Dedicated to the memory of Cabiria Andreian Cazacu,
a mathematician I much admired

WITTEN DEFORMATION AND THE SPECTRAL PACKAGE OF A RIEMANNIAN MANIFOLD

DAN BURGHELEA

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The Witten deformation associated to a Morse function f on a closed Riemannian manifold (M, g) via Rellich-Kato theorem (see section 3) relates analytically the spectral package of (M, g) (eigenvalues and eigenforms) to the Morse complex defined by (g, f) coupled with the "multivariable harmonic oscillators" associated to the critical points of f. We survey this relation and discuss some implications, including the finite subset of the spectral package referred to as the "virtually small spectral package".

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CONTENTS

1. INTRODUCTION

A smooth map $f : M \to \mathbb{R}$ on a closed Riemannian manifold (M, g), via Witten deformation, provides an analytic family of self adjoint operators $\Delta^q(t)$, with $\Delta^q(0) = \Delta^q$ the Laplace - Beltrami operator on q-differential forms on M, see section 3 below, for definition. This family is "holomorphic of type A" in the sense of Kato, cf [17].

Rellich-Kato theorem, cf [17] Theorem 3.9, organizes the eigenvalues and the eigenforms of $\Delta^q(t)$, for all $t \in \mathbb{R}$, as a *countable collection* $\mathcal{A}^q$ of analytic real-valued functions $\lambda^q_\alpha(t)$ and analytic q-differential form-valued maps $\omega^q_\alpha(t)$ s.t. for any t $\Delta^q(t)(\omega^q_\alpha(t)) = \lambda^q_\alpha(t)\omega^q_\alpha(t)$ and the collection $\{\lambda^q_\alpha(t), \omega^q_\alpha(t), \alpha \in \mathcal{A}^q\}$ represent the entire spectral package of $\Delta^q(t)$ (i.e. the collection of eigenvalues and eigenvectors of an operator, in this case $\Delta^q(t)$). We call $\lambda^q_\alpha(t)$s and $\omega^q_\alpha(t)$s branches of eigenvalues and eigenforms.

If f is a Morse function then the infinite collection $\mathcal{A}^q$ clusters as a disjoint union of finite collections $\mathcal{A}^q(k)$, $k \in \mathbb{Z}_{\geq 0}$, and the collection $\mathcal{A}^q(\infty)$ conjecturally empty. Each of these collections, but possibly $\mathcal{A}^q(\infty)$, are finite of cardinality depending only on the set of critical points x and the scalar product $g(x)$, the metric g provides on $T_x(M)$. If in the small neighborhood of each critical point the metric is flat then the cardinality depends only on the number of critical points of each index, as specified by Theorem 3.3. We expect this remains the same for any metric g.

For $\alpha \in \mathcal{A}^q(k)$ $\lim_{t\to\infty} \lambda^q_\alpha(t)/t = 2l_k$, $0 = l_0 < l_1 < l_2 < \cdots$, with $l_k \to \infty$ [1], and for $\alpha \in \mathcal{A}(\infty)$ $\lambda^q_\alpha(t)/t$ is unbounded in t. $\mathcal{A}^q(\infty) = \emptyset$ will imply that any $\lambda^q(t)/t$ is convergent to a finite number.

The cluster corresponding to $k = 0$ is in bijective correspondence to the set of critical points of index q and determines an analytic family in $t \in \mathbb{R}$ of cochain complexes with scalar product, all with the components and the cohomology of the same dimension. The cochain complex corresponding to $t = 0$ is isometrically embedded in $(\Omega^*(M), d^*)$ equipped with the scalar product defined by the metric g, and is left invariant by Δ^q. The cochain complex corresponding to t, when $t \to \infty$, is $O(1/t)$-isometric to the appropriately scaled geometric complex (= Morse complex) defined by the vector field $(-\operatorname{grad}_g f)$ and equipped with the scalar product making the characteristic functions of critical points orthonormal, cf. Theorem 6.1.

The restriction to $t = 0$ selects from the infinite spectral package of (M, g) a finite collection $\{\lambda^q_\alpha(0), \omega^q_\alpha(0),\ \alpha \in \mathcal{A}^q(0) \subset \mathcal{A}^q\}$ referred below as the *q-virtually small spectral package* of (M, g) associated with f which has remarkable topological and geometrical properties, cf.Theorem 6.2.

It is important to note that the collection of virtually small eigenvalues which is of cardinality $c_q = \sharp Cr_q(f)$ does not necessary consists of the smallest c_q eigenvalues of Δ^q, cf [9] for an example, however they are the restriction at $t = 0$ of the eigenvalue branches $\lambda^q_\alpha(t)$ whose restriction to t large enough exhaust the first smallest c_q eigenvalues of $\Delta^q(t)$. In different words, they become,

[1] One can ask what happens when $t \to -\infty$. The answer is similar; if one writes $\Delta^{q,f}(t)$ instead of $\Delta^q(t)$, in order to indicate the smooth function of concern, then $\Delta^{q,f}(-t) = \Delta^{q,-f}(t)$.

by analytic continuation in t, the smallest c_q eigenvalues. Note also that, in view of analyticity, the restriction of $\omega^q_\alpha(t)$ to an arbitrary small neighborhood of t_0 determines completely $\omega^q_\alpha(t)$ for any t and then $\omega^q_\alpha(0)$.

Observation: In view of Propositions 5.2 and 5.1, the restrictions at t of $\omega^q_\alpha(t)$ for $\alpha \in \mathcal{A}^q(0)$ and $\alpha \in \mathcal{A}^q \setminus \mathcal{A}^q(0)$ when t is very large are very different, as one can read off from Proposition (5.2) and Proposition (5.1) respectively; one expects this difference be manifest in some way for the restriction at $t = 0$ and provide a way to recognize the collection $\mathcal{A}^q(0)$.

The purpose of this largely expository paper is to present these observations as a starting point in exploring the geometric and topological applications/implications of the virtually small spectral package.

Theorem 6.2 item (2) suggests that at least one topological invariant involving the entire spectral package, as stated in Theorem 2.1 item (2), can be recovered from the virtually small spectral package. One expects that this is the case with many more geometric and topological invariants,

Theorems 3.3 and 6.2 can be regarded as new statements, while the others, Theorems 3.2, 4.1, 6.1 only as some improvements of results already published.

The paper contains in addition to introduction six more sections.

- Section 2 reviews the *spectral package* of a closed Riemannian manifold.
- Section 3 reviews the Witten deformation and Witten-Laplace operators $\Delta^q(t)$, the Rellich - Kato result on analytic branches of eigenvalues and eigenforms for $\Delta^q(t)$ and their clustering.
- Section 4 reviews the CW structure defined by a pair (g, f) and the associated geometric complex known as the Morse complex.
- Section 5 sketches the proof of Theorems 3.2 and 3.3.
- Section 6 discusses the cluster $\mathcal{A}^q(0)$ and sketches the proof of Theorems 6.1 and 6.2.
- Section 7 formulates a few conjectures and problems.

2. SPECTRAL PACKAGE OF A RIEMANNIAN MANIFOLD (M^n, g)

Let M^n be a closed differential manifold equipped with a Riemannian metric g. The differential structure on M provides

- $\Omega^r(M^n)$ - the vector space of differential forms of degree r,
- $\mathcal{X}(M)$ - the vector space of smooth vector fields,

  both modules over $\Omega^0(M) = \mathcal{D}(M)$, the commutative algebra of smooth functions,

- $\wedge : \Omega^r(M) \times \Omega^{r'}(M) \to \Omega^{r+r'}(M)$ - the exterior multiplication, which is $\Omega^0(M)$ - bilinear map,

- $\iota^r_X : \Omega^*(M) \to \Omega^{*-1}(M)$ - the contraction w.r. to $X \in \mathcal{X}(M)$, $\Omega^0(M)$ - linear map,

- $d^r : \Omega^r(M) \to \Omega^{r+1}(M)$ - the exterior differential, a differential operator of order one,

- $L^r_X : \Omega^r(M) \to \Omega^r(M)$ - the Lie derivative along X, $X \in \mathcal{X}(M)$ defined by $L^r_X := d^{r-1} \cdot \iota^r_X - \iota^{r+1}_X \cdot d^r$ a differential operator of order one.

