}\left(|u|^{q^+}_{q(x)s'(x)} + |u|^{q^-}_{q(x)s'(x)}\right). \tag{3.3}$$

Combining (3.1) and (3.3), we obtain

$$\int_\Omega V(x)|u|^{q(x)}dx \leq |V|_{s(x)}\left(c_3^{q^+}\|u\|_m^{q^+} + c_3^{q^-}\|u\|_m^{q^-}\right). \tag{3.4}$$

Hence, from (3.4), we deduce that for any $u \in X$, we have

$$\begin{aligned}\Phi(u) &= \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx + \int_{\partial\Omega} m(x)\frac{|u|^{p(x)}}{p(x)}dx - \lambda\int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}dx \\ &\geq \frac{1}{p^+}\alpha(\|u\|_m) - \frac{\lambda}{q^-}|V|_{s(x)}\left(c_3^{q^+}\|u\|_m^{q^+} + c_3^{q^-}\|u\|_m^{q^-}\right),\end{aligned}$$

where $\alpha : [0, +\infty[\to \mathbb{R}$ is defined by

$$\alpha(t) = \begin{cases} t^{p^+}, & \text{if } t \leq 1, \\ t^{p^-}, & \text{if } t > 1. \end{cases} \tag{3.5}$$

As $q^+ < p^-$, Φ is bounded from below and coercive, that is, $\Phi(u) \to \infty$ as $\|u\|_m \to \infty$.

Step 2. Φ is weakly lower semi-continuous. Let $u_n \rightharpoonup u$ weakly in X, by proposition 2.3, we obtain the following results

$$\begin{array}{rcl} X & \hookrightarrow & L^{p(x)}(\Omega) \\ u_n & \longrightarrow & u \quad in \quad L^{p(x)}(\Omega) \\ u_n & \longrightarrow & u \quad for\ a.e. \quad x \in \Omega \\ F(x, u_n(x)) & \longrightarrow & F(x, u(x)) \quad for\ a.e. \quad x \in \Omega. \end{array}$$

By Fatou's Lemma,

$$\lim_{n\to\infty} \sup \int_\Omega F(x, u_n(x))dx \le \int_\Omega F(x, u(x))dx.$$

Thus,

(3.6)

$$\begin{aligned} \lim_{n\to\infty} \inf \Phi(u_n) = \lim_{n\to\infty} \inf & \left(\int_\Omega \frac{1}{p(x)} |\Delta u_n|^{p(x)} dx + \int_{\partial\Omega} \frac{1}{p(x)} m(x)|u_n|^{p(x)} dx \right) \\ & - \lim_{n\to\infty} \sup \lambda \int_\Omega F(x, u_n(x))dx \\ & \ge \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx + \int_{\partial\Omega} \frac{1}{p(x)} m(x)|u|^{p(x)} dx \\ & - \lambda \int_\Omega F(x, u(x))dx = \Phi(u), \end{aligned}$$

hence, by the Weierstrass theorem, we deduce that there exists a global minimizer $u_0 \in X$ such that $\Phi(u_0) = \min_{u\in X} \Phi(u)$, because X is closed subspace of the space $W^{2,p(x)}(\Omega)$.

Step 3. We will show that there exists $\lambda_\star > 0$ such that for each $\lambda > \lambda_\star$, $\Phi(u_0) < 0$. We search $u_0 \in X$ in which Φ attains its infimum, hence u_0 represents a weak solution to problem (1.1). Furthermore, for all $\lambda > 0$,

$$\Phi(u_0) \le \Phi(u), \forall u \in X. \tag{3.7}$$

Given the ball B provided by hypothesis $\mathrm{H}_2(V)$, we can take $\varepsilon > 0$ sufficiently small such that

$$\overline{B_\varepsilon} := \overline{\{x \in \Omega | dist(x, B) \le \varepsilon\}} \subset \Omega.$$

Furthermore, let us define the following C_c^1 function

$$u_\varepsilon(x) = \begin{cases} t_0 & \text{when} \quad x \in B, \\ 0 & \text{when} \quad x \in \Omega \setminus B_\varepsilon. \end{cases}$$

