Dedicated to the memory of Cabiria Andreian Cazacu,
outstanding mathematician and brilliant person

COEFFICIENTS OF QUASICONFORMALITY OF GEODESIC RINGS

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We consider classes of homeomorphisms of smooth Riemannian manifolds depending on real parameters and define the mean coefficients of quasiconformality. The lower bounds for these coefficients and the corresponding extremal mappings are presented. The monotonicity of the above mean coefficients by real parameters allows to calculate the classical quasiconformality coefficients in $\mathbb{R}^n$.

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1. INTRODUCTION

The classical Riemann mapping theorem states that any simply connected domain in the complex plane $\mathbb{C}$ with more than two boundary points is conformally equivalent to the unit disk. By Liouville's theorem [17, p. 22, Theorem 4.1], for dimensions $n \geq 3$, a smooth enough conformal mapping $f : D \to D'$ where $D, D' \subset \mathbb{R}^n$ is of the form $f = g|D$ for some Möbius transformation g. Thus the only domains conformally equivalent to the unit ball are the balls and half spaces of $\mathbb{R}^n$, $n \geq 3$.

The idea to study conformal invariants defined by extremal considerations goes back to the classical paper by Ahlfors-Beurling [1], where the authors determine conformal moduli $M(\Gamma)$ of the families Γ of curves (in fact, the extremal length of Γ, $\lambda(\Gamma) = M^{-1}(\Gamma)$) and show the invariance of conformal moduli under conformal mappings in $\mathbb{C}$. In [5], the above work was generalized to higher dimensions and to the moduli of the families of k-dimensional surfaces. To survey these and other close results, we refer to [3].

The (inner and outer) coefficients of quasiconformality of domains in $\mathbb{R}^3$ were introduced by Gehring and Väisälä [8] as the suprema of ratios between the moduli of arc families Γ^* and Γ in the image and the inverse image. Then the corresponding coefficients for a domain $D \subset \mathbb{R}^3$ are defined as the infima over all homeomorphisms of D onto the unit ball $\mathbb{B}^3$. The existence of the extremal mappings (for which the corresponding infimum is attained) follows from the lower semicontinuity of distortion coefficients.

The next essential step in studying extremal mappings in higher dimensions was done by Cabiria Andreian Cazacu in [2]. Here the conformal moduli of arc (curve) families were extended to the families of k-dimensional surfaces in $\mathbb{R}^n$, $k = 1, \ldots, n-1$.

In [11] the above ideas and techniques were applied to the class of mappings quasiconformal in the mean and spherical rings. Chapter 12 of the monograph [13] presenting the results of the second author of this paper implies the further generalization of quasiconformality to the class of homeomorphisms with finite mean dilatations; see also [9].

In this paper we extend the above results to the Riemannian manifolds and geodesic rings. The monotonicity of the mean coefficients by real parameters and some illustrating examples are presented.

2. RIEMANNIAN MANIFOLDS AND RELATED NOTIONS

2.1. **A Riemannian manifold** $(\mathbb{M}^n, g)$ is defined as a smooth manifold endowed with a Riemannian metric, i.e., a positive definite symmetric tensor field $g = g_{ij}(x)$ defined on the coordinate charts and obeying the transition rule

$$'g_{ij}(x) = g_{kl}(y(x)) \frac{\partial y^k}{\partial x^i} \frac{\partial y^l}{\partial x^j},$$

where, as usual, $k, l = 1, \ldots, n$ are the so-called dummy indices over which the summation is performed. In what follows, $g_{ij}(x)$ are assumed to be smooth. Note that $\det g_{ij} > 0$, because g_{ij} is positive definite; see, e.g. [6].

The length element is determined by the invariant differential form

$$ds^2 = g_{ij} dx^i dx^j := \sum_{i,j=1}^{n} g_{ij} dx^i dx^j = (dx^1 \ldots dx^n) \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix} \begin{pmatrix} dx^1 \\ \vdots \\ dx^n \end{pmatrix},$$

where g_{ij} denotes the metric tensor and x^i are the local coordinates. Accordingly if $\gamma : [a, b] \to \mathbb{M}^n$ is a piecewise smooth curve and $x(t)$ is its parametric specification in the local coordinates, then the length of γ is calculated by

$$s_\gamma = \int_a^b \sqrt{g_{ij}(x(t)) \, \frac{dx^i}{dt} \frac{dx^j}{dt}} \, dt.$$

The geodesic distance $d(p_1, p_2)$ between two points p_1 and p_2 is defined as an infimum of lengths of all piecewise smooth curves which join p_1 and p_2 in $(\mathbb{M}^n, g)$. Any such curve connecting p_1 and p_2 on which this infimum is attained is called **geodesic**.

Recall also that **the volume element** on $(\mathbb{M}^n, g)$ is determined by an invariant form $dv = \sqrt{det g_{ij}}\, dx^1...dx^n$, and **the area element** of a smooth surface H on $(\mathbb{M}^n, g)$ is defined by an invariant form $d\mathcal{A} = \sqrt{det\ g^*_{\alpha\beta}}\, du_1...du_{n-1}$, where $g^*_{\alpha\beta}$ is a Riemannian metric on H induced by the initial Riemannian metric g_{ij} according to the rule

$$g^*_{\alpha\beta}(u) = g_{ij}(x(u)) \cdot \frac{\partial x^i}{\partial u^\alpha} \cdot \frac{\partial x^j}{\partial u^\beta}.$$

Here $x(u)$ is a smooth parametrization of the surface H for which $\nabla_u x$ vanishes nowhere. Thus, the metric tensor g on a Riemannian manifold generates the corresponding metric tensor g^* on any regular surface; see, e.g. [16, Sect. 88].

