This paper is dedicated to Cabiria Andreian Cazacu,
the scientist, the teacher, and over all a person with a warm heart

NON-COMMUTATIVE HARMONIC SPACES

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We present in this paper a non-commutative formulation of the now classical theory of Harmonic Spaces [2]. This construction starts with a fixed real W*-algebra E ([1, Definition 4.4.1.1]) and it is obtained by associating to every element of the classical theory either an element of E or an element of $\tau_0(E)$ ([1, Definition 2.3.5.1]). We want to draw the attention to Example 28, where it is solved an open problem of [2] page 312.

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NOTATION AND TERMINOLOGY

Throughout this paper we shall use the notation and the terminology of [2] without indicating the place where they can be found in this book. Throughout this paper we shall use the notation and the terminology of [1], but in contrast to [2] we shall indicate by the first use the place where the corresponding information may be found. A central part in this paper will play a real W*-algebra E ([1, Definition 4.4.1.1, Theorem 4.4.2.21]). In order to facilitate the lecture we shall write the elements connected with E either in fracture writing or marked with E.

In the sequel we give a list of notation and terminology used in this paper.

1. $\mathbb{N}$ denotes the set of natural numbers $0 \notin \mathbb{N}$ and $\mathbb{R}$ the field of real numbers. We put
$$\mathbb{R}_+ := \{\, \alpha \in \mathbb{R} \mid \alpha \geq 0 \,\}.$$

2. Let F be a C*-algebra ([1, Definition 4.1.1.1]). We denote by $Pr\,F$ the set of orthogonal projection of F ([1, Definition 4.1.2.18]), by $Prm\,F$ the set of minimal elements of $Pr\,F \setminus \{0\}$, and by $\tau_0(F)$ the pure space of F ([1, Definition 2.3.5.1]).

3. Let H be a real Hilbert space ([1, Proposition 5.1.1.2]). We denote by
$$H \times H \longrightarrow \mathbb{R}, \quad (\xi,\eta) \longmapsto \langle\, \xi \,|\, \eta \,\rangle$$
its scalar product, by
$$H \longrightarrow \mathbb{R}, \quad \xi \longmapsto \|\xi\| := \sqrt{\langle\, \xi \,|\, \xi \,\rangle}$$
its associated norm, and by $\mathcal{L}(H)$ the real W*-algebra of operators on H with $\mathcal{L}^1(H)$ as predual ($\mathcal{L}^1(H) \subset \mathcal{L}(H)$) ([1, Theorem 5.1.3.13, Theorem 5.6.3.5] with E replaced there by $\mathbb{R}$), and put for all $\xi,\eta \in H$
$$\xi\,\langle\, \cdot \,|\, \eta \,\rangle := H \longrightarrow H, \quad \zeta \longmapsto \langle\, \zeta \,|\, \eta \,\rangle\, \xi.$$

4. We denote by $\ddot{E}$ the pre-dual of E and by 1_E its unit ([1, Corollary 4.4.4.4]).

5. We denote by $\ddot{E}_+$ the set of positive elements of $\ddot{E}$ ([1, Theorem 4.2.1.1, Proposition 1.7.1.2, Proposition 1.7.1.9]) and put
$$\bar{E} := E \cup \{\pm\infty\}, \quad \bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\} \quad \ddot{E}_0 := \ddot{E}_+ \setminus \{0\}.$$

6. We put for every set A, every $f : A \longrightarrow \bar{E}$, and every $\mathfrak{a} \in \ddot{E}_0$
$$f_{\mathfrak{a}} : A \longrightarrow \bar{\mathbb{R}}, \quad x \longmapsto \begin{cases} \langle\, \mathfrak{a} \,|\, f(x) \,\rangle & \text{if} \quad f(x) \in E \\ \pm\infty & \text{if} \quad f(x) = \pm\infty \end{cases}$$