The pair $(\Omega^*, d^*), * = 0, 1, \cdots$ is a cochain complex, known as the de-Rham complex, with de-Rham cohomology $\mathcal{H}^r_{DR}(M) := \ker d^r / \operatorname{img} d^{r-1}$. The de-Rham theorem establishes a canonical isomorphism from de-Rham cohomology $\mathcal{H}^r_{DR}(M)$ to the singular cohomology with real coefficients of M, isomorphism obtained via integration of r - differential forms on singular smooth simplexes.

For simplicity we will assume M orientable. The Riemannian metric g on M provides the Hodge star-operator $\star^r : \Omega^r(M^n) \to \Omega^{n-r}(M)$ with $\star^{n-r} \cdot \star^r = (-1)^{r(n-r)}$ and then the scalar product

$$\langle \omega, \omega' \rangle := \int_M \omega \wedge \star^r \omega' \tag{2.1}$$

on each $\Omega^r(M)$. This in turn provides the differential operator $\delta^r : \Omega^r(M^n) \to \Omega^{r-1}(M)$, the formal adjoint of d^{r-1}, defined by $\delta^r := (-1)^{n(r-1)+1} \star^{n-r+1} \cdot d^{n-r} \cdot \star^r$ and then the operator $\Delta^r : \Omega^r(M) \to \Omega^r(M)$. This is a formally self adjoint, nonnegative definite differential operator of order two defined by

$$\Delta^r := \delta^{r+1} \cdot d^r + d^{r-1} \cdot \delta^r$$

called the r - Laplace - Beltrami operator or simpler the r - Laplacian.

Note that

$$\star^q \cdot \Delta^q = \Delta^{n-q} \cdot \star^{n-q}.$$

Note also that one has following commutative up to sign diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \Omega^q & \xrightarrow{d^q} & \Omega^{q+1} & \xrightarrow{d^{q+1}} & \Omega^{q+2} & \xrightarrow{d^{q+2}} & \cdots \\
 & & \downarrow{\scriptstyle \star^q} & & \downarrow{\scriptstyle \star^{q+1}} & & \downarrow{\scriptstyle \star^{q+2}} & & \\
\cdots & \longrightarrow & \Omega^{n-q} & \xrightarrow{\delta^{n-q}} & \Omega^{n-q-1} & \xrightarrow{\delta^{n-q-1}} & \Omega^{n-q-2} & \xrightarrow{\delta^{n-q-2}} & \cdots
\end{array} \tag{2.2}$$

with

$$\star^{q+1} \cdot d^q = (-1)^{q+1} \delta^{n-q} \cdot \star^q.$$

Since the homology of the second row of the diagram (2.2) is canonically identified via the metric g to the de-Rham homology of M based on currents, the $\star$-operator realizes the familiar Poincaré duality from $H^r_{DR}(M)$ to $H_{n-r}^{DR}(M)$.

Since M^n is a closed manifold then *formally adjoint* becomes *adjoint* on $L_2(\Omega^q(M))$ the L_2-closure of $\Omega^q(M)$ with respect to the scalar product (2.1).

Each Δ^r has spectrum consisting of the infinite sequence of eigenvalues, nonnegative real numbers λ^r_k, $0 = \lambda^r_0 < \lambda^r_1 < \cdots < \lambda^r_k < \lambda^r_{k+1} < \cdots$, each with finite multiplicity, which increase to ∞.

Denote by $H^r_k \subset \Omega^r(M)$ the eigenspace corresponding to the eigenvalues λ^k_r, $H^r_k := \ker(\Delta^r - \lambda^r_k Id) \subset \Omega^r(M)$ and note that $H^r_k \perp H^r_{k'}$ for $k \neq k'$.

Let $H^r_{0,\mathbb{Z}} \subset H^r_0 \subset \Omega^r(M)$ be the lattice of *integral harmonic r-forms*, i.e harmonic forms which take integer values when integrated on $\mathbb{Z}$-cycles of dimension r, and let $T_r := H^r_0/H^r_{0,\mathbb{Z}}$ be the compact torus equipped with the obvious Riemannian metric induced from the scalar product $\langle\ ,\ \rangle$ restricted to H^r_0.

Denote by $V^r := vol_g T_r$. Note that $V^0 = 1, V^n = vol_g M^n$ and let

$$\mathbb{V}(M,g) := \prod_i (V^i)^{(-1)^i}.$$

The table

$$\begin{array}{cccccc} 0 = \lambda^r_0 < & \lambda^r_1 < \cdots < & \lambda^r_k < & \lambda^r_{k+1} < & \cdots \\ H^r_0 & H^r_1 \ \cdots & H^r_k & H^r_{k+1} & \cdots \end{array}$$

with $H^r_i := \ker(\Delta^r - \lambda^r_i Id)$, is referred to as the *spectral package* of (M,g). Note that the spectral package corresponding to q identifies via $\star^q$-operator to the spectral package corresponding to $(n-q)$.

Sometimes H^r_i is specified by an orthonormal base. The table *spectral package* might also contain the numbers V^r's. Any finite part of the spectral package is, in principle, computable with arbitrary accuracy but not the entire spectral package.

Hodge theorem establishes a canonical decomposition

$$\Omega^r(M) = \Omega^r_+(M) \oplus \Omega^r_-(M) \oplus H^r_0$$

in mutually orthogonal subspaces $\Omega^r(M)_+ = \operatorname{img} d^{r-1}$, $\Omega^r(M)_- = \operatorname{img} \delta^{r+1}$ and $H^r_0 = \ker \Delta^r = \ker d^r \cap \ker \delta^r$, decomposition which diagonalizes Δ^r and satisfies $\star^r : \Omega^r(M)_\pm \to \Omega^{n-r}(M)_\mp$ and which implies the decomposition $H^r_k = H^r_{k,+} \oplus H^r_{k,-}$ with $H^r_{k,\pm} = H^r_k \cap \Omega^r(M)_\pm$. This shows that a Riemannian metric realizes de-Rham cohomology canonically as a subspace of differential forms referred to as *harmonic forms* and one has the morphisms of co-chain complexes

$$(H^*_0, 0) \xrightarrow{in} (\Omega^*(M), d^*) \xrightarrow{pr} (H^*_0, 0)$$

with composition the identity.

In view of the ellipticity and nonnegativity of Δ^r one has the well defined positive real numbers, the zeta-regularized determinants $\det \Delta^r_\pm, \det' \Delta^r$, which represent the $\zeta-$regularized product of the nonzero eigenvalues of $\Delta^r_\pm, \Delta^r$, [2] cf [12] and then

$$T_{an}(M,g) := \prod_r (\sqrt{\det{}'\Delta^r}\ )^{(-1)^{r+1}r} = \prod_r (\sqrt{\det\Delta^r_+}\ )^{(-1)^{r+1}}$$

referred to as the *analytic torsion.*

Recall that for any compact ANR X, in particular for any compact manifold, the integral homology $H_r(X;Z)$ is a finitely generated abelian group of a finite rank β_r whose set of finite order elements have finite cardinality $\mathbb{T}or_i$. The following two numbers are remarkable topological invariants

$$\chi(X) := \sum (-1)^i \beta_i(X) \text{ and } \mathbb{T}or(X) := \prod (\mathbb{T}or_i(X))^{(-1)^i}.$$

Note that when $X = M$ is a closed odd dimensional manifold $\chi(M) = 0$ and when X is an oriented even dimensional manifold $\mathbb{T}or(M) = 1$.

The following two familiar results are among the many relations between topology and the spectral packages.

THEOREM 2.1.

1. *(de-Rham - Hodge)* $\chi(M) = \sum_i (-1)^i \dim H^i_0$
2. *(Cheeger - Muller)* $\ln \mathbb{T}or(M) = -\sum_i (-1)^i \ln V^i + \ln T_{an}(M,g)$

Item (1) derives the integer $\chi(M)$ from the multiplicity of the eigenvalue 0 of Δ^rs while item (2) derives the integer $\mathbb{T}or(M)$ from the nonzero eigenvalues of Δ^rs and the numbers V^rs, cf. [12] Theorem 8.35.