Let ω be an open set in $\overline{\mathbb{R}^k}$ or, more generally, a k-dimensional manifold, where $k = 1, \dots, n-1$. Then, by the **k-dimensional surface** S on a Riemannian manifold $(\mathbb{M}^n, g)$, we call any continuous mapping $S : \omega \to \mathbb{M}^n$. The surfaces in $\mathbb{M}^n$ with dimension $k = n-1$ are usually called **hypersurfaces**.

Now we recall some fundamental facts following Lemma 5.10 and Corollary 6.11 in [12].

For any point of a Riemannian manifold, there exist its neighborhoods and the corresponding local coordinates in these neighborhoods for which the geodesic spheres centered at the indicated point are associated with Euclidean spheres of the same radii centered at the origin and a bundle of geodesic curves originating from this point is associated with the bundle of rays starting from the origin. Such neighborhoods and coordinates are called **normal**.

In the normal coordinates, the geodesic spheres have a natural smooth parametrization by the directional cosines of the corresponding rays passing from the origin. Moreover, the metric tensor in these coordinates coincides with the identity matrix at the origin; cf. [12, Proposition 5.11].

From now on, $n \geq 2$, D and D_* are two domains defined on smooth Riemannian manifolds $(\mathbb{M}^n, g)$ and $(\mathbb{M}^n_*, g^*)$ with geodesic distances d and d_*, respectively. Any Euclidean ball $\{x : r(x) < \varepsilon\}$ contained in a normal neighborhood U is a **geodesic ball** in $\mathbb{M}^n$; [12, Prop. 5.11]. Similarly to above we say that $A(x_0, r_1, r_2) = \{x \in \mathbb{M}^n : r_1 < d(x, x_0) < r_2\}$ and its boundaries $S(x_0, r_i) = \{x \in \mathbb{M}^n : d(x, x_0) = r_i\}$, $i = 1, 2$, which lie in a normal neighborhood $U(x_0)$, are a **geodesic ring** and **geodesic spheres**. Later on we assume that $A = A(x_0, r_1, r_2)$, $S(x_0, r_i)$ and the closed ball $\overline{B(x_0, \varepsilon)}$ lie in $U(x_0)$.

2.2 Recall the following definitions. Let U be an open set in $\mathbb{R}^n$ and $Q = \{x \in \mathbb{R}^n : a_i < x_i < b_i, i = 1, \dots, n\}$ be an open n-dimensional interval. A mapping $f : Q \to \mathbb{R}^n$ belongs to the **class ACL** (**absolutely continuous on lines**), if f is absolutely continuous on almost all linear segments in Q parallel to the coordinate axes. We say that the mapping $f : U \to \mathbb{R}^n$ belongs to the class ACL in U, when the restriction $f|_Q$ belongs to the class ACL for each interval $Q, \overline{Q} \subset U$. Note that, if $f \in$ ACL, then f has the first partial derivatives almost everywhere (a.e.). Due to [20], we say that $f \in$ **ACL$^{\mathbf{p}}$**, $p \geq 1$, if f is locally absolutely continuous on almost all segments parallel to the coordinate axes and its first partial derivatives are locally p-integrable in the local coordinates. Note that for a homeomorphism $f \in ACL^p$ is equivalent that f belongs to the Sobolev spaces $W^{1,p}_{\text{loc}}$; see e.g. [18, Prop 1.2].

By the Lebesgue theorem on differentiation of nonnegative semiadditive locally finite set functions **the generalized Jacobian** is well defined by

$$J(x, f) = \lim_{r \to 0} \frac{v_*(f(B(x, r)))}{v(B(x, r))} \qquad \text{a.e.},$$

cf. [15, III.2.4].

The quantity $K_O(x, f)$ is called **the outer dilatation** on $(\mathbb{M}^n, g)$, $n \geq 2$, and defined by

$$K_O(x, f) = \begin{cases} \frac{L^n(x,f)}{J(x,f)} & \text{if } J(x, f) \neq 0, \\ 1 & \text{if } L(x, f) = 0, \\ \infty & \text{otherwise}, \end{cases} \tag{1}$$

where

$$L(x, f) := \limsup_{y \to x} \frac{d_*(f(x), f(y))}{d(x, y)}. \tag{2}$$

Similarly $K_I(x, f)$ means **the inner dilatation** on $(\mathbb{M}^n, g)$, $n \geq 2$, and is determined by

$$K_I(x, f) = \begin{cases} \frac{J(x,f)}{l^n(x,f)} & \text{if } J(x, f) \neq 0, \\ 1 & \text{if } l(x, f) = 0, \\ \infty & \text{otherwise}, \end{cases} \tag{3}$$

where

$$l(x, f) := \liminf_{y \to x} \frac{d_*(f(x), f(y))}{d(x, y)}.$$

Recall also that $\|f'(x)\|$ denotes the matrix norm in $\mathbb{R}^n$ of the Jacobian matrix $f'(x)$ of mapping f at $x \in D$, $\|f'(x)\| = \sup\limits_{h \in \mathbb{R}^n, |h|=1} |f'(x) \cdot h|$, $J_f(x) = \det f'(x)$ is the Jacobian of f, and $K_O(x) = \|f'(x)\|^n / |J_f(x)|$, $K_I(x) = |J_f(x)| / l(f'(x))^n$ are the outer and inner dilatations of f at x in $\mathbb{R}^n$.