1. PRELIMINARIES

Definition 1. Let K be a compact space, μ a measure on K, and $f : K \longrightarrow \bar{E}$. We say that $\boldsymbol{f}$ **is** $\boldsymbol{\mu}$**-integrable** if for every $\mathfrak{a} \in \ddot{E}_0$, $f_{\mathfrak{a}}$ is μ-integrable and if there is an $\mathfrak{x}$ in E with
$$\int f_{\mathfrak{a}}\, \mathrm{d}\mu = \langle\, \mathfrak{a} \,|\, \mathfrak{x} \,\rangle$$
for every $\mathfrak{a} \in \ddot{E}_0$. We write in this case
$$\int \boldsymbol{f}\, \mathbf{d}\boldsymbol{\mu} := \mathfrak{x}.$$

Definition 2. Let U be on open set of X and $u : U \longrightarrow \bar{E}$ a lower semicontinuous map. We say that u is $\boldsymbol{E}$**-hyperharmonic** if for every $V \in \mathfrak{V}_r(\mathcal{H})$ and $x \in V$, u is ω_x^V-integrable and
$$\int u\, \mathrm{d}\omega_x^V \leq u(x).$$
We put for every open set U of X,
$$\boldsymbol{\mathcal{H}^E(U)} := \left\{\, u : U \longrightarrow \bar{E} \;\middle|\; u \text{ is hyperharmonic} \,\right\}.$$
It is easy to see that $\mathcal{H}^E$ is a sheaf.

PROPOSITION 3. *Let U be an open set of X and $u : U \longrightarrow \bar{E}$ a lower semi-continuous map. Then u is E-hyperharmonic iff $u_{\mathfrak{a}}$ is hyperharmonic for every $\mathfrak{a} \in \ddot{E}_0$.*

Proof. The assertion follows immediately from the definition of an E-hyperharmonic function. □

Definition 4. Let U be an open set of X and $u \in \mathcal{H}^E(U)$. We say that u is an ***E*-potential** (resp. an ***E*-harmonic function**, resp. an ***E*-superharmonic function**) if $u_{\mathfrak{a}}$ is a potential (resp. a harmonic function, resp. a superharmonic function) for every $\mathfrak{a} \in \ddot{E}_0$. We denote for every open set U of X by $\boldsymbol{\mathcal{P}^E(U)}$, resp. $\boldsymbol{\mathcal{H}_+^E(U)}$, resp. $\boldsymbol{\mathcal{S}^E(U)}$ the set of E-potentials, resp. positive E-harmonic functions, resp. positive E-superharmonic functions. We put for every open set U of X

$$\left[\boldsymbol{\mathcal{S}^E(U)}\right] := \mathcal{S}^E(U) - \mathcal{S}^E(U).$$

Let $u \in \mathcal{S}^E(U)$ and let V be a relatively compact resolutive set of U. We denote by $\boldsymbol{u_V}$ the function of $\mathcal{S}^E(U)$ for which

$$(u_V)_{\mathfrak{a}} = (u_{\mathfrak{a}})_V$$

for every $\mathfrak{a} \in \ddot{E}_0$.

$U \mapsto \mathcal{S}^E(U)$ and $U \mapsto \mathcal{H}_+^E(U)$ are sheaves.

COROLLARY 5. *The set of positive E-superharmonic (*resp. *E-potentials,* resp. *positive E-harmonic functions) on X is a covex cone and a prevector lattice such that every upper bounded subset has a supremum.*

Proof. The assertion follows from Proposition 3 and [2], page 194. □

PROPOSITION 6. *If u is a positive E-superharmonic function on X then there are uniquely an E-potential p on X and a positive E-harmonic function u on X such that $u = p + h$.*

Proof. Let $\mathfrak{a} \in \ddot{E}_0$. By [2, Proposition Theorem 2.2.2] there are uniquely a potential $p^{\mathfrak{a}}$ on X and a positive harmonic function $h^{\mathfrak{a}}$ on X with $u_{\mathfrak{a}} = p^{\mathfrak{a}} + h^{\mathfrak{a}}$. By Proposition 3 (and Definition 4) there are an E-potential p on X and a positive E-harmonic function h on X such that $p_{\mathfrak{a}} = p^{\mathfrak{a}}$ and $h_{\mathfrak{a}} = h^{\mathfrak{a}}$. We get $u_{\mathfrak{a}} = p_{\mathfrak{a}} + h_{\mathfrak{a}}$. Since $\mathfrak{a}$ is arbitrary it follows $u = p + h$. The uniqueness is obvious. □

2. THE DIRICHLET PROBLEM

Throughout this section X denotes a $\mathfrak{P}$-harmonic space.