3. WITTEN DEFORMATION AND THE SPECTRAL PACKAGE OF A TRIPLE (M, g, f)

Let $f : M \to \mathbb{R}$ be a smooth function. The *Witten deformation* is a one parameter family of scaled de-Rham complexes, precisely the family of cochain complexes $(\Omega^*(M), d^*(t)),\ t \in \mathbb{R}$, with the differential

$$d^q(t)\omega = e^{-tf} d^q(e^{tf}\omega) = d^q\omega + tdf \wedge \omega,\ \omega \in \Omega^q(M).$$

[2] $\Delta^r_\pm = \Delta^r|_{\Omega^r_\pm(M)}$,

The Laplacians $\Delta^q(t) := d^{q-1}(t) \cdot \delta^q(t) + \delta^{q+1}(t) \cdot d^q(t)$ with

$$\delta^q(t) := (-1)^{n(q-1)+1} \star^{n-q+1} \cdot e^{th} d^{n-q} e^{-th} \cdot \star^q$$

referred below as the *Witten Laplacians* can be expressed in terms of the Laplacians Δ^q as follows:

$$\Delta^q(t) = \Delta^q + t(L^q_X + \mathcal{L}^q_X) + t^2||X||^2. \tag{3.1}$$

These operators remain self adjoint, nonnegative elliptic operators on $L_2(\Omega^q(M))$, the L_2- completion of $\Omega^q(M)$. Here $\mathcal{L}^q_X$ its formal adjoint of L^q_X given by $\mathcal{L}^q_X := (-1)^{(n+1)q+1} \star^{n-q} \cdot L^{n-q}_X \cdot \star^q$.

Hodge decomposition continues to hold for any t, namely

$$\Omega^q(M) = \Omega^r_+(M)(t) \oplus \Omega^r_-(M)(t) \oplus H^r_0(t)$$

with mutually orthogonal $\Omega^r_+(M)(t) = \operatorname{img} d^{r-1}(t)$, $\Omega^r_-(M)(t) = \operatorname{img} \delta^{r+1}(t)$ and $H^r_0(t) = \ker \Delta^r(t) = \ker d^r(t) \cap \ker \delta^r(t)$, decomposition which diagonalizes $\Delta^r(t)$. Note that in view of the isomorphism of $(\Omega^*(M), d^*)$ and $(\Omega^*(M), d^*(t))$ we have the isomorphism of their cohomology and then the equality

$$\dim H^r_0 = \dim H^r_0(t). \tag{3.2}$$

Note that the diagram 2.2 can be enhanced by replacing d^* by $d^*(t)$ and δ^* by $\delta^*(t)$.

As a consequence of (3.1), by a result of Rellich - Kato, Theorem 3.9, chapter 7, in [17] one has Theorem 3.1 below.

THEOREM 3.1 (Rellich - Kato). *There exists a collection of nonnegative real-valued functions $\lambda^q_\alpha(t)$ unique up to permutation and a collection of norm one $q-$differential form-valued maps $\omega^q_\alpha(t) \in \Omega^q(M)$, analytic in $t \in \mathbb{R}$, indexed by $\alpha \in \mathcal{A}$, $\mathcal{A}$ countable set, each with holomorphic extension to a neighborhood of the real line $\mathbb{R} \subset \mathbb{C}$* [3] *s.t.*

1. $\Delta^q(t)\omega^q_\alpha(t) = \lambda^q_\alpha(t)\omega^q_\alpha(t)$,
2. *for any t the collection $\lambda^q_\alpha(t)$ represent all repeated eigenvalues of $\Delta^q(t)$ and the collection $\omega^q_\alpha(t)$ form a complete orthonormal family of associated eigenvectors for the operator $\Delta^q(t)$,*
3. *exactly $\beta_q = \dim H^q(M;\mathbb{R})$ eigenvalue functions $\lambda^q_\alpha(t) = 0$ for any t, all other are strictly positive for any t.*

[3]holomorphic extension means extensions $\lambda^q(z) \in \mathbb{C}$, $\omega^q(z) \in \Omega(M) \otimes C$ for z in a neighborhood of $\mathbb{R}$ in $\mathbb{C}$ which for $t \in \mathbb{R}$ is a real number and $\omega^q(t) \in \Omega(M) \otimes 1$

The analytic functions $\lambda^q_\alpha(t)$ and the analytic differential form-valued maps $\omega^q_\alpha(t)$ are called *eigenvalue branches* and *eigenform branches* and the collection $\{\lambda^q_\alpha(t), \omega^q_\alpha(t), \alpha \in \mathcal{A}^q\}$ is called the *spectral package of* (M, g, f).

We are interested in these branches when $f : M \to \mathbb{R}$ is a Morse function on M. This means that for any $x \in Cr(f) := \{x \in M \mid df_x = 0\}$ in some local chart $\varphi_x : U_x \to \mathbb{R}^n, U_x$ open neighborhood of x, φ_x open embedding with $\varphi_x(x) = (0, , 0, \cdots 0)$ one has

$$f \cdot \varphi_x^{-1}(x_1, x_2, \cdots x_n) = f(x) - 1/2 \sum_{1 \leq i \leq k} x_i^2 + 1/2 \sum_{k+1 \leq i \leq n} x_i^2. \tag{3.3}$$

In this case x is called *critical point of Morse index* k. Denote by $Cr_k(f) \subset Cr(f)$ the set of critical points of Morse index k. A chart φ with such property for a critical point is called *Morse chart* in which case the coordinates $(x_1, x_2, \cdots x_n)$ are called *Morse coordinates.*

From now on, mostly for simplicity in estimates, we will assume that in a small neighborhood of critical points there exists a Morse chart φ s.t. the metric g is given by $g_{i,j}(x_1, \cdots x_n) = \delta_{i,j}$. Such metric is called *flat near critical points.* We believe this assumption can be removed or at least weakened.

The following result is credited to E. Witten; a proof will be sketched in Section 5 following [7] section 5 where a similar statement under more general hypotheses is treated.

THEOREM 3.2. *Suppose that* (M, g) *is a closed Riemannian manifold and* $f : M \to \mathbb{R}$ *a Morse function with* c_q *critical points of Morse index* q. *There exists constants* C_1, C_2, C_3 *and* T_0 *depending on* (M, g, f) *s.t. for any* $t > T_0$

1. *Spect* $\Delta_q(t) \cap (C_1 e^{-C_2 t}, C_3 t) = \emptyset$ *and* $1 \in (C_1 e^{-C_2 t}, C_3 t)$,

2. *for* $t > T_0$ *the number of eigenvalue branches* $\lambda^q_\alpha(t)$ *that satisfy* $\lambda^q_\alpha(t) \in [0, C_1 e^{-C_2 t})$, *counted with multiplicity, is exactly* c_q, *the number of critical points of Morse index* q,

3. *any eigenform branch* $\omega^q_\alpha(t)$ *with eigenvalue branch* $\lambda^q_\alpha(t) < 1$ *for* $t > T_0$ *localizes at the set* $Cr(f)_q$, *i.e.* $\lim_{t\to\infty} ||\omega^q_\alpha(t)||_{L_2(M\setminus U)} = 0$ *for any open neighborhood* U *of* $Cr_q(f)$.

As a consequence exactly c_q eigenvalue branches $\lambda^q_\alpha(t)$ go exponentially fast to 0 when $t \to \infty$, in particular $\lim_{t\to\infty} \lambda^q_\alpha(t)/t = 0$ and all other go more than linearly fast to ∞, in particular $\lim_{t\to\infty} \lambda^q_\alpha(t) = \infty$. It can be shown that $\lambda^q_\alpha(t) \leq c_1 + c_2 t + c_3 t^2$ cf [15]. Also, as a consequence, if $\lambda^q(t)$ is bounded for $t > 0$ (say $\lambda^q(t) < 1$) then $\lambda^q(t)$ goes exponentially fast to 0 when $t \to \infty$.

Denote by $\mathcal{A}^q_{vs} = \mathcal{A}^q(0) := \{\alpha \in \mathcal{A}^q | \lim_{t\to\infty} \lambda^q_\alpha(t)/t = 0\}$ which is the same as the set of branches which go exponentially fast to *zero*. These c_q eigenvalue branches are called *virtually small* and the collection

$$\{\lambda^q_\alpha(t), \omega^q_\alpha(t), \alpha \in \mathcal{A}_{vs}\}$$

the *virtually small spectral package of* (M, g, f).

Since $\Delta^q(-t)$ for f is the same as $\Delta^q(t)$ for $-f$, when $t \to -\infty$ exactly β_{n-q} eigenvalue branches go exponentially fast to zero the other more than linearly fast to ∞.

