Dedicated to the memory of Cabiria Andreian-Cazacu

A SURVEY ON LOEWNER CHAINS AND RELATED PROBLEMS FOR BOUNDED BALANCED PSEUDOCONVEX DOMAINS IN $\mathbb{C}^N$

HIDETAKA HAMADA, MIHAI IANCU, and GABRIELA KOHR

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Let $\Omega \subset \mathbb{C}^n$ be a bounded balanced pseudoconvex domain with C^1 plurisubharmonic defining functions. In this paper, we study properties of the Carathéodory family $\mathcal{M}(\Omega)$, and prove that it is compact with respect to the topology of locally uniform convergence on $H(\Omega)$. Then we survey some results in the theory of Loewner chains on Ω, which are extensions to bounded balanced pseudoconvex domains of corresponding results on the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$. A special interest in our discussion is played by the family $S^0(\Omega)$ of normalized univalent mappings which have parametric representation on Ω. Certain questions and conjectures related to the family $S^0(\Omega)$ are also considered.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{C}^n$ be the space of n complex variables $z = (z_1, \dots, z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$ and the Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. Also, let $\mathbb{B}^n = \{z \in \mathbb{C}^n : \|z\| < 1\}$ be the Euclidean unit ball in $\mathbb{C}^n$ and let $\mathbb{B}^1 = \mathbb{U}$ be the unit disc. If $G \subseteq \mathbb{C}^n$ is a domain, we denote by $H(G)$ the family of holomorphic mappings from G to $\mathbb{C}^n$. If $0 \in G$ and $f \in H(G)$, we say that f is normalized if $f(0) = 0$ and $Df(0) = I_n$, where $Df(z)$ is the Fréchet derivative of f at $z \in G$, and I_n is the identity operator in the space $L(\mathbb{C}^n)$ of linear operators from $\mathbb{C}^n$ into $\mathbb{C}^n$. Also, let $S(G)$ be the family of normalized biholomorphic mappings on G, and let $S^*(G)$ be the subset of $S(G)$ consisting of starlike mappings on G, where a mapping $f \in S(G)$ is said to be starlike if $f(G)$ is a starlike domain in $\mathbb{C}^n$ with respect to the origin. The family $S(\mathbb{U})$ is denoted by S.

Definition 1.1 (see [20]). Let $D \subseteq \mathbb{C}^n$ be a domain. We say that D has C^1 plurisubharmonic defining functions if for every $w \in \partial D$, there exists a neighborhood V of w in $\mathbb{C}^n$ and a C^1 plurisubharmonic function $r : V \to \mathbb{R}$ such that $D \cap V = \{z \in V : r(z) < 0\}$.

Note that if $D \subseteq \mathbb{C}^n$ is a domain which has C^1 plurisubharmonic defining functions, then D is pseudoconvex.

Definition 1.2. Let $D \subseteq \mathbb{C}^n$ be a domain. We say that D is balanced if $\lambda z \in D$, whenever $z \in D$ and $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$. In this case, the Minkowski function h of D is given by:

$$h(z) = \inf \left\{ t > 0 : \frac{z}{t} \in D \right\}, \quad \forall\, z \in \mathbb{C}^n.$$

The following known results are of special interest in our consideration.

Proposition 1.3 (see [8, Lemma 6.2]). *Let D be a balanced domain in $\mathbb{C}^n$. Then D is pseudoconvex if and only if the associated Minkowski function h of D is plurisubharmonic on $\mathbb{C}^n$.*

Proposition 1.4 (see e.g. [8] and [25]). *If D is a bounded balanced domain in $\mathbb{C}^n$, then D is convex if and only if the associated Minkowski function h of D is a norm on $\mathbb{C}^n$.*

The following remarks provide the motivation for our work in the case of bounded balanced pseudoconvex domains Ω in $\mathbb{C}^n$ such that $\Omega \neq \mathbb{B}^n$.