Passing to the local coordinates one can conclude that at the points of differentiability of f the definition (2) of $L(x,f)$ agrees with the matrix norm $\|f'(x)\|$ in $\mathbb{R}^n$. The same conclusion holds for the outer dilatations $K_O(x,f)$ and $K_O(x)$, for the inner dilatations $K_I(x,f)$ and $K_I(x)$, and for the generalized Jacobian $J(x,f)$ and $J_f(x)$ in $\mathbb{R}^n$ as well. Note also that each quantity $K_O(x)$ and $K_I(x)$ remains an invariant with respect to changes of the local coordinates. Thus, taking into account their presentations in the normal coordinates both dilatations $K_O(x,f)$ and $K_I(x,f)$ can be calculated a.e. in terms of $K_O(x)$ and $K_I(x)$, respectively, in any local coordinates for mappings mentioned above. The Euclidean distance and the Riemannian (geodesic) distance are equivalent in the local coordinates; see [12, Lemma 6.2].

Recall the **geometric interpretation of dilations** following [20]. Suppose that $h:\ \mathbb{R}^n \to \mathbb{R}^n$ is a linear bijection. The image of the unit ball B^n under h is an ellipsoid $E(h)$. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the semi-axes of $E(h)$. Then

$$L(h) = \lambda_1, \quad l(h) = \lambda_n, \quad |\det h| = \lambda_1 \cdot \ldots \cdot \lambda_n,$$

and we can also write

$$K_I(h) = \frac{\lambda_1 \cdot \ldots \cdot \lambda_{n-1}}{\lambda_n^{n-1}}, \quad K_O(h) = \frac{\lambda_1^{n-1}}{\lambda_2 \cdot \ldots \cdot \lambda_n}.$$

Replacing n by a real parameter $\alpha \geq 1$ in (1) and (3), we consider the so-called α-inner and α-outer dilatations $K_{I,\alpha}(x,f)$ and $K_{O,\alpha}(x,f)$, respectively. At a regular point x (of nondegenerate differentiability of f)

$$K_{O,\alpha}(x,f) = \frac{L^\alpha(x,f)}{J(x,f)}, \qquad K_{I,\alpha}(x,f) = \frac{J(x,f)}{l^\alpha(x,f)}.$$

Recall the following definition. Let a mapping $f:\ D \to D_*$ be differentiable at a point $x \in D$. Then there exists a linear mapping $f'(x):\ \mathbb{R}^n \to \mathbb{R}^n$ called **the (strong) derivative of the mapping** f at x such that

$$f(x+h) = f(x) + f'(x)h + \omega(x,h)|h|,$$

where $\omega(x,h) \to 0$ as $h \to 0$.

2.3. We say that a Borel function $\rho : \mathbb{M}^n \to [0,\infty]$ is **admissible** for a family Γ of k-dimensional surfaces S in $\mathbb{M}^n$, $k = 1, \ldots, n-1$, and write $\rho \in \operatorname{adm}\Gamma$ if

$$\int\limits_S \rho^k \, d\mathcal{A} \geq 1 \quad \forall\, S \in \Gamma.$$

Here $d\mathcal{A}$ denotes the k-dimensional Hausdorff measure H^k on the manifold $\mathbb{M}^n$, and the integral over the surface S is understanding in the following sense:

$$\int\limits_S \rho\, d\mathcal{A} := \int\limits_{\mathbb{M}^n} \rho(y) N(S,y)\, dH^k,$$

where $N(S,y)$ means the multiplicity of the covering of y by S; cf. [5].

Given $p \in (0,\infty)$, the **p-modulus** of the family Γ is defined as

$$M_p(\Gamma) := \inf_{\rho\in \operatorname{adm}\Gamma} \int\limits_{\mathbb{M}^n} \rho^p\ dv.$$

In the case $p = n$, this quantity is called the conformal modulus and the subscript n is usually omitted, i.e. $M(\Gamma)$.

3. INTEGRAL DILATATION COEFFICIENTS

Consider homeomorphisms $f : D \to D_*$ and rings A and $A_* = f(A)$ in D and D_*, respectively. For fixed real numbers $\alpha, \beta, \gamma, \delta$ such that

$$1 \le \alpha < \beta < \infty \quad \text{and} \quad 1 \le \gamma < \delta < \infty,$$

and for the mapping f we define the **inner** and the **outer mean dilatations** $I_{\alpha,\beta}(f)$ and $O_{\gamma,\delta}(f)$ by

$$(4)\qquad I_{\alpha,\beta}(f) = \left(\sup \frac{M_\alpha^\beta(\Gamma^*)}{M_\beta^\alpha(\Gamma)}\right)^{\frac{1}{\beta-\alpha}}, \quad O_{\gamma,\delta}(f) = \left(\sup \frac{M_\gamma^\delta(\Gamma)}{M_\delta^\gamma(\Gamma^*)}\right)^{\frac{1}{\delta-\gamma}},$$

where the suprema are taken over all families Γ of surfaces S in $\mathbb{M}^n$, located in A such that the numerator and denominator in each above fraction cannot be equal to 0 or ∞ simultaneously. Here $\Gamma^* = f(\Gamma)$. Obviously,

$$(5)\qquad I_{\alpha,\beta}(f^{-1}) = O_{\alpha,\beta}(f), \quad O_{\alpha,\beta}(f^{-1}) = I_{\alpha,\beta}(f).$$