Then

$$\Phi(u_\varepsilon) \le \Phi_1(u_\varepsilon) - \lambda \int_B F(x, t_0)dx - \lambda \int_{B_\varepsilon \setminus B} F(x, u_\varepsilon)dx.$$

By the definition of F, we are able to fix ε_0 sufficiently small such that there exists a positive constant c_0 with the property that

$$\Phi(u_{\varepsilon_0}) \leq \Phi_1(u_{\varepsilon_0}) - \lambda c_0 \int_B F(x, t_0)dx.$$

Now, by taking

$$\lambda_{\star\star} := \frac{\Phi_1(u_{\varepsilon_0})}{c_0 \displaystyle\int_B F(x, t_0)dx} > 0, \tag{3.8}$$

we conclude that $\Phi(u_{\varepsilon_0}) < 0$ for all $\lambda > \lambda_{\star\star}$. By choosing $u = u_{\varepsilon_0}$ in (3.7) we obtain that u_0 is nontrivial for all $\lambda > \lambda_\star$ since $\Phi(0) = 0$, and we have completed our proof.

Proof of Theorem 1.2

Firstly, we need to prove the following lemmas.

LEMMA 3.1.

(i) *There exists* $\lambda^{\star\star} > 0$ *such that for any* $\lambda \in (0, \lambda^{\star\star})$, *there exist* $\rho > 0$ *and* $\alpha > 0$ *for which* $\Phi(u) \geqslant \alpha$ *for any* $u \in X$ *with* $\|u\|_m = \rho$.

(ii) *There exists* $\psi \in X, \psi \neq 0$ *such that* $\lim_{t\to+\infty} \Phi(t\psi) = -\infty$.

(iii) *There exists* $\phi \in X, \phi \neq 0$ *such that* $\Phi(t\phi) < 0$ *for any* $t > 0$ *small enough.*

Proof. We will prove the lemma for the case where the former condition of (Q_2'') holds true. While the case (Q_2'') can be made by similar arguments.

(i) Let us define the functions q_1 and q_2 as follows

$$q_1 : \overline{B}_r(x_0) \longrightarrow (1, +\infty), q_1(x) = q(x) \quad \text{for any } x \in \overline{B}_r(x_0)$$

and

$$q_2 : \overline{\Omega\backslash B_r(x_0)} \longrightarrow (1, +\infty), q_2(x) = q(x) \quad \text{for any } x \in \overline{\Omega\backslash B_r(x_0)}.$$

For simplicity, we denote

$$q_1^- = \min_{x\in \overline{B_r(x_0)}} q_1(x), \quad q_1^+ = \max_{x\in \overline{B_r(x_0)}} q_1(x),$$

$$q_2^- = \min_{x\in \overline{\Omega\backslash B_R(x_0)}} q_2(x), \quad q_2^+ = \max_{x\in \overline{\Omega\backslash B_R(x_0)}} q_2(x).$$

By the conditions (Q_1) and (Q_2),

$$1 < q_1^- \leqslant q_1^+ < p^- \leqslant p^+ < q_2^- \leqslant q_2^+ < p^\star(x) \quad \text{for all } x \in \overline{\Omega}. \tag{3.9}$$

It follows that X is continuously embedded in $L^{q_i^\pm}(\Omega)$ for $i = 1, 2$. Then there exists a positive constant C_1 such that

$$\int_\Omega |u|^{q_i^\pm} dx \leqslant C_1 \|u\|_m^{q_i^\pm}, \forall u \in X, i = 1, 2. \tag{3.10}$$

From (3.10), we have

$$\begin{aligned}
\int_{B_r(x_0)} |u|^{q_1(x)} dx &\leq \int_{B_r(x_0)} |u|^{q_1^-} dx + \int_{B_r(x_0)} |u|^{q_1^+} dx \\
&\leq \int_\Omega |u|^{q_1^-} dx + \int_\Omega |u|^{q_1^+} dx \\
&\leq C_1 \left(\|u\|_m^{q_1^-} + \|u\|_m^{q_1^+} \right), \forall u \in X,
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
\int_{\Omega \backslash B_R(x_0)} |u|^{q_2(x)} dx &\leq \int_{\Omega \backslash B_R(x_0)} |u|^{q_2^-} dx + \int_{\Omega \backslash B_R(x_0)} |u|^{q_2^+} dx \\
&\leq \int_\Omega |u|^{q_2^-} dx + \int_\Omega |u|^{q_2^+} dx \\
&\leq C_1 \left(\|u\|_m^{q_2^-} + \|u\|_m^{q_2^+} \right), \forall u \in X.
\end{aligned} \tag{3.12}$$