Remark 1.5. Let $n \geq 2$. There exist non-convex bounded balanced pseudoconvex domains in $\mathbb{C}^n$ with C^1 plurisubharmonic defining functions.

Proof. Without loss of generality, we assume that $n = 2$. Let

$$\Omega = \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 |z_2|^2 + |z_1|^2 + |z_2|^2 < 20 \right\}.$$

Clearly, Ω is a bounded balanced pseudoconvex domain with C^1 plurisubharmonic defining functions (cf. [32, Proposition II.4.6]). We note that the points $(1, 3)$ and $(3, 1)$ belong to Ω, but $(2, 2) \notin \Omega$, and thus Ω is not convex. □

Remark 1.6 (see [5, Corollary]; cf. [8, Proposition 6.5]). Let D be a convex domain in $\mathbb{C}^n$ and let $F : D \to \mathbb{C}^n$ be a univalent mapping. If $F(D)$ is balanced, then $F(D)$ is also a convex domain.

Remark 1.7. Let $n \geq 2$. There exist univalent mappings $F : \mathbb{B}^n \to \mathbb{C}^n$ such that $F(\mathbb{B}^n)$ is starlike, but not convex, and thus not balanced.

Proof. Let $n \geq 2$ and $f : \mathbb{B}^n \to \mathbb{C}^n$ be given by

$$f(z) = \left(\frac{z_1}{1-z_1}, \ldots, \frac{z_n}{1-z_n} \right), \quad \forall\, z = (z_1, \ldots, z_n) \in \mathbb{B}^n.$$

Then $f \in S^*(\mathbb{B}^n)$ (see e.g. [36]), however, if $n \geq 2$ then $f(\mathbb{B}^n)$ is not a convex domain in $\mathbb{C}^n$ (see [33], [19, Example 11.1.1]), and thus $f(\mathbb{B}^n)$ is not a balanced domain in $\mathbb{C}^n$, in view of Remark 1.6. □

Throughout this paper, we consider the following assumption:

Assumption 1.8. Let $\Omega \subset \mathbb{C}^n$ be a bounded balanced pseudoconvex domain with C^1 plurisubharmonic defining functions, and let h be the Minkowski function associated with Ω.

The following proposition is very useful in our considerations (see [8, Lemma 6.2], [20, Proposition 1] and [25, Remark 2.2.1, Proposition 3.1.11]). Applications of this result may be found in [24], [38], [39] (see also [19]).

Proposition 1.9. *Let Ω and h satisfy Assumption* 1.8. *Then the following statements hold:*

(i) *$h(z) = 0$ if and only if $z = 0$.*

(ii) *$h(\zeta z) = |\zeta| h(z)$, for $z \in \mathbb{C}^n$, $\zeta \in \mathbb{C}$.*

(iii) *h is of class C^1 on $\mathbb{C}^n \setminus \{0\}$.*

(iv) *h is continuous and plurisubharmonic on $\mathbb{C}^n$.*

(v) *$\Omega = \big\{z \in \mathbb{C}^n : h(z) < 1\big\}$ and $\partial\Omega = \big\{z \in \mathbb{C}^n : h(z) = 1\big\}$.*

(vi) *If $r \in (0,1)$ and $\Omega_r = \{z \in \Omega : h(z) < r\}$, then $\overline{\Omega}_r = \{z \in \Omega : h(z) \leq r\}$ and it is a compact set. In particular, $\partial\Omega_r = \{z \in \Omega : h(z) = r\}$.*

(vii) *There exist $c_1, c_2 > 0$ such that $c_1\|z\| \leq h(z) \leq c_2\|z\|$, for all $z \in \mathbb{C}^n$.*

(viii) *$h(F(z)) \leq h(z)$, for $z \in \Omega$, where $F : \Omega \to \Omega$ is holomorphic with $F(0) = 0$.*

We also need the following version of the Schwarz lemma on bounded balanced pseudoconvex domains in $\mathbb{C}^n$. The following result is a generalization of Proposition 1.9 (viii) (cf. [19, Lemma 6.1.28] in the case $\Omega = \mathbb{B}^n$).