THEOREM 3.3. *If the metric g is flat near the critical points of the Morse function f then:*

1. *the set $\mathcal{A}^q \setminus \mathcal{A}^q(\infty)$ is in bijective correspondence with the collection of symbols*
   $\{\alpha = (x, I, P) \in Cr(f) \times \mathcal{I}^q \times (Z_{\geq 0})^n \mid o(\alpha) \geq 0\}$ *where*
   $x \in Cr(f), I \in \mathcal{I}^q = \{(j_1, j_2, \cdots j_q) \mid 1 \leq j_1 < j_2 < \cdots j_q \leq n\}$,
   $P = (p_1, \cdots p_n), p_i \in \mathbb{Z}_{\geq 0}$ *and*
   $o(\alpha) = \sum_i p_i + q + index(x) - 2N_{index(x)}(I)$ *with* $N_k(I) := \sharp\{j \leq k \mid j \in I\}$.
2. *each $\lambda^q_\alpha(t)$ satisfies*
   $$\lim_{t\to\infty} \lambda^q_\alpha(t)/t = 2o(\alpha)$$
3. *each $\omega^q_\alpha(t)$ localizes at x.*

Recall that the conjecturally empty set $\mathcal{A}^q(\infty)$ was defined by

$$\mathcal{A}^q(\infty) := \{\alpha \in \mathcal{A}^q \mid \lambda^q_\alpha(t)/t \text{ unbounded}\}.$$

A proof of Theorem 3.3 can probably be derived from [18] Theorem 1.1, or from [11], section 11.5 formula (11.36). A proof on the lines of the proof of Theorem 3.2 is suggested in Section 5 and will be detailed in a paper in preparation.

4. CW-COMPLEX STRUCTURE ASSOCIATED WITH (M, g, f) AND THE GEOMETRIC COMPLEX

For (M^n, g, f) with (M^n, g) a closed Riemannian and $f : M \to \mathbb{R}$ a Morse function denote by $\gamma_y(t)$ the trajectory of the vector field $X = -\operatorname{grad}_g f$ with $\gamma_y(0) = y$.

For $x \in Cr(f)$ the set $W^\pm_x := \{y \in M \mid \lim_{t\to\pm\infty} \gamma_y(t) = x\}$ is called the stable/unstable set of x and is a submanifold diffeomorphic to $\mathbb{R}^{n-k}/\mathbb{R}^k$ with k the index of x.

One says that the vector field X defined above is *Morse-Smale* if for any $x, y \in Cr(f)$ the unstable set W_x^- and the stable set W_y^+ are transversal which implies that $\mathcal{T}(x,y) = (W_x^- \cap W_y^+)/\mathbb{R}$ [4], the space of trajectories from x to y, is a manifold of dimension $\mathrm{ind}(x) - \mathrm{ind}(y) - 1$. For any string of critical points $x = y_0, y_1, \cdots, y_k$ with

$$\mathrm{ind}(y_0) > \mathrm{ind}(y_1) >, \cdots, > \mathrm{ind}(y_k)$$

consider the smooth manifold of dimension $\mathrm{ind}(y_0) - k$,

$$\mathcal{T}(y_0, y_1) \times \cdots \mathcal{T}(y_{k-1}, y_k) \times W_{y_k}^-,$$

and the smooth map

$$i_{y_0,y_1,\cdots,y_k} : \mathcal{T}(y_0, y_1) \times \cdots \times \mathcal{T}(y_{k-1}, y_k) \times W_{y_k}^- \to M,$$

defined by $i_{y_0,y_1,\cdots,y_k}(\gamma_1, \cdots, \gamma_k, y) := i_{y_k}(y)$, where $i_x : W_x^- \to M$ denotes the inclusion of W_x^- in M.

THEOREM 4.1.

1) *For any critical point $x \in Cr(h)$ the smooth manifold W_x^- has a canonical compactification $\hat{W}_x^-$ to a compact manifold with corners* [5] *and the inclusion i_x has a smooth extension $\hat{i}_x : \hat{W}_x^- \to M$ so that :*
(a) *the $k-$ boundary is $(\hat{W}_x^-)_k = \bigsqcup_{(x,y_1,\cdots,y_k)} \mathcal{T}(x, y_1) \times \cdots \times \mathcal{T}(y_{k-1}, y_k) \times W_{y_k}^-$,*
(b) *the restriction of $\hat{i}_x$ to $\mathcal{T}(x, y_1) \times \cdots \times \mathcal{T}(y_{k-1}, y_k) \times W_{y_k}^-$ is $i_{x,y_1\cdots,y_k}$.*

2) *$\hat{W}_x^-$ is homeomorphic to the compact disc of dimension index(x).*

A proof of this result is contained in [3] see also [8]. As a consequence one has

PROPOSITION 4.2. *If M is a closed manifold then the following holds true:*

1. *for any x, y critical points with $\mathrm{ind}(x) - \mathrm{ind}(y) = 1$ the set $\mathcal{T}(x, y)$ is finite,*

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[4] $\mathbb{R}$ acts freely by translation along the flow defined by $-\mathrm{grad}_g f$

[5] Recall that an $n-$dimensional manifold X with corners is a paracompact Hausdorff space equipped with a maximal smooth atlas with charts $\varphi : U \to \varphi(U) \subseteq \mathbb{R}_+^n$ with $\mathbb{R}_+^n = \{(x_1, x_2, \cdots x_n)|x_i \geq 0\}$. The collection of points of X which correspond (by some and then any chart) to points in $\mathbb{R}^n$ with exactly k coordinates equal to zero is a well defined subset of X, called the $k-$boundary, and denoted by X_k. It has a structure of a smooth $(n-k)-$dimensional manifold. The subset $\partial X = X_1 \cup X_2 \cup \cdots X_n$ is a closed subset which is a topological manifold of dimension $(n-1)$ and the pair $(X, \partial X)$ is a topological manifold with boundary ∂X. A compact smooth manifold with corners with interior diffeomorphic to the Euclidean space, will be called a compact smooth cell and is homeomorphic to the compact unit disc of the same dimension. When $\dim W_x^- = 3$ in order to establish this result one needs the recently proven Poincaré conjecture.

2. *the canonical embedding* $\iota_x : W_x^- \to M$ *extends to a smooth map* $\hat{\iota}_x : \hat{W}_x^- \to M$ *where* $\hat{W}_x^-$ *is a compact smooth manifold with corners (see Theorem 4.1 above) which makes* $M = \sqcup_{x \in Cr(f)} W_x^-$ *a CW complex with open cells* W_x^-, $x \in Cr(f)$,

3. *if for any* $x \in Cr(f)$ *one chooses an orientation* O_x *of* W_x^- *then to any* $\gamma \in \mathcal{T}(x, y)$, *with* $\operatorname{ind} x - \operatorname{ind} y = 1$, *a sign* $\epsilon(\gamma) = \pm 1$ *can by defined* [6] *as explained in [3]. The incidence* $I(x, y)$ *of the cell* W_x^- *and* W_y^- *is exactly* $\sum_{\gamma \in \mathcal{T}(x,y)} \epsilon(\gamma)$.

Note that changing f into $-f$ one obtains an other CW structure of M whose cels are the *stable sets* of the critical points x of f, i.e. of dimension $n - \operatorname{ind} x$ with $I^{(-f)}(y, x) = (-)^{q(n-1-q)} I^f(x, y)$, $q = \operatorname{ind}(x)$.

The geometric complex associated with $-\operatorname{grad}_g f$ **and integration maps.**

Define $C^k := Maps(Cr_k(f), \mathbb{R})$ with a base provided by the characteristic functions E_x, $x \in Cr_k(f)$, [7] and the linear maps $\partial^k : C^k \to C^{k+1}$ defined by

$$\partial^k(E_x) = \sum_{x \in Cr_k(f)} I(x, y) E_y,$$

$x \in Cr_{k+1}(f)$, $y \in Cr_k(f)$.

In view of Proposition 4.2 one has $\partial^{k+1} \cdot \partial^k = 0$. In view of Theorem 4.1 for $\omega \in \Omega^r(M)$ and $\operatorname{ind} x = r$ one has $\int_{W_x^-} \iota^*(\omega) < \infty$ and then one obtains the linear maps $Int^r : \Omega^r(M) \to C^r$.