By the relations (3.11) and (3.12), we have

$$\begin{aligned}
\Phi(u) &= \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx + \int_{\partial\Omega} \frac{m(x)}{p(x)} |u|^{p(x)} dx - \lambda \int_\Omega \frac{V(x)}{q(x)} |u|^{q(x)} dx \\
&\geq \frac{1}{p^+} \|u\|_m^{p^+} - \lambda \int_{B_r(x_0)} \frac{V(x)}{q(x)} |u|^{q(x)} dx - \lambda \int_{\Omega \backslash B_R(x_0)} \frac{V(x)}{q(x)} |u|^{q(x)} dx \\
&\geq \frac{1}{p^+} \|u\|_m^{p^+} - \frac{\lambda C_1 |V|_\infty}{q^-} \left(\|u\|_m^{q_1^-} + \|u\|_m^{q_1^+} + \|u\|_m^{q_2^-} + \|u\|_m^{q_2^+} \right) \\
&\geq \left[\frac{1}{2p^+} - \frac{\lambda C_1 |V|_\infty}{q^-} \left(\|u\|_m^{q_1^- - p^+} + \|u\|_m^{q_1^+ - p^+} \right) \right] \|u\|_m^{p^+} \\
&+ \left[\frac{1}{2p^+} - \frac{\lambda C_1 |V|_\infty}{q^-} \left(\|u\|_m^{q_2^- - p^+} + \|u\|_m^{q_2^+ - p^+} \right) \right] \|u\|_m^{p^+},
\end{aligned}$$

for all $u \in X$ with $\|u\|_m < 1$. Let us define $h : [0, 1] \longrightarrow \mathbb{R}$ by

$$h(t) = \frac{1}{2p^+} - \frac{\lambda C_1}{q^-} |V|_\infty t^{q_2^+ - p^+} - \frac{\lambda C_1}{q^-} |V|_\infty t^{q_2^- - p^+},$$

then for all $\lambda > 0$, there exists $\rho \in (0,1)$ such that $h(\rho) > 0$. Setting

$$\lambda^{\star\star} = \min\left\{1, \frac{q^-}{4p^+ C_1 |V|_\infty} \min\left\{\rho^{p^+ - q_1^-}, \rho^{p^+ - q_1^+}\right\}\right\} > 0, \tag{3.13}$$

we can conclude that for any $\lambda \in (0, \lambda^{\star\star})$ and any $u \in X$ with $\|u\|_m = \rho$, we have

$$\begin{aligned}
\Phi(u) &\geqslant \left[\frac{1}{2p^+} - \frac{\lambda C_1}{q^-}|V|_\infty \left(\rho^{q_1^- - p^+} + \rho^{q_1^+ - p^+}\right)\right]\rho^{p^+} \\
&+ \left[\left[\frac{1}{2p^+} - \frac{\lambda C_1}{q^-}|V|_\infty \left(\rho^{q_2^- - p^+} + \rho^{q_2^+ - p^+}\right)\right]\rho^{p^+}\right] \\
&\geqslant \left[\frac{1}{2p^+} - \frac{\lambda C_1}{q^-}|V|_\infty \left(\rho^{q_1^- - p^+} + \rho^{q_1^+ - p^+}\right)\right]\rho^{p^+} + h(\rho)\rho^{p^+} \\
&\geqslant \frac{1}{4p^+}\rho^{p^+}. \qquad (3.14)
\end{aligned}$$

Then, we deduce that for any $\lambda \in (0, \lambda^{\star\star})$, there exists $\alpha > 0$ such that for any $u \in X$ with $\|u\|_m = \rho$ we have $\Phi(u) \geqslant \alpha$.