Proposition 1.10. *Let $\Omega \subset \mathbb{C}^n$ and h satisfy Assumption* 1.8. *Let $f : \Omega \to \Omega$ be a holomorphic mapping such that $f(0) = 0$, $Df(0) = 0, \ldots,$ $D^{k-1}f(0) = 0$, where $k \in \mathbb{N}$ and $D^k f(0)$ is the k-th Fréchet derivative of f at $z = 0$. Then $h(f(z)) \leq h^k(z)$, for all $z \in \Omega$.*

Proof. Let $z \in \Omega \setminus \{0\}$ be fixed and let

$$g(\zeta) = \begin{cases} \frac{1}{\zeta^k} f(\zeta z), & 0 < |\zeta| < \frac{1}{h(z)} \\ \frac{1}{k!} D^k f(0)(z^k), & \zeta = 0. \end{cases}$$

Then g is holomorphic on $\frac{1}{h(z)}\mathbb{U}$. Since h is plurisubharmonic on $\mathbb{C}^n$, we deduce that $h \circ g$ is subharmonic on $\frac{1}{h(z)}\mathbb{U}$ (cf. [25, B.4.9]). In view of the maximum principle for subharmonic functions, we have for every $r \in \left(0, \frac{1}{h(z)}\right)$ that

$$\frac{h(f(\zeta z))}{|\zeta|^k} \leq \max_{|\zeta|=r} \frac{h(f(\zeta z))}{r^k} \leq \frac{1}{r^k}, \quad 0 < |\zeta| \leq r.$$

Letting $r \to \frac{1}{h(z)}$ and $\zeta = 1$, we get $h(f(z)) \leq h^k(z)$. □

Next, we recall the definition of the Carathéodory family $\mathcal{M}(\Omega)$.

Definition 1.11. (see [24]) Let $\Omega \subset \mathbb{C}^n$ and h satisfy Assumption 1.8. Let

$$\mathcal{M}(\Omega) = \left\{ g \in H(\Omega) : g(0) = 0, Dg(0) = I_n, \Re\left\langle g(z), \frac{\partial h^2}{\partial \overline{z}}(z) \right\rangle > 0, z \in \Omega \setminus \{0\} \right\}$$

be the family of the Carathéodory mappings on Ω, where

$$\frac{\partial h^2}{\partial \overline{z}}(z) = \left(\frac{\partial h^2}{\partial \overline{z}_1}(z), \ldots, \frac{\partial h^2}{\partial \overline{z}_n}(z) \right), \quad \forall\, z = (z_1, \ldots, z_n) \in \Omega \setminus \{0\}.$$

Note that if $\Omega = \mathbb{B}^n$, then $h(z) = \|z\|$, for all $z \in \mathbb{C}^n$, and thus (see [27])

$$\mathcal{M}(\mathbb{B}^n) = \left\{ g \in H(\mathbb{B}^n) : g(0) = 0, Dg(0) = I_n, \Re\langle g(z), z \rangle > 0, z \in \mathbb{B}^n \setminus \{0\} \right\}.$$

In particular, if $n = 1$, then $g \in \mathcal{M}(\mathbb{U})$ if and only if $\zeta \mapsto g(\zeta)/\zeta \in \mathcal{P}$, where

$$\mathcal{P} = \left\{ p \in H(\mathbb{U}) : p(0) = 1,\, \Re p(\zeta) > 0,\, \zeta \in \mathbb{U} \right\}.$$

Applications of this family in the study of univalence in $\mathbb{C}$ and $\mathbb{C}^n$ may be found in e.g. [11], [19], [28], and the references therein.