In view of Stokes theorem and Theorem 4.1 one has $\partial^k \cdot Int^k = Int^{k+1} \cdot d^k$ which makes $Int^* : (\Omega^*(M), d^*) \to (C^*, \partial^*)$ a morphism of cochain complexes which by de-Rham theorem is a quasi-isomorphism. In the diagram below all arrows are quasi-isomorphisms (i.e. induce isomorphisms in cohomology).

$$\begin{tikzcd} (\Omega^*_{vs}(M)(t), d^*(t)) \arrow[r, "\subset"] \arrow[rrr, bend left, "Int^r_{vs}(t)"] & (\Omega^*(M), d^*(t)) \arrow[r, "\cdot e^{tf}"] \arrow[rr, bend right, "Int^r(t)"'] & (\Omega^*(M), d^*) \arrow[r, "Int^*"] & (C^*, \partial^*) \end{tikzcd}$$

For future needs one considers the scaling $\mathcal{S}^*(t)$ of the geometric complex, precisely the isomorphism $\mathcal{S}^*(t) : (C^*, \partial^*) \to (C^*, \partial^*(t))$ with $\mathcal{S}^q(t) : C^q = Maps(Cr_q(f), \mathbb{R}) \to C^q = Maps(Cr_q(f), \mathbb{R})$, defined by

$$\mathcal{S}^q(t)(E_x) := (\pi/t)^{(n-2q)/4} e^{-tq} E_x \tag{4.1}$$

[6] by comparing the orientation Q_x to the orientation defined by the tangent to γ plus the orientation O_y

[7] given by $E_x(z) = \delta_{x,z}, x, z \in Cr_k(f)$,

where E_x is the characteristic function of $x \in Cr_q(f)$ and $\partial^q(t) = \mathcal{S}^{q+1}(t) \cdot \partial^q \cdot (\mathcal{S}^q(t)^{-1}$. Clearly $(C^*, \partial^*(t))$ is an analytic family of cochain complexes in $t \in \mathbb{R}$.

5. SKETCH OF THE PROOF OF THEOREMS 3.2 AND 3.3

The proof of Theorems 3.2 and 3.3 is based on the "mathematics of harmonic oscillator", in our case of *multivariable harmonic oscillators* cf. Propositions 5.1 and 5.2 and on a criterion for detecting a *gap in the spectrum* of a nonnegative self adjoint operator in a Hilbert space H, cf. Lemma 5.3 below.

Recall from [14] that the *classical harmonic oscillator*, is the operator

$$-\frac{d^2}{dx^2} + x^2 : L^2(R) \to L^2(R) \tag{5.1}$$

which when considered as an unbounded self adjoint operator with domain $\mathcal{S}(R)$, the space of rapidly decaying function, has the pure spectrum with eigenvalues $\{2j+1, j \in \mathbb{Z}_{\geq 0}\}$ and corresponding eigenfunction $H_j(x)e^{-x^2}$ with $H_j(x)$ the $j-$th Hermite polynomial.

This permits to derive the eigenvalues and the eigenfunctions for

$$-\frac{d^2}{dx^2} + a + bx^2 : L^2(R) \to L^2(R). \tag{5.2}$$

We call the operators of the form

$$\Delta^q + A + b\sum_i x_i^2 \tag{5.3}$$

on $\Omega^q(\mathbb{R}^n) = \{\omega = \sum_I a_I(x_1, x_2, \cdots x_n)dx_I\}$ *multivariable harmonic oscillator.* Here $I = (j_1, j_2, \cdots j_q)$, $1 \leq j_1 < j_2, \cdots < j_q \leq n$, $dx_I = dx_{j_1} \wedge \cdots \wedge dx_{j_q}$ with

$$\Delta^q\omega = -\sum_I \sum_{1\leq i\leq n} \frac{\partial^2}{\partial x_i^2} a_I(x_1, \cdots x_n)dx_I, \quad A\omega = \sum_I \epsilon_I a_I(x_1, \cdots x_n)dx_I,$$
$$\epsilon_I \in \mathbb{R}, \quad b \in \mathbb{R}_{>0}.$$

As an unbounded self adjoint operator on $L^2(\mathcal{S}^q(R^n))$ with domain $\mathcal{S}^q(R^n) = \{\omega \in \Omega^q(R^n) \mid a_I(x_1, x_2, \cdots x_n) \in \mathcal{S}(\mathbb{R}^n)\}$ the operator (5.3) is globally elliptic in the sense of Shubin [19], and can be decomposed as tensor product of operators of the form (5.2) hence its eigenvalues and eigenforms can be calculated explicitly.

In this paper we consider the operator $\Delta^{q,k}(t)$ on $L_2(\mathcal{S}^q(R^n))$ of the form (5.3) with $\epsilon_I^{q,k} = (-n + 2k - 2q + 4\sharp\{j \in I \mid k+1 \leq j \leq q\})t$, $b = t^2$ and notice that this is exactly the Witten Laplacian for the function $f_k : \mathbb{R}^n \to \mathbb{R}$

$$f_k(x_1, \cdots x_n) = c - 1/2 \sum_{1\leq i\leq k} x_i^2 + 1/2 \sum_{k+1\leq i\leq n} x_i^2. \tag{5.4}$$

For $I = (j_1, j_2, \cdots j_q)$ with $1 \leq j_1 < j_2 < \cdots j_q \leq n$ and $P = (p_1, p_2, \cdots p_n)$ with $p_i \in \mathbb{Z}_{\geq 0}$ define

$$o^k(I, P) = \sum p_i + q + k - 2\sharp\{j \in I \mid 1 \leq j \leq k\}. \tag{5.5}$$

Since $\Delta^{q,k}(t)$ is of the form (5.3), one can calculate its eigenvalues and eigenforms using the eigenvalues and the eigenfunctions of (5.2) and obtain

PROPOSITION 5.1 (cf [6]). *The eigenvalues $\lambda_\beta^{q,k}(t)$ and their corresponding eigenforms $\omega_\beta^{q,k}(t)(x)$ abbreviated $\omega_\beta^{q,k}(t)$ of the globally elliptic operator $\Delta^{q,k}(t)$ are indexed by pairs $\beta = (I, P) \in \mathcal{I}^q \times (\mathbb{Z}_{\geq 0})^n$ with $o^k(I, P) \geq 0$ and are exactly*

1. $\lambda_\beta^{q,k}(t) = 2t \cdot o^k(\beta)$

2. $\omega_\beta^{q,k}(t) = H_{p_1}(\sqrt{t}x_1) \cdots H_{p_n}(\sqrt{t}x_n)e^{-t|x|^2/2}dx_I$.

In view of the above one has.

PROPOSITION 5.2. *For any q integer between 0 and n and $N \in \mathbb{Z}_{\geq 0}$*

1. *the collection of pairs (I, P) with $o^k(I, P) = N$, is finite,*

2. *if $N = 0$ then $k = q$, hence $\ker \Delta^q(t) = \ker \Delta^{q,q}(t)$, and there is only one pair (I_0, P_0), $I_0 = (1, 2, \cdots q)$ and $P_0 = (0, 0, \cdots, 0)$ s.t. $\lambda^q(t) := \lambda_{I_0,P_0}^{q,q} = 0$ with the corresponding eigenform*

$$\omega^q(t) := \omega_{I_0,P_0}^{q,q}(t) = (t/\pi)^{n/4} e^{-t\sum_i x_i^2/2} dx_1 \wedge \cdots \wedge dx_q.$$

The main criterion to recognize a gap in the spectrum is the following lemma whose proof can be found in [5] Lemma 1.2.

LEMMA 5.3. *Let $A : H \to H$ be a densely defined (not necessarily bounded) self adjoint nonnegative operator in a Hilbert space $(H, <,>)$ and a, b two real numbers so that $0 < a < b < \infty$. Suppose that there exist two closed subspaces H_1 and H_2 of H with $H_1 \cap H_2 = 0$ and $H_1 + H_2 = H$ such that*
(1) $< Ax_1, x_1 > \leq a||x_1||^2$ for any $x_1 \in H_1$,
(2) $< Ax_2, x_2 > \geq b||x_2||^2$ for any $x_2 \in H_2$.
Then $\mathrm{spect} A \bigcap (a, b) = \emptyset$.