PROPOSITION 7. *Let U be an open set of X and let*

$$f : \partial(U) \longrightarrow \bar{E}.$$

We define $\bar{\mathcal{U}}_f^{U,E}$ and $\bar{H}_f^{U,E}$ as usually. Then for every $\mathfrak{a} \in \ddot{E}_0$,

$$u \in \bar{\mathcal{U}}_f^{U,E} \Leftrightarrow u_{\mathfrak{a}} \in \bar{\mathcal{U}}_{f_{\mathfrak{a}}}^{U}, \qquad (\bar{H}_f^{U,E})_{\mathfrak{a}} = \bar{H}_{f_{\mathfrak{a}}}^{U}.$$

COROLLARY 8. *If U is an open set of X and $f \in \mathcal{K}^E(\partial U)$ then*

$$\boldsymbol{H_f^{(U,E)}} := \bar{H}_f^{U,E} = \underline{H}_f^{U,E}.$$

Proof. The assertion follows from Proposition 7 and [2, Proposition 2.4.6]. □

With the notation of the preceding Corollary, let y be a regular point of $\partial(U)$. From this Corollary it follows that $H_f^{U,E}$ converges weakly to $f(y)$. In Proposition 9 we shall give a sufficient condition such that this convergence holds strongly.

PROPOSITION 9. *Let U be an open set of X, $y \in \partial U$, $\mathfrak{F}$ an ultrafilter on U converging to y, and u a barrier for $\mathfrak{F}$, i.e. $u \in \mathcal{S}^E(U)_+$ such that*

$$\lim_{x,\mathfrak{F}} u(x) = 0$$

and $u(x) \neq 0$ for every $x \in U$. Then for every $f \in \mathcal{K}^E(\partial U)$,

$$\lim_{x \to y} H_f^{U,E} = f(y).$$

Proof. By Corollary 8, $H_f^{U,E}$ is well-defined. Let $\varepsilon > 0$, let $V \in \mathfrak{V}_r$ with $y \in V$ and $\bar{V}$ compact, let $V' \in \mathfrak{V}_r$ with $\bar{V} \subset V'$, and let $h \in \mathcal{H}(V')$ with $h(y) = 1$ and such that $f \leq (f(y) + \varepsilon 1_E)h$ on $\bar{V}$. Put

$$g' : \partial V \longrightarrow \mathrm{I\!R}, \quad x \longmapsto \begin{cases} 1 & \text{if} \quad x \in U \\ 0 & \text{if} \quad x \notin U \end{cases},$$

and let $g \in \mathcal{K}(\partial V)_+$ such that $Supp\ g \subset U \cap \partial V$ and $H_{g'-g}^V(y) < \varepsilon$. Choose $\bar{u} \in \bar{\mathcal{U}}_{g'-g}^V$ and put

$$\gamma := \|f(u)\| \sup_{x \in \bar{V}} h(x) + \sup_{x \in U \cap V'} \left\| H_f^{U,E}(x) \right\|,$$

$$f' : \partial(U\cap V) \longrightarrow E\,, \quad x \longmapsto \begin{cases} f(x) & \text{if} \quad x\in\partial U \\ H_f^{U,E}(x) & \text{if} \quad x\in U\cap\partial V \end{cases}.$$