For a homeomorphism f which is differentiable almost everywhere (a.e.) in D and for the given real numbers α, β such that $1 \le \alpha < \beta < \infty$, we introduce the integrals

$$(6)\quad HI_{\alpha,\beta}(f) = \int\limits_D K_{I,\alpha}^{\frac{\beta}{\beta-\alpha}}(f'(x))\, dv(x), \quad HO_{\alpha,\beta}(f) = \int\limits_D K_{O,\beta}^{\frac{\alpha}{\beta-\alpha}}(f'(x))\, dv(x).$$

We call these values the **inner** and the **outer integral dilatations** of the mapping f in domain D, respectively.

Now we define for the given fixed real numbers $\alpha, \beta, \gamma, \delta$ such that $1 \le \alpha < \beta < \infty$, $1 \le \gamma < \delta < \infty$, the **class** $\mathcal{F}(D, D_*)$ of homeomorphisms $f :\ D \to D_*$ which satisfy:

(i) $f \in W^{1,p}_{\rm loc}(D)$ with some $p > n-1$ and $J(x,f) > 0$ a.e. in D,
(ii) $f^{-1} \in W^{1,q}_{\rm loc}(D_*)$ with some $q > n-1$ and $J(y,f^{-1}) > 0$ a.e. in D_*,
(iii) the inner and outer integral dilatations $HI_{\alpha,\beta}(f)$ and $HO_{\gamma,\delta}(f)$ are finite.

Remark 1. It follows that for any mapping $f \in \mathcal{F}(D,D_*)$ we have:

$$HI_{\alpha,\beta}(f^{-1}) = HO_{\alpha,\beta}(f), \qquad HO_{\alpha,\beta}(f^{-1}) = HI_{\alpha,\beta}(f).$$

Remark 2. Note that the conditions (i)–(ii) ensure for any $f \in \mathcal{F}(D,D_*)$ and its inverse f^{-1} to be differentiable a.e. in D and D_*, respectively; see, e.g. [10, Corol. 2.25]. Moreover, any such f possesses the Lusin (N) and (N^{-1})-properties in the normal coordinates, namely f preserves any sets of zero measure in the image and preimage. More precisely, $J(x,f) > 0$ a.e. guarantees the Lusin (N^{-1})-property, and, correspondingly, $J(y,f^{-1}) > 0$ a.e. provides the Lusin (N)-property; cf. [14].

Remark 3. For a restricted range of parameters in the definition of $\mathcal{F}(D,D_*)$ the differentiability and the Lusin (N) and (N^{-1})-properties can be reached under a weaker condition than (i)–(ii); cf. [4].

PROPOSITION 1. *Any mapping $f : D \to D_*$ from the class $\mathcal{F}(D,D_*)$ is of finite distortion. The same is true for its inverse f^{-1}.*

This result follows from [10, Theorem 5.8] since both f and f^{-1} belong to $W^{1,n-1}_{\rm loc}$.

PROPOSITION 2. *Assume that $f : D \to D_*$ belongs to the Sobolev class $W^{1,n-1}_{\rm loc}(D)$ and $HI_{\alpha,\beta}(f) < \infty$ with $n-1 < \alpha \le n$. Then f is differentiable a.e. in D.*

The assertion of this statement is a conclusion from [19, Theorem 5.1]. To this end we have to show that $K_{I,\alpha}(x,f)$ is locally integrable under $n-1 < \alpha \le n$. The latter follows directly by applying the Hölder inequality in the local coordinates,

$$\int\limits_E K_{I,\alpha}(x,f)\,dv(x) \le \left(\int\limits_E K_{I,\alpha}^{\frac{\beta}{\beta-\alpha}}(x,f)\,dv(x)\right)^{\frac{\beta-\alpha}{\beta}} (v(E))^{\frac{\alpha}{\beta}}$$
$$= HI_{\alpha,\beta}^{\frac{\beta-\alpha}{\beta}}(f)\,(v(E))^{\frac{\alpha}{\beta}} < \infty,$$

where E is an open subset of D.

PROPOSITION 3. *Suppose that $f : D \to D_*$ and $f^{-1} : D_* \to D$ belong to the Sobolev class $W^{1,n-1}_{\rm loc}$ in D and D_*, respectively, and condition (iii) holds with $\beta, \delta \le n$. Then f and f^{-1} possess the Lusin (N) and (N^{-1})-properties.*