Lemma 5.3 will be applied to $H = L_2(\Omega^q(M))$, $A = \Delta^q(t)$ for the proof of Theorem 3.2 and $A = 1/t \cdot \Delta^q(t), t > 0$ for the proof of Theorem 3.3 with H_1 described explicitly and $H_2 := H_1^{\perp}$.

For Theorem 3.2 the subspace H_1 will be of dimension $c_q = \sharp Cr_q(f)$. To construct H_1 we first construct for any $\eta > 0$ the form $\tilde{\omega}^{q,\eta}(t) \in \Omega^q_c(\mathbb{R}^n)$ with support in the ball $\{|x| \leq \eta\}$, $|x| = \sqrt{\sum_i x_i^2}$, which agrees with $\omega^q(t)$ given above on the ball $\{|x| \leq \eta/2\}$ and satisfies $< \tilde{\omega}^{q,\eta}(t), \tilde{\omega}^{q,\eta}(t) >= 1$[8] and place this form as a form $\overline{\omega}^q_y(t)$ on M in a neighborhood of each critical point $y \in Cr_k(f)$.

Choose a smooth function $\gamma_\eta(u)$, $\eta \in (0, \infty)$, $u \in \mathbb{R}$ s.t.

$$\gamma_\eta(u) = \begin{Bmatrix} 1 \text{ if } u \leq \eta/2 \\ 0 \text{ if } u > \eta \end{Bmatrix}. \tag{5.6}$$

and define first $\tilde{\gamma}_\eta : \mathbb{R}^n \to \mathbb{R}$ by $\tilde{\gamma}_\eta(x) = \gamma_\eta(|x|)$ and then $\tilde{\omega}^{q,\eta}(t) \in \Omega^q_c(\mathbb{R}^n)$ by

$$\tilde{\omega}^{q,\eta}(t) = (1/\beta_q(t)) \cdot \tilde{\gamma}_\eta \cdot \omega^q(t)$$

with

$$\beta_q(t) = (t/\pi)^{n/4}\left(\int_{\mathbb{R}^n} \gamma^2_\eta(|x|)e^{-t|x|^2} dx_1 \cdots dx_n\right)^{1/2}. \tag{5.7}$$

The following proposition can be obtained by elementary calculations in coordinates in view of the explicit formula of $\Delta^{q,k}(t)$, its eigenforms and of Propositions 5.1 and 5.2(cf. [7], Appendix 2)[9].

Proposition 5.4. *For a fixed $r \in \mathbb{Z}_{\geq 0}$ there exist positive constants C, C', C'', T_0, and ϵ_0 so that $t > T_0$ and $\epsilon < \epsilon_0$ imply*

(1) $|\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \Delta^{q,q}(t)\tilde{\omega}^{q,\epsilon}(t)(x)| \leq Ce^{-C't}$ *for any $x \in \mathbb{R}^n$ and multiindex $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq r$*

(2) $< \Delta^{q,k}(t)\tilde{\omega}^{q,\epsilon}(t)(x), \tilde{\omega}^{q,\epsilon}(t)(x) > \geq 2t|q-k|$

(3) *If $\omega \perp \tilde{\omega}^{q,\epsilon}(t)$ with respect to the scalar product $< .,. >$ then*

$$< \Delta^{q,q}\omega, \omega > \geq C''t||\omega||^2.$$

In order to place $\tilde{\omega}^{q,\eta}(t)$ on M one proceeds as follows.

By hypothesis, in a neighborhood U_y of each $y \in Cr_q(f)$, one chooses Morse charts $\varphi_y : U_y \to \mathbb{R}^n$ s.t. in these coordinates f has the form (5.4) and the metric g given by $g_{ij} = \delta_{ij}$. Choose $\epsilon > 0$ small enough s.t. the inverse image

[8] The scalar product $\langle\ ,\ \rangle$ on $\mathcal{S}^q(\mathbb{R}^n)$ is induced by the Euclidean metric on $\mathbb{R}^n$.

[9] A conceptual derivation of this proposition can be also obtained in view of Propositions 5.1 and 5.2

of the open discs of radius slightly larger than 2ϵ are disjoint neighborhoods of the critical points and consider the smooth forms $\overline{\omega}^q_y(t) \in \Omega^q(M)$ defined by

$$\overline{\omega}^q_y(t)|_{M\setminus\varphi_y^{-1}(D_{2\epsilon})} := 0, \quad \overline{\omega}^q_y(t)|_{\varphi_y^{-1}(D_{2\epsilon})} := \varphi_y^*(\tilde{\omega}^{q,\epsilon}(t)). \tag{5.8}$$

For any given t the forms $\overline{\omega}^q_y(t) \in \Omega^q(M),\ y \in Cr_q(f)$, are orthonormal. Indeed, if $y, z \in Cr_q(f),\ y \neq z,\ \ \overline{\omega}^q_y(t)$ and $\overline{\omega}^q_z(t)$ have disjoint support, hence are orthogonal, and because the support of $\overline{\omega}^q_y(t)$ is contained in charts as specified above satisfy $< \overline{\omega}^q_y(t), \overline{\omega}^q_y(t) >= 1$.

One considers the linear map $J^q(t) : C^q \to \Omega^q(M)$ defined by $J^q(t)(E_y) = \overline{\omega}^q_y, (t)$. Clearly $J_q(t)$ is an isometry, thus injective.

Proof of Theorem 3.2: (sketch). Take $H_1 := J^q(t)(C^q)$ and $H_2 = H_1^\perp$, let T_0, C, C', C'' be given by Proposition 5.4 and define

$$\begin{aligned} C_1 &:= \inf_{z\in M'} ||grad_g f(z)||, \\ C_2 &:= \sup_{z\in M} ||(L_{-grad_g f}(z) + \mathcal{L}_{-grad_g f}(z)|| \end{aligned} \tag{5.9}$$

for $M' = M \setminus \bigcup_{y\in Cr_q(\alpha)} \varphi_y^{-1}(D_\epsilon)$.

Here $||grad_g f(z)||$ resp. $||(L_{-grad_g f}(z) + \mathcal{L}_{-grad_g f}(z)||$ denote the norm of $grad_g f(z) \in T_z(M)$ resp. of the linear map $(L_{-grad_g f} + \mathcal{L}_{-grad_g f})(z) : \Lambda^q T^\sharp_z(M) \to \Lambda^q T^\sharp_z(M)$ with respect to the scalar product induced in $T^\sharp_z(M)$, the dual to $T_z(M)$. Note that if X is a vector field then $L_X + \mathcal{L}_X$ is a differential operator of order zero, hence an endomorphism of the bundle $\Lambda^q T^\sharp(M) \to M$, whose space of smooth sections is $\Omega^q(M)$.

We can use the constants $T_0, C, C', C'', C_1, C_2$ to construct C''' and ϵ_1 so that for $t > T_0$ and $\epsilon < \epsilon_1$, one has $< \Delta^q(t)\omega, \omega >\geq C_3 t < \omega, \omega >$ for any $\omega \in H_2$ (cf. [7], page 808-810).

One applies Lemma 5.3 for $a = Ce^{-C't}, b = C'''t$ with $t > T_0$. This concludes the first part of Theorem 3.2. and also establishes that c_q is larger or equal to the number of eigenvalues $\lambda^q_\alpha(t) \in [0, Ce^{-C't})$ for $t > T_0$.

In order to check that c_q is smaller or equal to this number let $Q^q(t)$ denote the orthogonal projection in H on the span of the eigenvectors corresponding to the eigenvalues smaller than 1. In view of the ellipticity of $\Delta^q(t)$ all these eigenvectors are smooth $q-$forms.

Let $\mathcal{I}^q(t) := Q^q(t) \cdot J^q(t)$. By decreasing ϵ and increasing T_0 one can insure (via rather technical estimates, cf. Proposition 5.4 in [7]) that $Q^q(t) \cdot J^q(t)$ is so closed to $J^q(t)$ that it is also injective for t very large; hence c_q is smaller or equal to the number of eigenvalues $\lambda^q_\alpha(t) \in [0, 1]$ which for t large enough equals to the number eigenvalues $\lambda^q_\alpha(t) \in [0, Ce^{-C't}]$.