Put

$$\beta := \inf_{x\in Supp\, g} \|u(x)\|\,.$$

Then

$$(f(y)+\varepsilon 1_E)h+\beta u+\gamma\bar{u}1_E \in \bar{\mathcal{U}}_{f'}^{U\cap V,E},$$

so

$$H_{f'}^{U\cap V,E} \le (f(y)+\varepsilon 1_E)h+\beta u+\gamma\bar{u}$$

on $U\cap V$. Since $\bar{u}$ is arbitrary we get

$$H_{f'}^{U\cap V,E} \le (f(y)+\varepsilon 1_E)h+\beta u+\gamma H_{g'-g}^{V}$$

on $U\cap V$. Thus ([2, Proposition 2.4.4]),

$$\lim_{x,\mathfrak{F}} H_f^{U\cap V,E} = \lim_{x,\mathfrak{F}} H_{f'}^{U\cap V,E} \le (f(y)+\varepsilon 1_E)+\gamma\varepsilon 1_E.$$

Since ε is arbitrary

$$\lim_{x,\mathfrak{F}} H_f^{U\cap V,E} \le f(y).$$

It follows

$$-\lim_{x,\mathfrak{F}} H_f^{U\cap V,E} = \lim_{x,\mathfrak{F}} H_{-f}^{U\cap V,E} \le -f(y),$$

$$\lim_{x,\mathfrak{F}} H_f^{U\cap V,E} \ge f(y), \quad \lim_{x,\mathfrak{F}} H_f^{U\cap V,E} = f(y).$$

□

PROPOSITION 10. *Let U be an open set of X and let $p\in\mathcal{P}^E(U)$. Then for every $x\in U$ there is an $u\in\mathcal{S}^E(U)$* (called **the Evans function of *p* at *x***) *which is finite at x and such that $\alpha p(y)\le u(y)$ for every $\alpha\in]0,\infty[$ and every $y\in U$.*

Proof. Let $\mathcal{W}$ be the set of open, relatively compact, resolutive subsets of U and let $\mathcal{V}$ by the set of functions of the form $(\dots((p_{V_1})_{V_2})\dots{}_{V_n})$, where $V_1, V_2, \dots, V_n$ belong to $\mathcal{W}$. The limit (=infimum) of $\mathcal{V}(U)$ is an E-harmonic function. Being a minorant of an E-potential it is equal to 0. Thus there is a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{V}(U)$ the sum of which is finite at x. Put

$$u := \sum_{n\in\mathbb{N}} u_n.$$

Let $\alpha\in]0,\infty[$. If we take $n\in\mathbb{N}, n\ge\alpha$, then

$$u(y) \ge \sum_{m=1}^{m=n} u_m(y) \ge \alpha p(y)$$

for every $y\in U$. Thus $\alpha p(y)\le u(y)$ for every $\alpha\in]0,\infty[$ and every $y\in U$. □

3. $\tau_0(E)$ AND $\mathcal{S}^E$

PROPOSITION 11. *If E is finite-dimensional then $\tau_0(E)$ is a finite set and so it is compact.*

Proof. The assertion follows immediately from [1, Corollary 6.3.6.5]. □

PROPOSITION 12. *Let H be a real Hilbert space. The following are equivalent for all $\xi \in H$:*

a) $\|\xi\| = 1$.

b) $\xi \langle \cdot | \xi \rangle \in Prm\, \mathcal{L}(H)$.

c) $\xi \langle \cdot | \xi \rangle \in \tau_0(\mathcal{L}(H))$.

Proof. $b \Rightarrow a$, $c \Rightarrow a$, and $c) \Rightarrow \xi \langle \cdot | \xi \rangle \in Pr\, \mathcal{L}(H) \cup \mathcal{L}^1(H)$ are easy to see.

$b \Rightarrow c$ Let $(\alpha_\iota, p_\iota)_{\iota \in I}$ be a finite family in $\mathbb{R}_+ \times (Pr\, \mathcal{L}(H) \setminus \{0\})$ such that

$$\sum_{\iota \in I} \alpha_\iota\, p_\iota \leq \xi \langle \cdot | \xi \rangle .$$

Then for every $\iota \in I$, $p_\iota \leq \xi \langle \cdot | \xi \rangle$ and so by b), $p_\iota = \xi \langle \cdot | \xi \rangle$. It follows

$$\sum_{\iota \in I} \alpha_\iota \leq 1.$$

Let $z \in \mathcal{L}(H)$, $0 \leq z \leq \xi \langle \cdot | \xi \rangle$. By the above and [1, Theorem 4.4.1.8 l)], there is an $\alpha \in [0, 1]$ with $z = \alpha(\xi \langle \cdot | \xi \rangle)$ so $\xi \langle \cdot | \xi \rangle \in \tau_0(\mathcal{L}(H))$.