First note that by [10, Theorem 5.8] both f and f^{-1} are of finite distortion. Next we show that $K_O(x,f)$ is $L^{\frac{1}{n-1}}$-integrable in any arbitrary open $E \subset D$. Indeed, for $1 < \delta \leq n$, applying the Hölder inequalities in the local coordinates,

$$\int\limits_E K_O^{\frac{1}{n-1}}(x,f)\, dv(x) = \int\limits_E \frac{L^{\frac{n}{n-1}}(x,f)}{J^{\frac{1}{n-1}}(x,f)}\, dv(x)$$

$$= \int\limits_E \left(\frac{L^{\frac{\gamma\delta}{\delta-\gamma}}(x,f)}{J^{\frac{\gamma}{\delta-\gamma}}(x,f)}\right)^{\frac{(\delta-\gamma)n}{\gamma\delta(n-1)}} J^{\frac{(n-\delta)}{\delta(n-1)}}(x,f)\, dv(x)$$

$$\leq \left(\int\limits_E K_{O,\delta}^{\frac{\gamma}{\delta-\gamma}}(x,f)\, dv(x)\right)^{\frac{(\delta-\gamma)n}{\gamma\delta(n-1)}} \left(\int\limits_E J^{\frac{(n-\delta)\gamma}{\tau}}(x,f)\, dv(x)\right)^{\frac{\tau}{\gamma\delta(n-1)}}$$

$$\leq (HO_{\gamma,\delta}(f))^{\frac{(\delta-\gamma)n}{\gamma\delta(n-1)}} (v_*(E_*))^{\frac{n-\delta}{\delta(n-1)}} (v(E))^{\frac{(\gamma-1)n}{\gamma(n-1)}} < \infty,$$

where $\tau = \gamma\delta(n-1) - (\delta-\gamma)n$ and $E_* = f(E)$.

Now due to [10, Theorem 4.13], one implies that f satisfies the Lusin (N^{-1})-property. Repeating the same arguments for $K_{I,\beta}(x,f) = K_{O,\beta}(y,f^{-1})$ we obtain that f possesses the Lusin (N)-property locally in the normal coordinates.

Now we present the following type of K_I and K_O inequalities with the integral dilatation coefficients for mappings on Riemannian manifolds and the general range of the parameters α, β, γ and δ.

Theorem 1. *Let $f:\ D \to D_*$ belong to the class $\mathcal{F}(D,D_*)$. Then for all fixed values $\alpha,\beta,\gamma,\delta$ such that $1 \leq \alpha < \beta < \infty$, $1 \leq \gamma < \delta < \infty$, and for any rings $A = A(x_0,r_1,r_2) \subset D$ the following inequalities hold:*

$$M_\alpha^\beta(\Gamma^*) \leq HI_{\alpha,\beta}^{\beta-\alpha}(f) M_\beta^\alpha(\Gamma), \tag{7}$$

$$M_\gamma^\delta(\Gamma) \leq HO_{\gamma,\delta}^{\delta-\gamma}(f) M_\delta^\gamma(\Gamma^*). \tag{8}$$

Proof. Let $A = A(x_0,r_1,r_2) \subset \overline{B(x_0,\varepsilon)} \subset D$, $0 < r_1 < r_2 < \varepsilon$ and ρ be an admissible function for Γ. Recall that within any geodesic ball around $x_0 \in D$, the radial distance function $r(x)$ defined by $r(x) := \left(\sum\limits_i (x^i)^2\right)^{1/2}$ is equal to the Riemannian distance from x_0 to x, see for example Corollary 6.11 in [12]. Denote by $\mu_k(x)$ the minimal distortion of k-dimensional measures at x under f, i. e.,

$$\mu_k(x) = \lambda_n \cdot \lambda_{n-1} \cdot \ldots \cdot \lambda_{n-k+1}.$$

Note that the inequality $\mu_k(x) \geq l^k(f'(x))$ holds locally for a.e. $x \in D$. Define in $A_* = f(A)$ the function

$$\rho^*(y) = \frac{\rho(x)}{\big[\mu_k(x)\big]^{1/k}},$$

where $x = f^{-1}(y)$. It is easy to check that $\rho^*(y)$ is an admissible function for Γ^*.

Since $d\mathcal{A}^* \geq \mu_k(x) d\mathcal{A}$ we have

$$\int\limits_{\mathcal{S}^*} \rho^{*^k}(y)\, d\mathcal{A}^* \geq \int\limits_{\mathcal{S}} \frac{\rho^k(x)}{\mu_k(x)} \mu_k(x)\, d\mathcal{A} = \int\limits_{\mathcal{S}} \rho^k(x)\, d\mathcal{A} \geq 1$$

for every surface $\mathcal{S}^* \in \Gamma^*$.

One concludes from (i)–(ii) that f and f^{-1} satisfy the Lusin (N)-property in normal coordinates of D and D_*, respectively. Using [20, Theorem 13.2] one obtains that

$$K_{I,\alpha}(f'(x)) = K_{O,\alpha}((f^{-1}(y))')$$

for a.e. $x \in D$ and $y \in D_*$. Applying Hölder's inequality in each $\overline{B(x_0, \varepsilon)} \subset U(x_0)$ and the properties of f and f^{-1} in local coordinates, one obtains

$$\int\limits_{A_*} \rho^{*^\alpha}(y)\, dv_*(y) = \int\limits_{A} \frac{\rho^\alpha(x)}{\big[\mu_k(x)\big]^{\alpha/k}} J(x,f)\, dv(x) \leq \int\limits_{A} \rho^\alpha(x) \frac{J(x,f)}{l^\alpha(x,f)}\, dv(x)$$

$$\leq \Big(\int\limits_{A} \rho^\beta(x)\, dv(x)\Big)^{\alpha/\beta} \Big(\int\limits_{A} K_{I,\alpha}^{\frac{\beta}{\beta-\alpha}}(f'(x))\, dv(x)\Big)^{(\beta-\alpha)/\beta}.$$

Taking the infima over all such $\rho(x)$ yields (7). Replacing $f: D \to D_*$ by $f^{-1}: D_* \to D$ in (7), one obtains the inequality (8). □

4. MEAN COEFFICIENTS OF QUASICONFORMALITY

We introduce also the quantities

$$I_{\alpha,\beta}(A, A_*) = \inf_f I_{\alpha,\beta}(f), \quad O_{\gamma,\delta}(A, A_*) = \inf_f O_{\gamma,\delta}(f),$$

where the infima are taken over all mappings of the class $\mathcal{F}(D, D_*)$. These quantities are called the **inner** and the **outer mean characteristics** of the rings A and A_*. The mappings on which the infima are attained are said to be **extremal** for the corresponding mean coefficients.