About the proof of Theorem 3.3

A proof of Theorem 3.3 can be obtained in similar way using Lemma 5.3 applied to the same Hilbert space H, operator $A = (1/t)\cdot\Delta^q(t)$, and H_1 the finite dimensional vector space spanned by the forms $\overline{\omega}^q_\alpha(t)$, $\alpha = (x,\beta) = (x, I, P)$, with $x \in Cr(f)$ and $0 \le o(\alpha) = o^{\mathrm{ind}\, x}(\beta) \le N$ derived from $\tilde\omega^{q,\mathrm{ind}(x)}_{I,P}(t)$ in a similar manner the forms $\overline{\omega}^q_y(t)$ were derived from the forms $\tilde\omega^q(t)$. As before one choses $H_2 := H_1^{\perp}$ and one produces the constants $C_1, C_2, \epsilon_1, \epsilon_2, T$ such that for $t > T$ the hypotheses of Lemma 5.3 are satisfied for $t > T$ with $a = 2N + C_1\cdot t^{-\epsilon_1}, b = 2N + 2 - C_2\cdot t^{-\epsilon_2}$. One establishes first an analogue of Proposition 5.4 where the forms $\omega^{q,\epsilon}(t)$ are replaced by $\omega^{q,k;\epsilon}_{I,P}(t)$ with $o^k(I,P) \le 2N$ and inequalities (1), (2), (3) able to insure that for $t > T$, strictly below the line $x(t) = t, y(t) = t(2k+1)$ in the complex plane, there are exactly $\sharp\{(i,P) \mid o(I,p) = N\}$ eigenvalue branches and all other eigenvalue branches are strictly above this line. Moreover each eigenvalue branch (below this line) has $\lim_{t\to\infty}\lambda^q(t)/t$ convergent to $2k', k' \le N$. Details will be provided in a paper in preparation. So far we are unable to decide if there exist branches with $\lambda^q(t)/t$ unbounded.

6. VIRTUALLY SMALL SPECTRAL PACKAGE OF (M, g, f)

For each q denote by $\mathcal{A}^q_{vs}, \mathcal{A}^q_{la}, \mathcal{A}^q_{vs,0}, \mathcal{A}^q_{vs,+}$ the subsets of $\mathcal{A}^q$ defined by

1. $\mathcal{A}^q_{vs} := \{\alpha \in A^q \mid o(\alpha) = 0\}$,
2. $\mathcal{A}^q_{la} := \{\alpha \in A^q \mid o(\alpha) \neq 0\}$,
3. $\mathcal{A}^q_{vs,0} := \{\alpha \in A^q_{vs} \mid \lambda^q_\alpha(t) = 0\}$,
4. $\mathcal{A}^q_{vs,+} := \{\alpha \in A^q_{vs} \mid \lambda^q_\alpha(t) \neq 0\}$.

Note that $\mathcal{A}^q_{vs}$ is the same as the cluster $\mathcal{A}^q(0)$ considered in the introduction. Clearly $\mathcal{A}^q_{vs} = \mathcal{A}^q_{vs,0} \sqcup \mathcal{A}^q_{vs,+}$.

Let $\Omega^q(M)_{vs}(t)$, $\Omega^q(M)_{vs,0}(t)$, $\Omega^q(M)_{vs,+}(t)$ be the span of $\omega^q_\alpha(t)$ with $\alpha \in \mathcal{A}^q_{vs}, \mathcal{A}^q_{vs,0}, \mathcal{A}^q_{vs,+}$, respectively; let $\Omega^q(M)_{la}(t)$ be the orthogonal complement of $\Omega^q(M)_{vs}(t)$ in $\Omega^q(M)$. The first three are finite dimensional subspaces of $\Omega^q(M)$ of dimension c_q,$\beta_q(M)$, $c_q - \beta_q(M)$ and all four are equipped with a scalar product, the restriction of the scalar product on $\Omega^q(M)$. All these subspaces are preserved by $d^*(t)$ and left invariant by $\Delta^q(t)$ and in view of Hodge decomposition for $\Delta^q(t)$, $\Omega^q(M) = \Omega^q_{vs,0}(M)(t) \oplus \Omega^q_{vs,+}(M)(t) \oplus \Omega^q_{la}(M)(t)$.

In particular, one has

$$(\Omega^*_{vs}(M)(t), d^*(t)), (\Omega^*_{la}(M)(t), d^*(t)) \subset (\Omega^*(M), d^*(t))$$

and

$$(\Omega^*_{vs,0}(M)(t), 0),\ (\Omega^*_{vs,+}(M)(t), d^*(t)) \subset (\Omega^*(M)(t)_{vs}, d^*(t))$$

inclusions of sub complexes and each is an analytic family in t of cochain complexes. One has

$$\begin{aligned} (\Omega^*_{vs,0}(M)(t), 0) \oplus (\Omega^*_{vs,+}(M)(t), d^*(t)) =&(\Omega^*(M)(t)_{vs}, d^*(t)) \\ (\Omega^*_{vs}(M)(t), d^*(t)) \oplus (\Omega^*_{la}(M)(t), d^*(t)) =&(\Omega^*(M), d^*(t)). \end{aligned} \tag{6.1}$$

Consider the compositions

$$\xymatrix{ (\Omega^*_{vs}(M)(t), d^*(t)) \ar[r]^{\subset} \ar@/^2pc/[rr]^{Int^*_{vs}(t)} \ar@/^3pc/[rrr]^{L(t)^*} & (\Omega^*(M), d^*(t)) \ar[r]^{Int^*(t)} & (C^*, \partial^*) \ar[r]^{S^*(t)} & (C^*, \partial^*(t)) }$$

with $L^q(t) = S^q(t) \cdot Int^q_{vs}(t)$ and $S^q(t)$ given by (4.1). All these complexes are equipped with scalar product; on C^q one considers the scalar product which makes the base $E_x \in C^q$ orthonormal.

If $\varphi : (V, \langle,\rangle_V) \to (W, \langle,\rangle_W)$ is a linear map between two finite dimensional vector spaces with scalar products define $Vol(\varphi) := (\det(\varphi^\sharp \cdot \varphi))^{1/2}$ which is different from zero iff φ is injective. Define

$$a^q(t) := Vol(Int^q_{vs}(t))$$

analytic function in t which has holomorphic extension to a neighborhood of $\mathbb{R}$ in $\mathbb{C}$.

Theorem 6.1.
If the vector field $(-\operatorname{grad}_g f)$ is Morse-Smale and the metric g is flat near the critical points then there exists T s.t. for $t > T$ the following holds true.

1. *$Int^q_{vs}(t)$ is an isomorphism of cochain complexes,*
2. *there exists a family of isometries $R^q(t) : C^q \to \Omega^q(M)_{vs}(t)$ of finite dimensional vector spaces so that $L^q(t) \cdot R^q(t) = Id + O(1/t)$, hence $L^q(t)$ is an $O(1/t)-$isometry.*

Because of item (1) $a^q(t)$ is an analytic function with finitely many zeros and then $a(t) := \prod a^q(t)^{(-1)^q}$ is a priory a meromorphic function with finitely many zero and poles.

About the proof of Theorems 6.1

Item (1) follows from item (2) in view of the definition of $L^q(t)$. The isometry $R^q(t)$ is given by :

$$R^q(t) := \mathcal{I}^q(t) \cdot (\mathcal{I}^q(t)^\sharp \cdot \mathcal{I}^q(t))^{-1/2},\ \mathcal{I}^q(t) = Q^q(t) \cdot J^q(t) \tag{6.2}$$

where $\mathcal{I}^q(t)^\sharp$ denotes the adjoint of $\mathcal{I}^q(t)$. Note that $\mathcal{I}^q(t)$, for $t > T$, is a linear bijective map between f.d. vector spaces with scalar product.

The verification of item (2) involves a number of estimates and is quite technical. It is a particular case of Theorem 5.5 item 5. in [7]. The result can be also recovered from the work of Helffer an Sjöstrandt [16] but the proof in [7] is different.

The following result is due to Y. Lee and the present author, cf. [9].

Theorem 6.2.
If $-\mathrm{grad}_f$ is Morse-Smale and the metric g is flat near critical points then the following holds.