Proposition 1.12 (see [20, Theorem 3 and (3.2)]). *Let $\Omega \subset \mathbb{C}^n$ be a bounded balanced pseudoconvex domain with C^1 plurisubharmonic defining functions. If $f : \Omega \to \mathbb{C}^n$ is a normalized locally biholomorphic mapping, then $f \in S^*(\Omega)$ if and only if $g \in \mathcal{M}(\Omega)$, where $g(z) = (Df(z))^{-1} f(z)$, $\forall\, z \in \Omega$.*

In Section 2, we shall prove that if $\Omega \subset \mathbb{C}^n$ is a bounded balanced pseudoconvex domain with C^1 plurisubharmonic defining functions, then $\mathcal{M}(\Omega)$ is a compact subset of $H(\Omega)$ with respect to the topology of locally uniform convergence on $H(\Omega)$. To this end, we need to state some preliminary results. First, we recall the notion of the kernel convergence in $\mathbb{C}^n$ (cf. [3, Definition 3.3]; see [10, Definition 2.16]; see also [28], for $n = 1$).

Definition 1.13. Let $\{G_k\}_{k\in\mathbb{N}}$ be a sequence of domains in $\mathbb{C}^n$ such that $0 \in G_k$, $k \in \mathbb{N}$. If 0 is an interior point of $\bigcap_{k\in\mathbb{N}} G_k$, we define the kernel G of $\{G_k\}_{k\in\mathbb{N}}$ to be the largest domain in $\mathbb{C}^n$ which contains 0, such that if K is a compact subset of G, then K is contained in all but finitely many G_k. If 0 is not an interior point of $\bigcap_{k\in\mathbb{N}} G_k$, we define the kernel to be $\{0\}$.

We say that $\{G_k\}_{k\in\mathbb{N}}$ kernel converges to G, and write $G_k \to G$, if each subsequence of $\{G_k\}_{k\in\mathbb{N}}$ has the same kernel G.

Using arguments similar to those in [10, Theorem 2.17] and [3, Theorem 3.5] (cf. [9, Theorem 5.2]), in view of Proposition 1.9, we deduce the following kernel convergence result for biholomorphic mappings on Ω.

PROPOSITION 1.14. *Let $\Omega \subset \mathbb{C}^n$ be a bounded balanced pseudoconvex domain in $\mathbb{C}^n$ with C^1 plurisubharmonic defining functions. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of biholomorphic mappings on Ω with $f_k(0) = 0$, for $k \in \mathbb{N}$. If $(f_k)_{k\in\mathbb{N}}$ converges locally uniformly on Ω to a biholomorphic mapping f, then $(f_k(\Omega))_{k\in\mathbb{N}}$ kernel converges to $f(\Omega)$ and $f_k^{-1} \to f^{-1}$ locally uniformly on $f(\Omega)$ as $k \to \infty$.*

2. COMPACTNESS AND COEFFICIENT BOUNDS FOR $\mathcal{M}(\Omega)$

In this section, we shall prove that if $\Omega \subset \mathbb{C}^n$ is a bounded balanced pseudoconvex domain with C^1 plurisubharmonic defining functions, then the family $\mathcal{M}(\Omega)$ is a compact subset of $H(\Omega)$. For this aim, we need to mention some auxiliary results of independent interest.

The following lemma is related to [31, Lemma 4.1] and is an immediate consequence of [24, Proposition 1.2] (cf. [19, Theorem 8.1.3]).

LEMMA 2.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded balanced pseudoconvex domain in $\mathbb{C}^n$ with C^1 plurisubharmonic defining functions, and let $g \in \mathcal{M}(\Omega)$. Then, for every $z \in \Omega$, the initial value problem*

$$\frac{\partial v}{\partial t}(z,t) = -g(v(z,t)), \ \text{a.e. } t \geq 0, \quad v(z,0) = z, \tag{2.1}$$

has a unique locally absolutely continuous solution $v(z,\cdot)$ on $[0,\infty)$, and, for every $t \geq 0$, $v(\cdot,t)$ is a biholomorphic Schwarz mapping with $Dv(0,t) = e^{-t}I_n$.