(ii) Let $\psi \in C_0^\infty(\Omega), \psi \geqslant 0$ and there exist $x_1 \in \Omega \setminus B_R(x_0)$ and $\varepsilon > 0$ such that for any $x \in B_\varepsilon(x_1) \subset \Omega \setminus B_R(x_0)$ we have $\psi(x) > 0$. For $1 < t$, we have

$$\begin{aligned}
\Phi(t\psi) &= \int_\Omega \frac{1}{p(x)}|\Delta t\psi|^{p(x)}dx + \int_{\partial\Omega} \frac{m}{p(x)}|t\psi|^{p(x)}dx - \lambda \int_\Omega \frac{V(x)}{q(x)}|t\psi|^{q(x)}dx \\
&\leqslant \frac{t^{p^+}}{p^-}\left(\int_\Omega |\Delta\psi|^{p(x)}dx + \int_{\partial\Omega} m(x)|\psi|^{p(x)}dx\right) \\
&- \lambda t^{q_2^-}\int_{\Omega\setminus B_R(x_0)} \frac{V(x)}{q(x)}|\psi|^{q(x)}dx \quad \longrightarrow -\infty, \qquad (3.15)
\end{aligned}$$

since $p^+ < q_2^-$.

(iii) Let $\phi \in C_0^\infty(\Omega), \phi \geq 0$ and there exist $x_2 \in B_r(x_0)$ and $\varepsilon > 0$ such that for any $x \in B_\varepsilon(x_2) \subset B_r(x_0)$ we have $\phi(x) > 0$. Letting $0 < t < 1$, we get

$$\begin{aligned}
\Phi(t\phi) &= \int_\Omega \frac{1}{p(x)}|\Delta t\phi|^{p(x)}dx + \int_{\partial\Omega} \frac{m}{p(x)}|t\phi|^{p(x)}dx - \lambda \int_\Omega \frac{V(x)}{q(x)}|t\phi|^{q(x)}dx \\
&\leqslant \frac{t^{p^-}}{p^-}\left(\int_\Omega |\Delta\phi|^{p(x)}dx + \int_{\partial\Omega} m(x)|\phi|^{p(x)}dx\right) \\
&- \lambda t^{q_1^+}\int_{B_r(x_0)} \frac{V(x)}{q(x)}|\phi|^{q(x)}dx. \qquad (3.16)
\end{aligned}$$

Obviously, we have $\Phi(t\phi) < 0$ for any $0 < t < \delta^{\frac{1}{p^- - q_1^+}}$, where

$$0 < \delta < \min\left\{1, \frac{\lambda \int_{B_r(x_0)} \frac{V(x)}{q(x)} |\phi|^{q(x)} dx}{\int_\Omega |\Delta\phi|^{p(x)} dx + \int_{\partial\Omega} m(x)|\phi|^{p(x)} dx}\right\}.$$

The proof is complete. □

LEMMA 3.2. *The functional Φ satisfies the Palais-Smale condition in X.*

Proof. Let (u_n) in X be such that

$$\Phi(u_n) \longrightarrow c, \Phi'(u_n) \longrightarrow 0 \quad \text{in} \quad X^\star \quad \text{as} \quad n \to \infty, \tag{3.17}$$

where $X^{\star\star}$ is the dual space of X.

We will show that (u_n) is bounded in X. We assume by contradiction that passing if necessary to a subsequence, still denoted by (u_n), we have $\|u_n\|_m \longrightarrow \infty$ as $n \longrightarrow \infty$. By (3.17), for n large enough and $\lambda \in (0, \lambda^{\star\star})$, we have