$c \Rightarrow b$ Let $p \in Pr\, \mathcal{L}(H) \setminus \{0\}$ with $p \leq \xi \langle \cdot | \xi \rangle$. By c), there is an $\alpha \in]0, 1]$ with $p = \alpha(\xi \langle \cdot | \xi \rangle)$. This imply $\alpha = 1$ and so $p = \xi \langle \cdot | \xi \rangle$. Thus $\xi \langle \cdot | \xi \rangle \in Prm\, \mathcal{L}(H)$. □

COROLLARY 13. *If H is a real Hilbert space then $\tau_0(\mathcal{L}(H))$ is compact iff H is finite-dimensional.*

PROPOSITION 14. *Let T be a hyperstonian space* (see [1], Definition 1.7.2.12), *T_0 the set of isolated points of T, and for every $t \in T$ put*

$$p_t : T \longrightarrow \mathbb{R}, \quad s \longmapsto \begin{cases} 1 & \text{if} \quad s = t \\ 0 & \text{if} \quad s \neq t \end{cases},$$

$$x'_t : \mathcal{C}(T) \longrightarrow \mathbb{R}, \quad f \longmapsto f(t).$$

Then

$$\{\, p_t \mid t \in T_0 \,\} = Prm\, \mathcal{C}(T), \quad \left\{\, x'_t \mid t \in T \,\right\} = \tau_0(\mathcal{C}(T)).$$

Thus $Prm\, \mathcal{C}(T)$ may by identified with T_0 and $\tau_0(\mathcal{C}(T))$ may by identified with T.

Proof. The inclusions from left to right are obvious. If $p \in Prm\ \mathcal{C}(T)$ (resp. $x' \in \tau_0(\mathcal{C}(T))$) then the set

$$\{\, s \in T \mid p(s) \neq 0 \,\} \quad (\text{resp.} \{\, s \in T \mid x'(s) \neq 0 \,\})$$

has to be an one-point set, say $\{t\}$, and then $p = p_t$ (resp. $x' = x'_t$). □

COROLLARY 15. *Let T be a set endowed with the discret topology and let $\bar{T}$ be its Stone-Čech compactification. Then $\mathcal{C}(\bar{T})$ is a W^*-algebra and* (by using the identifications of Proposition 14)

$$\tau_0(\mathcal{C}(\bar{T})) = \bar{T}, \quad Prm\ \mathcal{C}(\bar{T}) = T.$$

Proof. By [1, Example 1.7.2.15 b)], $\mathcal{C}(\bar{T})$ is a W*-algebra and the assertion follows from Proposition 14. □

PROPOSITION 16. *If $(E_\iota)_{\iota \in I}$ is a finite family of W^*-algebras* ([1, Example 2.3.1.4]) *then $\prod_{\iota \in I} E_\iota$ is a W^*-algebra and*

$$Prm \left(\prod_{\iota \in I} E_\iota \right) = \prod_{\iota \in I} (Prm\ E_\iota), \quad \tau_0 \left(\prod_{\iota \in I} E_\iota \right) = \prod_{\iota \in I} \tau_0(E_\iota).$$

Proof. The assertion follows from [1, Proposition 4.4.4.21]. □

PROPOSITION 17. *If $u \in \mathcal{S}^E(X)$ and $f \in \mathcal{C}(X)_+$ then* (with the notation of [2] pages 188-189)

$$\curlyvee_{(\delta \in \Delta_u)} \delta_*(f) = \curlywedge_{(\delta \in \Delta_u)} \delta^*(f).$$