Note that Theorem 1 gives the next relationship between the integral dilatations and the mean ones,

$$I_{\alpha,\beta}(f) \leq HI_{\alpha,\beta}(f), \qquad O_{\gamma,\delta}(f) \leq HO_{\gamma,\delta}(f).$$

Similarly to the case of classical quasiconformality coefficients $K_I(D, D_*)$ and $K_O(D, D_*)$, the problem *to determine the mean characteristics* $I_{\alpha,\beta}(D, D_*)$ *and* $O_{\gamma,\delta}(D, D_*)$ *for ring domains* A *and* A_* is fairly difficult (cf. [8, p. 6]). To obtain upper bounds for given ring domains A and A_*, it is only necessary to construct an appropriate mapping f from A onto A_* and calculate the dilatations of f by (6). The problem of obtaining significant lower bounds is much more difficult, since one must find lower bounds for the various dilatations of all homeomorphisms of class $\mathcal{F}(D, D_*)$. We do this by considering what happens to certain families of k-dimensional surfaces under each homeomorphism f and then appealing to (4).

Now we present the extremal mappings which realize the minimum for the integral dilatations $HI_{\alpha,\beta}(f)$ and $HO_{\gamma,\delta}(f)$ over the class $\mathcal{F}(A, A_*)$ in the n-dimensional Euclidean space, i.e. $\mathbb{M}^n = \mathbb{M}^n_* = \mathbb{R}^n$. We take the spherical rings in $\mathbb{R}^n$, denoting by

$$D(r_0) = \{x \in \mathbb{R}^n : 0 < r_0 < 1\}, \qquad D(\rho_0) = \{y \in \mathbb{R}^n : 0 < \rho_0 < 1\},$$

and calculate the coefficients $I_{\alpha,\beta}(D(r_0), D(\rho_0))$ and $O_{\gamma,\delta}(D(r_0), D(\rho_0))$ for the case $0 < \rho_0 \le r_0 < 1$.

For $1 < \alpha < \beta < \infty$ we obtain by (4) the following bound

$$I_{\alpha,\beta}(D(r_0), D(\rho_0)) \ge \left(\frac{M_\alpha^\beta(\Sigma_D^*)}{M_\beta^\alpha(\Sigma_D)}\right)^{\frac{1}{\beta-\alpha}},$$

where Σ_D denotes a family of all piecewise smooth $(n-1)$-dimensional surfaces which separate the boundary components of $D(r_0)$ in $D(r_0)$ and $\Sigma_D^* = f(\Sigma_D)$. For $\alpha \ne n$ and $\beta \ne n$, we get

$$I_{\alpha,\beta}(D(r_0), D(\rho_0)) \ge \omega_{n-1}\left(\frac{1-\rho_0^{n-\alpha}}{n-\alpha}\right)^{\frac{\beta}{\beta-\alpha}}\left(\frac{n-\beta}{1-r_0^{n-\beta}}\right)^{\frac{\alpha}{\beta-\alpha}},$$

where ω_{n-1} denotes the $(n-1)$-dimensional Lebesgue measure of the unit $(n-1)$-dimensional sphere. Here we have substituted the explicit expressions for $M_\alpha(\Sigma_D^*)$ and $M_\beta(\Sigma_D)$ taken into account the relation between the p-moduli of families $(n-1)$-dimensional surfaces and p-capacities; cf. [21]. For calculation of p-capacities we refer e.g. to [7].

Now consider two spherical systems of coordinates (r, φ_i) and (ρ, ψ_i), $i = \overline{1, n-1}$ on $D(r_0)$ and $D(\rho_0)$, respectively. It is easy to see that the mapping

$$f_1 = \left(\rho = \left[1 + \frac{\rho_0^{n-\alpha}-1}{r_0^{n-\beta}-1}(r^{n-\beta}-1)\right]^{\frac{1}{n-\alpha}}, \psi_i = \varphi_i, 0 \le \varphi_i < \pi, i = \overline{1, n-2},\right.$$

$$\left. 0 \le \varphi_{n-1} < 2\pi, r_0 < r < 1\right)$$

transforms $D(r_0)$ onto $D(\rho_0)$. From (6) we have

$$HI_{\alpha,\beta}(f_1) = \omega_{n-1}\left(\frac{1-\rho_0^{n-\alpha}}{n-\alpha}\right)^{\frac{\beta}{\beta-\alpha}}\left(\frac{n-\beta}{1-r_0^{n-\beta}}\right)^{\frac{\alpha}{\beta-\alpha}}.$$