$$\begin{aligned}
1 + c + \|u_n\|_m &\geq \Phi(u_n) - \frac{1}{q_2^-}\left\langle \Phi'(u_n), u_n \right\rangle \\
&= \int_\Omega \frac{1}{p(x)} |\Delta u_n|^{p(x)} dx + \int_{\partial\Omega} \frac{m(x)}{p(x)} |u_n|^{p(x)} dx - \lambda \int_\Omega \frac{V(x)}{q(x)} |u_n|^{q(x)} dx \\
&- \frac{1}{q_2^-} \int_\Omega |\Delta u_n|^{p(x)} dx + \frac{1}{q_2^-} \int_{\partial\Omega} m(x)|u_n|^{p(x)} dx + \frac{\lambda}{q_2^-} \int_\Omega V(x)|u_n|^{q(x)} dx \\
&\geq \left(\frac{1}{p^+} - \frac{1}{q_2^-}\right) \left(\int_\Omega |\Delta u_n|^{p(x)} dx + \int_{\partial\Omega} m(x)|u_n|^{p(x)}\right) \\
&- \lambda \int_{B_r(x_0)} V(x) \left(\frac{1}{q_1(x)} - \frac{1}{q_2^-}\right) |u_n|^{q_1(x)} dx \\
&\geq \left(\frac{1}{p^+} - \frac{1}{q_2^-}\right) \|u_n\|_m^{p^-} - C_1 |V|_\infty \lambda^\star \left(\frac{1}{q_1^-} - \frac{1}{q_2^-}\right) \left(\|u_n\|_m^{q_1^-} + \|u_n\|_m^{q_1^+}\right).
\end{aligned}$$

Dividing the above inequality by $\|u_n\|_m^{p^-}$, passing to the limit as $n \to \infty$, we obtain a contradiction. It follows that (u_n) is bounded in X. Since X is a reflexive Banach space, there exists a subsequence, still denoted by (u_n), it converges weakly to u in X. Then $(\|u_n - u\|_m)$ is bounded. By $(Q1)$, the embedding $X \hookrightarrow L^{q(x)}(\Omega)$ is compact. Thus, using the Hölder inequality, we have

$$\left| \int_\Omega V(x)|u_n|^{q(x)-2} u_n(u_n - u) dx \right| \leq |V|_\infty \int_\Omega |u_n|^{q(x)-1} |u_n - u| \, dx$$

$$\leq 2|V|_\infty \left| |u_n|^{q(x)-1} \right|_{\frac{q(x)}{q(x)-1}} |u_n - u|_{q(x)}$$

$$(3.18) \qquad \to 0 \quad \text{as} \quad n \to \infty,$$

and then,

$$(3.19) \qquad \lim_{\Omega} V(x)|u_n|^{q(x)-2} u_n(u_n - u)dx = 0.$$

Using relation (3.17) and the boundedness of $(u_n - u)$ in X, we find that $\left\langle \Phi'(u_n), u_n - u \right\rangle \to 0$ as $n \to \infty$. Combining this with (3.19), it follows that

$$\lim_{n\to\infty} \left(\int_\Omega |\Delta u_n|^{p(x)-2} \Delta u_n \left(\Delta u_n - \Delta u\right) dx + \int_{\partial\Omega} m(x)|u_n|^{p(x)-2} u_n \left(u_n - u\right) dx \right) = 0,$$

or

$$(3.20) \qquad \lim_{n\to\infty} \Phi_1'(u_n)(u_n - u) = 0.$$

Finally, by (3.20) and Proposition 2.3, (u_n) converges strongly to u in X and the functional Φ satisfies the Palais-Smale condition. □

Now, we will give the proof of theorem 1.2.

From lemma 3.1, for any $u \in X$ with $\|u\|_m = \rho$ we have $\Phi(u) \geq \alpha > 0$ and there exists $e \in X$ with $\|e\|_m > \rho$ such that $\Phi(e) < 0$. Let

$$\Gamma := \{\gamma \in C\left([0,1], X\right) : \gamma(0) = 0, \gamma(1) = e\}$$

and define

$$\overline{c} = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J\left(\gamma(t)\right).$$

Since $\|e\|_m > \rho$, every path $\gamma \in \Gamma$ intersects the sphere $\|u\| = \rho$, so we have

$$\overline{c} \geq \inf_{\|u\|_m=\rho} \Phi(u) \geq \alpha > 0.$$

According to Lemma 3.1 and Lemma 3.2, all assumptions of the mountain pass theorem in [4] are satisfied. Then we deduce a function $u_1 \in X$ as a nontrivial critical point of the functional Φ with $\Phi(u_1) = c > 0$ and thus a non-trivial weak solution of problem (1.1).