We put

$$\boldsymbol{f \cdot u} = \curlyvee_{(\delta \in \Delta_u)} \delta_*(f) = \curlywedge_{(\delta \in \Delta_u)} \delta^*(f).$$

Proof. The proof of [2, Proposition 8.1.5] works in this case too. □

COROLLARY 18. *If $u \in \mathcal{S}^E(X)$ and $f \in \mathcal{C}_+(X)$ then $(f \cdot u)_{\mathfrak{a}} = f \cdot u_{\mathfrak{a}}$ for every $\mathfrak{a} \in \ddot{E}_0$.*

Proof. The assertion follows from the given definitions. □

COROLLARY 19. (We use in the sequel the notation of [2], pages 286-287.) *For all $x \in X$, $(V, f) \in I_x$, $u \in \mathcal{S}^E(X)$, and $\mathfrak{a} \in \ddot{E}_0$,*

$$(A_{(V,f)} u)_{\mathfrak{a}} = A_{(V,f)} u_{\mathfrak{a}}$$

Proof. By Corollary 18,

$$(A_{(V,f)} u)_{\mathfrak{a}} = \mu^V (f \cdot u)_{\mathfrak{a}} + (((1 - f) \cdot u))|_V)_{\mathfrak{a}} =$$
$$= \mu^V (f \cdot u_{\mathfrak{a}}) + ((1 - f) \cdot u_{\mathfrak{a}})|_V = A_{(V,f)} u_{\mathfrak{a}}.$$

□

PROPOSITION 20. *For $x \in X$ and $u \in \mathcal{S}^E(X)$,*

$$u(x) = \lim_{(V,f),\mathfrak{I}_x} A_{(V,f)} u(x).$$

Proof. The assertion follows from Corollary 19 and [2, Proposition 11.2.2 c)]. $\square$

COROLLARY 21. *Let $u, v \in \mathcal{S}^E(X)$. If $f \cdot u = f \cdot v$ on $X \setminus Supp f$ for every $f \in \mathcal{C}_+(X)$ then $u = v$.*

Proof. By Corollary 19 and Proposition 20, $u(x) = v(x)$. $\square$

Definition 22. Let $f \in \mathcal{C}_+(X_0)$ and let K be a compact subset of $X \setminus Supp\, f$. We set

$$\mathcal{V}^E(f, K) := \left\{ u \in \left[\mathcal{S}^E(X)\right] \mid \sup_{\mathfrak{a} \in \ddot{E}_0, \|\mathfrak{a}\| \leq 1} \sup_{x \in K} |(f \cdot u_{\mathfrak{a}})(x)| \right\}.$$

Finite intersections of sets of the form $\mathcal{V}^E(f, K)$ form a system of neighbourhoods of the origin of $\mathcal{S}^E(X) - \mathcal{S}^E(X)$ for a locally convex topology on $\mathcal{S}^E(X) - \mathcal{S}^E(X)$, which will be denoted by $\boldsymbol{\mathcal{T}^E}$.

By Corollary 21, $\mathcal{T}^E$ is Hausdorff.

PROPOSITION 23. *If U is an open set of X then every uniformly bounded set of $\mathcal{H}_+^E(U)$ is equicontinuous.*

Proof. The assertion follows from [2, Theorem 11.1.1. d)⇒ *b*)]. $\square$

PROPOSITION 24. *On a bounded set of $\left[\mathcal{S}^E(X)\right]$ the uniform structure induced by the topology $\mathcal{T}^E$ coincides with the uniform structure induced by the weak topology of $\mathcal{T}^E$.*

Proof. The proof of [2, Proposition 11.2.9] works in this case too. $\square$

PROPOSITION 25. *$\mathcal{S}^E(X)$ is a complete convex cone of $\left[\mathcal{S}^E(X)\right]$.*

Proof. The very long proof can be done similarly as in [2, Corollary 11.2.2]. $\square$

COROLLARY 26. *Every bounded set of $\mathcal{S}^E(X)$ is relatively compact. Hence the closed convex hull of a compact set of $\mathcal{S}^E(X)$ is a compact set of $\mathcal{S}^E(X)$.*