Thus the mapping $f_1 : D(r_0) \to D(\rho_0)$ is extremal for the coefficient $I_{\alpha,\beta}(D(r_0), D(\rho_0))$, and

$$I_{\alpha,\beta}(D(r_0), D(\rho_0)) = \omega_{n-1}\left(\frac{1-\rho_0^{n-\alpha}}{n-\alpha}\right)^{\frac{\beta}{\beta-\alpha}}\left(\frac{n-\beta}{1-r_0^{n-\beta}}\right)^{\frac{\alpha}{\beta-\alpha}}.$$

In the cases $\alpha = n$ and $\beta = n$ the corresponding inner mean characteristics have the forms

$$I_{n,\beta}(D(r_0), D(\rho_0)) = \omega_{n-1}\left(\ln\frac{1}{\rho_0}\right)^{\frac{\beta}{\beta-n}}\left(\frac{n-\beta}{1-r_0^{n-\beta}}\right)^{\frac{n}{\beta-n}}$$

and

$$I_{\alpha,n}(D(r_0), D(\rho_0)) = \omega_{n-1}\left(\frac{1-\rho_0^{n-\alpha}}{n-\alpha}\right)^{\frac{n}{n-\alpha}}\left(\ln\frac{1}{r_0}\right)^{\frac{\alpha}{n-\alpha}}.$$

In the same way one obtains from (4) the following estimate for $1 < \gamma < \delta < \infty$

$$O_{\gamma,\delta}(D(r_0), D(\rho_0)) \geq \left(\frac{M_\gamma^\delta(\Gamma_D)}{M_\delta^\gamma(\Gamma_D^*)}\right)^{\frac{1}{\delta-\gamma}},$$

where Γ_D is a family of curves which join the boundary components of $D(r_0)$ in $D(r_0)$. Substituting into the right-hand side of the last bound the well-known expressions for $M_\gamma(\Gamma_D)$ and $M_\delta(\Gamma_D^*)$ ([7]) for $\gamma \neq n$ and $\delta \neq n$, we have

$$O_{\gamma,\delta}(D(r_0), D(\rho_0)) \geq \omega_{n-1}\left(\frac{n-\gamma}{\gamma-1}\right)^{\frac{(\gamma-1)\delta}{\delta-\gamma}}\left(\frac{\delta-1}{n-\delta}\right)^{\frac{(\delta-1)\gamma}{\delta-\gamma}}\frac{\left(r_0^{\frac{\gamma-n}{\gamma-1}}-1\right)^{\frac{(1-\gamma)\delta}{\delta-\gamma}}}{\left(\rho_0^{\frac{\delta-n}{\delta-1}}-1\right)^{\frac{(1-\delta)\gamma}{\delta-\gamma}}}.$$

The mapping

$$f_2 = \left(\rho = \left[1 + \frac{\rho_0^{\frac{\delta-n}{\delta-1}}-1}{r_0^{\frac{\gamma-n}{\gamma-1}}-1}\left(r^{\frac{\gamma-n}{\gamma-1}}-1\right)\right]^{\frac{\delta-1}{\delta-n}}, \psi_i = \varphi_i, 0 \leq \varphi_i < \pi, i = \overline{1, n-2},\right.$$
$$\left. 0 \leq \varphi_{n-1} < 2\pi, r_0 < r < 1\right)$$

transforms $D(r_0)$ onto $D(\rho_0)$. By (6),

$$HO_{\gamma,\delta}(f_2) = \omega_{n-1}\left(\frac{n-\gamma}{\gamma-1}\right)^{\frac{(\gamma-1)\delta}{\delta-\gamma}}\left(\frac{\delta-1}{n-\delta}\right)^{\frac{(\delta-1)\gamma}{\delta-\gamma}}\frac{\left(r_0^{\frac{\gamma-n}{\gamma-1}}-1\right)^{\frac{(1-\gamma)\delta}{\delta-\gamma}}}{\left(\rho_0^{\frac{\delta-n}{\delta-1}}-1\right)^{\frac{(1-\delta)\gamma}{\delta-\gamma}}}.$$

This yields that the mapping $f_2: D(r_0) \to D(\rho_0)$ is extremal for the mean coefficient $O_{\gamma,\delta}(D(r_0), D(\rho_0))$ and

$$O_{\gamma,\delta}(D(r_0), D(\rho_0)) = \omega_{n-1}\left(\frac{n-\gamma}{\gamma-1}\right)^{\frac{(\gamma-1)\delta}{\delta-\gamma}}\left(\frac{\delta-1}{n-\delta}\right)^{\frac{(\delta-1)\gamma}{\delta-\gamma}}\frac{\left(r_0^{\frac{\gamma-n}{\gamma-1}}-1\right)^{\frac{(1-\gamma)\delta}{\delta-\gamma}}}{\left(\rho_0^{\frac{\delta-n}{\delta-1}}-1\right)^{\frac{(1-\delta)\gamma}{\delta-\gamma}}}.$$

In the case $\gamma = n$, the outer mean characteristic is

$$O_{n,\delta}(D(r_0), D(\rho_0)) = \omega_{n-1}\left(\frac{n-\gamma}{\gamma-1}\right)^{\frac{(\gamma-1)\delta}{\delta-\gamma}}\left(\ln\frac{1}{r_0}\right)^{\frac{(1-n)\delta)}{\delta-n}}\left(\rho_0^{\frac{\delta-n}{\delta-1}}-1\right)^{\frac{(\delta-1)n}{\delta-n}}.$$

The case $\delta = n$ has been studied in [11].