Now, we prove that there exists a second weak solution $u_2 \in X$ such that $u_2 \neq u_1$. Indeed, let $\lambda^{\star\star}$ as in the proof of Lemma 3.1(i) and assume that $\lambda \in (0, \lambda^{\star\star})$. By Lemma 3.1(i), it follows that on the boundary of the ball centered at the origin and of radius $\rho \in X$, denoted by $B_{\rho(0)} = \{u \in X : \|u\|_m < \rho\}$, we have

$$\inf_{u\in\partial B_{\rho(0)}} \Phi(u) > 0.$$

On the other hand, by Lemma 3.1(ii), there exists $\varphi \in X$ such that $\Phi(t\varphi) < 0$ for all $t > 0$ small enough.

Moreover, from the proof of Lemma 3.1, the functional Φ is bounded from below on $B_\rho(0)$. It follows that

$$-\infty < c_- := \inf_{u \in \overline{B}_\rho(0)} \Phi(u) < 0.$$

Now we can apply the Ekeland variational principle in [12] to the functional $\Phi : B_\rho(0) \longrightarrow \mathbb{R}$, it follows that there exists $u_\varepsilon \in B_\rho(0)$ such that

$$\Phi(u_\varepsilon) < \inf_{u \in \overline{B_\rho(0)}} \Phi(u) + \varepsilon,$$

$$\Phi(u_\varepsilon) < \Phi(u) + \varepsilon\|u - u_\epsilon\|, u \neq u_\epsilon.$$

Let us choose $\varepsilon > 0$ such that

$$0 < \varepsilon < \inf_{u \in \overline{\partial B_\rho(0)}} \Phi(u) - \inf_{u \in \overline{B_\rho(0)}} \Phi(u),$$

then we deduce that $\Phi(u_\varepsilon) < \inf_{u \in \overline{\partial B_\rho(0)}} \Phi(u)$ and thus, $u_\varepsilon \in B_\rho(0)$.

Now, we define the functional $\overline{J}_\lambda : \overline{B}_\rho(0) \longrightarrow \mathbb{R}$ by $\overline{J}_\lambda(u) = \Phi(u) + \varepsilon\|u - u_\varepsilon\|_m$. It is clear that u_ε is a minimum point of $\overline{J}_\lambda$ and thus

$$\frac{\overline{J}_\lambda(u_\varepsilon + tv) - \overline{J}_\lambda(u_\varepsilon)}{t} \geqslant 0$$

for all $t > 0$ small enough and all $v \in B_\rho(0)$. It follows that

$$\frac{\Phi(u_\varepsilon + tv) - \Phi(u_\varepsilon)}{t} + \varepsilon\|v\|_m \geqslant 0.$$

Letting $t \to 0^+$, we obtain

$$\left\langle J'_\lambda(u_\varepsilon), v \right\rangle \geqslant -\varepsilon\|v\|_m.$$

Also, it should be noticed that $-v$ also belongs to $B_{\rho_1}(0)$, so replacing v by $-v$, we get

$$\left\langle \Phi'(u_\varepsilon) - v \right\rangle \geqslant -\varepsilon\| - v\|_m \text{ or } \left\langle J'_\lambda(u_\varepsilon), v \right\rangle \leqslant \varepsilon\|v\|_m,$$

which implies that $\|\Phi(u_\varepsilon)\|_{X^\star} \leqslant \varepsilon$. Thus, there exists a sequence $(u_n) \subset B_\rho(0)$ such that

$$\Phi(u_n) \longrightarrow \underline{c} = \inf_{u \in \overline{B}_\rho(0)} J(u) < 0, \quad \text{and} \quad J'_\lambda(u_n) \longrightarrow 0 \in X, \quad \text{as} \quad n \to \infty. \tag{3.21}$$

From Lemma 3.2, the sequence (u_n) converges strongly to $u_2 \in X$ as $n \to \infty$. In addition, since $\Phi \in C_1(X, \mathbb{R})$, by (3.21) it follows that $\Phi(u_2) = \underline{c}$ and $J'_\lambda(u_2) = 0$. Thus, u_2 is a non-trivial weak solution of problem (1.1). Finally, we point out the fact that $u_1 \neq u_2$ since $\Phi(u_1) = \overline{c} > 0 > \underline{c} = \Phi(u_2)$. Moreover, since $\Phi(u) = \Phi(|u|)$, problem (1.1) has at least two non-negative non-trivial weak solutions. This concludes the proof of Theorem 1.2 .

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