1. *The meromorphic function $a(t) = \prod (a^q(t))^{(-1)^q}$ has no zero and no poles and has a holomorphic extension to a neighborhood of $\mathbb{R}$ in $\mathbb{C}$.*

2. *If M^n is a closed odd dimensional manifold then*
$$\ln Tor(M) = 1/2 \sum_q (-1)^{q+1} q(\sum_{\alpha \in \mathcal{A}^q_{vs,+}} \ln \lambda^q_\alpha(0)) + \ln a(0) - \sum (-1)^i \ln V^i.$$

Proof. Observe that:

1. If $\varphi(t) : (V(t), \langle , \rangle_{V(t)}) \to (W(t), \langle , \rangle_{W(t)})$ is continuous/analytic family of isomorphisms between finite dimensional vector spaces equipped with scalar products[10] then the function $Vol(\varphi(t))$ is continuous/analytic in t.

2. For a cochain complex $\mathcal{C} = (C^*, d^*)$ of finite dimensional vector spaces equipped with scalar products
$$\mathcal{C}: \quad 0 \longrightarrow (C^0, \langle\ \rangle_0) \xrightarrow{d^0} (C^1, \langle\ \rangle_1) \xrightarrow{d^1} \cdots \qquad (C^n, \langle\ \rangle_n) \longrightarrow 0$$
one denotes by $\Delta^q_{\mathcal{C}} := \delta^{q+1} \cdot d^q + d^{q-1} \cdot \delta^q$, δ the adjoint of d, and by $\det' \Delta^q_C \neq 0$ be the product of nonzero eigenvalues of Δ^q_C. The product
$$T(\mathcal{C}) := \prod (\det' \Delta^q_C)^{(-1)^{q+1} q/2}$$
is referred to as the torsion of $\mathcal{C}$. For a continuous/analytic family of cochain complexes $\mathcal{C}(t) = (C^*(t), d^*(t))$ such that $\dim C^q(t)$ and $\dim H^q(\mathcal{C}(t))$ are constant in t for any q the function $T(\mathcal{C}(t))$ is continuous/analytic in t.
The verifications of (1) and (2) are straightforward from definitions.

[10] for example $V(t)$ resp.$W(t)$ appear as images in $\mathcal{V}$ resp.$\mathcal{W}$, of an analytic/continuous family of bounded projectors $P(t) : \mathcal{V} \to \mathcal{V}$ resp. $Q(t) : \mathcal{W} \to \mathcal{W}$ for $\mathcal{V}$ resp $\mathcal{W}$ topological vector spaces; this gives meaning to "analytic family".

3. Suppose that $\varphi : \mathcal{C}_1 \to \mathcal{C}_2$, is a morphism of cochain complexes of finite dimensional vector spaces with scalar products,. $\mathcal{C}_i = (C^*_i, d^*_i), i = 1, 2$, $\varphi = \{\varphi^q : C^q_1 \to C^q_2\}$. Suppose that for any q, φ^q is an isomorphism. Then φ induces the isomorphism $H^q(\varphi) : H^q(\mathcal{C}_1) \to H^q(\mathcal{C}_2)$ between vector spaces equipped with induced scalar product. Let

$$Vol(\varphi) := \prod (vol(\varphi^q))^{(-1)^q}$$

and

$$Vol(H(\varphi)) := \prod vol(H^q(\varphi))^{(-1)^q}.$$

As verified in [4] Proposition 2.5 one has

$$T(\mathcal{C}_2)/T(\mathcal{C}_1) = Vol(H(\varphi))/Vol(\varphi). \tag{6.3}$$

4. For a continuous/analytic family of isomorphisms $\varphi(t) : \mathcal{C}_1(t) \to \mathcal{C}_2(t)$, $t \in \mathbb{R}$, with $\dim C^q_1(t) = \dim C^q_2(t)$ and $\dim H^q(C_1(t)) = \dim H^q(C_2(t)$ constant in t, the real-valued functions $T(C_1(t))$, $T(C_1(t)$, $Vol(\varphi(t))$, $Vol(H(\varphi(t)))$ are nonzero and continuous/analytic.

We consider $\varphi(t) = Int^*(t) : (\Omega^*_{vs}(M)(t), d^*(t) \to (C^*, \partial^*)$ with $* = 0, 1, \cdots, \dim M$. In view of (4) the function

$$\frac{T(\Omega^*_{vs}(M)(t), d^*(t))}{T(C^*, \partial^*)} \cdot Vol(H(\varphi(t))$$

is a strictly positive analytic function and in view of (3) agrees with $a(t)$ for all t but the finite collection which might be a zero or a pole for $a(t)$. Hence the meromorphic function $a(t)$ has no zeros and no poles. This establishes item (1) in Theorem 6.2. Together with (6.3) it also implies

$$\frac{T(\Omega^*_{vs}(M)(t), d^*(t))}{a(t)} \cdot Vol(H(\varphi(t)) = T(C^*, \partial^*).$$

Evaluation at $t = 0$ combined with the observation that $\mathbb{T}or(M) = T(C^*, \partial^*)$ implies

$$\frac{T(\Omega^*_{vs}(M), d^*)}{a(0)} \cdot Vol(H(\varphi(0))) = \mathbb{T}or(M).$$

Taking "ln" one derives item (2) in Theorem 6.2. □

7. CONJECTURES, QUESTIONS, PROBLEMS

Conjecture 1. *The set $\mathcal{A}^q(\infty)$ is empty.*

Conjecture 2. *For a generic family of pairs (g, f) the positive eigenvalue - branches and therefore their corresponding eigenform - branches have multiplicity one .*

A stronger version of this conjecture asks for the same conclusion (multiplicity one for eigenvalue branches) for a generic set of smooth functions f given any metric g.

In view of [2], which implies that for generic continuous f, $H^r(M;\mathbb{R})$ in the presence of a scalar product has a canonical orthonormal base determined by f, a positive answer to Conjecture 1 implies that for f generic with this base completed by $\omega^q_\alpha(0), \alpha \in \mathcal{A}^q \setminus \mathcal{A}^q_{vs,0}$, one obtains a canonical orthonormal Hilbert space base for $L_2(\Omega^r(M))$. This can be used to reduce PDE problems involving differential forms to "manipulation of sequences ", as it is the case of the torus equipped with the flat metric (via Fourier series theory).

Conjecture 3. *All numbers $a^q \neq 0$.*

If Conjecture 3 holds true, not only the cohomology has a canonical realization by differential forms (the harmonic forms), as Hodge theorem in Riemannian geometry states, but the entire geometric complex (C^*, ∂^*) associated to (g, f), at least in the case the hypotheses in Theorem 6.1 are satisfied, can be canonically realized as a subcomplex $(\Omega^*_{vs}(M), d^*)$ of the de-Rham complex. This provides a substantial and consequential generalization of Hodge theorem.

Question 1. *We expect that for a fixed metric g (under mild hypotheses) the virtually small spectral package $\{\lambda^q_\alpha(0), \omega^q_\alpha(0), \alpha \in \mathcal{A}^q_{vs}\}$, depends on the Morse function f only up to homotopy by Morse functions.*

Such robustness conclusion is necessary in order to calculate the virtually small spectral package with arbitrary accuracy by effective numerical methods.

Question 2. *Can* **Observation** *in Section 1 be used to determine $\mathcal{A}^q_{vs}$ inside $\mathcal{A}^q$?*

Problem 1. *The works of Cheeger and Buser cf. [13] and [10] provide bounds for the first nonzero eigenvalue of Δ^0. One hopes to extend this to the virtually small spectral package and also derive bounds for the eigenvalues of the virtually small spectral package.*

The following two problems concern the spectral package of a Riemannian manifold (M, g).

Problem 2. *Find a spectral description of the numbers V^r.*

For V^n we have the famous *Weyl law* which evaluates the volume of M in terms of the growth of the number of eigenvalues of the Laplacian Δ^0 the same as of Δ^n.

Problem 3. *A result of K. Uhlenbeck [20] shows that for a generic Riemannian metric g on a connected closed Riemannian manifold the eigenvalues of the Laplacian Δ^0 and then of Δ^n are all simple. Then for a generic metric on an oriented 2−dimensional manifold the multiplicity of nonzero eigenvalues of $\Delta^q$is one for $q = 0, 2$ and 2 for $q = 1$. Clarify the multiplicity of nonzero eigenvalues of Δ^q for a generic metric on a closed manifold of arbitrary dimension.*

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Ohio State University
Department of Mathematics
Columbus, OH 43210, U.S.A.
burghele@math.ohio-state.edu