It follows from (5) that in the case $0 < r_0 \le \rho_0 < 1$, the mapping f_2^{-1} is extremal for the inner mean coefficient. Then the mapping f_1^{-1} is extremal for the outer mean coefficient.

Now we present the monotonicity of the appropriate mean dilatations by real parameters on Riemannian manifolds. Let

$$IM^*_{\alpha,\beta}(A, A_*) = \left(\frac{I_{\alpha,\beta}(A, A_*)}{v_*(A_*)}\right)^{\frac{\beta-\alpha}{\alpha}}, \quad IM_{\alpha,\beta}(A, A_*) = \left(\frac{I_{\alpha,\beta}(A, A_*)}{v(A)}\right)^{\frac{\beta-\alpha}{\beta}},$$

$$OM_{\alpha,\beta}(A, A_*) = \left(\frac{O_{\alpha,\beta}(A, A_*)}{v(A)}\right)^{\frac{\beta-\alpha}{\alpha}}, \quad OM^*_{\alpha,\beta}(A, A_*) = \left(\frac{O_{\alpha,\beta}(A, A_*)}{v_*(A_*)}\right)^{\frac{\beta-\alpha}{\beta}}.$$

Theorem 2. (a) *If s, α, β are the real numbers such that $1 < s < \alpha < \beta < \infty$ then $IM^*_{s,\beta}(A, A_*) \le IM^*_{\alpha,\beta}(A, A_*)$ and $OM_{s,\beta}(A, A_*) \le OM_{\alpha,\beta}(A, A_*)$.*

(b) *If α, β, t are the real numbers such that $1 < \alpha < \beta < t < \infty$ then $IM_{\alpha,t}(A, A_*) \le IM_{\alpha,\beta}(A, A_*)$ and $OM^*_{\alpha,t}(A, A_*) \le OM^*_{\alpha,\beta}(A, A_*)$.*

Proof. We will prove only the first inequalities in each part of the theorem. Applying Hölder's inequality to $IM^*_{s,\beta}(A, A_*)$ on any ball $\overline{B(x_0, \varepsilon)} \subset U(x_0)$,

we have: $$IM^*_{s,\beta}(A,A_*) = \inf \frac{1}{\big(v_*(A_*)\big)^{\frac{\beta-s}{s}}}\left(\sup \frac{M_s^\beta(\Gamma^*)}{M_\beta^s(\Gamma)}\right)^{\frac{1}{s}}$$
$$\le \inf \frac{1}{\big(v_*(A_*)\big)^{\frac{\beta-s}{s}}}\left(\sup \frac{\big(v_*(A_*)\big)^{\frac{(\alpha-s)\beta}{\alpha}} M_\alpha^{\frac{s\beta}{\alpha}}(\Gamma^*)}{M_\beta^s(\Gamma)}\right)^{\frac{1}{s}}$$
$$= \inf \frac{1}{\big(v_*(A_*)\big)^{\frac{\beta-\alpha}{\alpha}}}\left(\sup \frac{M_\alpha^\beta(\Gamma^*)}{M_\beta^\alpha(\Gamma)}\right)^{\frac{1}{\alpha}} = IM^*_{\alpha,\beta}(A,A_*).$$

Similarly for the part (b),
$$IM_{\alpha,t}(A,A_*) = \inf \frac{1}{\big(v(A)\big)^{\frac{t-\alpha}{t}}}\left(\sup \frac{M_\alpha^t(\Gamma^*)}{M_t^\alpha(\Gamma)}\right)^{\frac{1}{t}}$$
$$\le \inf \frac{1}{\big(v(A)\big)^{\frac{t-\alpha}{t}}}\left(\sup \frac{\big(v(A)\big)^{\frac{(t-\beta)\alpha}{\beta}} M_\alpha^t(\Gamma^*)}{M_\beta^{\frac{\alpha t}{\beta}}(\Gamma)}\right)^{\frac{1}{t}}$$
$$= \inf \frac{1}{\big(v(A)\big)^{\frac{\beta-\alpha}{\alpha}}}\left(\sup \frac{M_\alpha^\beta(\Gamma^*)}{M_\beta^\alpha(\Gamma)}\right)^{\frac{1}{\beta}} = IM_{\alpha,\beta}(A,A_*),$$

which yields the desired relations. □

The assertions of the last theorem allow us to calculate the classical coefficients of quasiconformality $K_I(D(r_0),D(\rho_0))$ and $K_O(D(r_0),D(\rho_0))$ (cf. [20]), since the quasiconformality relates to the case when all real parameters are equal to n. So, for example, $0<\rho_0<r_0<1$,
$$K_I(D(r_0),D(\rho_0)) = \lim_{\alpha\to n} IM^*_{\alpha,n}(D(r_0),D(\rho_0))$$
$$= \lim_{\beta\to n} IM_{n,\beta}(D(r_0),D(\rho_0)) = \frac{\ln\frac{1}{\rho_0}}{\ln\frac{1}{r_0}},$$
$$K_O(D(r_0),D(\rho_0)) = \lim_{\alpha\to n} OM_{\alpha,n}(D(r_0),D(\rho_0))$$
$$= \lim_{\beta\to n} OM^*_{n,\beta}(D(r_0),D(\rho_0)) = \left(\frac{\ln\frac{1}{\rho_0}}{\ln\frac{1}{r_0}}\right)^{n-1}.$$

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