Proof. The assertion follows from Proposition 25 and Proposition 24. $\square$

Example 27. Denote by X the harmonic space obtained by endowing $[0,1[$ with the harmonic sheaf of decreasing lower bounded lower semi-continuous functions. For all $a,b \in [0,1]$, $a<b$, the open set $]a,b[$ is resolutive and $\omega_x^{]a,b[}$ is equal to the Dirac measure at the point b for all $x \in]a,b[$. It follows that the harmonic functions on X are constant and a positive superharmonic function u on X is a potential iff

$$\lim_{x\to 1} u(1) = 0.$$

Thus X is a $\mathfrak{P}$-space, P_X is a one-point set, and R_X is empty. In particular R_X is not dense in M_X.

Since there are many examples of $\mathfrak{P}$-harmonic spaces X where R_X is dense in M_X the above example answers the open question of [2], page 312.

4. THE MARTIN SPACE

Throughout this section X denotes a $\mathfrak{P}$-harmonic space.

We endow $\left[\mathcal{S}^E(X)\right]$ with the topology $\mathcal{T}^E$. Then $\mathcal{S}^E(X)$ is a closed convex cone of $\left[\mathcal{S}^E(X)\right]$ with vertex 0 such that $\mathcal{S}^E(X) \cap \mathcal{S}^E(X) = \{0\}$. If $\mathcal{A}$ is a bounded set of $\mathcal{S}^E(X)$ then it is relatively compact (Corollary 26) and if we endow $\mathcal{A}$ with the topology $\mathcal{T}^E$ or with the weak topology of $\mathcal{T}^E$ we get the same measure on $\mathcal{A}$ (Proposition 24).

Definition 28. We put

$$\boldsymbol{M_X^E} := M_X \times \tau_0(E), \quad \boldsymbol{R_X^E} := R_X \times \tau_0(E), \quad \boldsymbol{P_X^E} := P_X \times \tau_0(E).$$

We have

$$M_X^E = R_X^E \cup P_X^E, \quad R_X^E \cap P_X^E = \emptyset.$$

Definition 29. Let A be a subset of M_X^E. An **$\boldsymbol{E}$-Riesz-Martin kernel on $\boldsymbol{A}$** is a function

$$k : X \times A \longrightarrow [0,\infty]$$

such that for every $\mathfrak{a} \in \ddot{E}_0$ the map

$$k_{\mathfrak{a}} : X \times A_{\mathfrak{a}} \longrightarrow [0,\infty]$$

is a Riesz-Martin kernel on X.

THEOREM 30. *Assume* $\tau_0(E)$ *compact* (many examples are given in the preceding section), *let A be a subset of M_X^E, and let k be an E-Riesz-Martin kernel on A. For every $\mathfrak{a} \in \ddot{E}_0$ we denote by $k_{\mathfrak{a}}$ the Riesz-Martin kernel on X,*

$$k : X \times A_{\mathfrak{a}} \longrightarrow [0,\infty].$$

Then the restriction of k to the set

$$\{\, (x,\xi) \in X \times A \mid x \neq \pi'(\xi) \,\}$$

is continuous iff for every $\mathfrak{a} \in \ddot{E}_0$, $k_{\mathfrak{a}}$ is a regular Riesz-Martin kernel on X.

Proof. Since $\tau_0(E)$ is compact the set $\{\, k_{\mathfrak{a}} \mid \mathfrak{a} \in \tau_0(E) \,\}$ is equally upper bounded. The last assertion is equivalent to the assertion that for every $\mathfrak{a} \in \ddot{E}_0$ the restriction of $k_{\mathfrak{a}}$ to the set

$$\{\, (x,\xi) \in X \times A \mid x \neq \pi'(\xi) \,\}$$

is continuous and this is obviously to the assertion that the restriction of k to the set

$$\{\, (x,\xi) \in X \times A \mid x \neq \pi'(\xi) \,\}$$

is continuous. $\square$

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