The next lemma is closely related to [31, Lemma 4.2 and Theorem 4.3] (cf. [24, Lemmas 2.1 and 2.3], [19, Theorem 8.1.5]). The proof can be carried out in the same way as in [31, Section 4], by using the properties presented in Proposition 1.9 (cf. [11], [19], [24]).

LEMMA 2.2. *Let $\Omega \subset \mathbb{C}^n$ be as in Lemma 2.1, and let $g \in \mathcal{M}(\Omega)$. Let $v = v(z,t)$ be the unique locally absolutely continuous solution on $[0,\infty)$ of the* *initial value problem* (2.1) *associated with* g. *Then the limit* $f = \lim_{t\to\infty} e^t v(\cdot, t)$ *exists locally uniformly on* Ω. *Moreover,* f *is the unique normalized biholomorphic mapping on* Ω *such that* $Df(z)g(z) = f(z)$, $\forall\, z \in \Omega$, *and thus* $f \in S^*(\Omega)$.

The following lemma provides a useful characterization of $\mathcal{M}(\Omega)$ in terms of starlike mappings on Ω (see [31, Theorem 4.5] and [37, Remark 3.2], in the case of the unit ball $\mathbb{B}^n$; see also [26, Corollary 2], in the case of bounded balanced pseudoconvex domains in $\mathbb{C}^n$ with C^2 plurisubharmonic defining functions).

Lemma 2.3. *Let* $\Omega \subset \mathbb{C}^n$ *be a bounded domain as in Lemma* 2.1. *Then*

$$\mathcal{M}(\Omega) = \Big\{ g \in H(\Omega) : \exists\, f \in S^*(\Omega) \text{ with } g(z) = (Df(z))^{-1} f(z), \forall\, z \in \Omega \Big\}.$$

Proof. The equality follows from Lemma 2.2 and Proposition 1.12. □

Next, we recall the following growth result for the family $S^*(\Omega)$ (see [20, Theorem 4]; see also [13] and the references therein; see e.g. [19, Chapter 7], in the case $\Omega = \mathbb{B}^n$).

Proposition 2.4. *Let* Ω *and* h *satisfy Assumption* 1.8. *If* $f \in S^*(\Omega)$, *then*

$$\frac{h(z)}{(1+h(z))^2} \le h(f(z)) \le \frac{h(z)}{(1-h(z))^2}, \quad z \in \Omega. \tag{2.2}$$

This result is sharp in the case $\Omega = \mathbb{B}^n(p_1, \ldots, p_n)$, *where*

$$\mathbb{B}^n(p_1, \ldots, p_n) = \Big\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^{p_j} < 1 \Big\}, \tag{2.3}$$

with $p_j > 1$, $j = 1, \ldots, n$ *is the complex ellipsoid.*

In view of Proposition 2.4 and the minimum principle for harmonic mappings, we deduce the following compactness result for the family $S^*(\Omega)$ with respect to the topology of locally uniform convergence in $H(\Omega)$ (see e.g. [19, Chapter 7] and the references therein, in the case $\Omega = \mathbb{B}^n$).

Theorem 2.5. *Let* Ω *be as in Lemma* 2.1. *Then* $S^*(\Omega)$ *is compact in* $S(\Omega)$.

Proof. Since Ω is bounded, we deduce in view of Proposition 1.9 (vii) and Proposition 2.4 (cf. [24]) that $S^*(\Omega)$ is a locally uniformly bounded family. On the other hand, if $(f_k)_{k\in\mathbb{N}}$ is a sequence in $S^*(\Omega)$, which converges locally uniformly on Ω to a mapping f, then f is a normalized holomorphic mapping on Ω, by Weierstrass' theorem for holomorphic mappings. Moreover, f is also univalent on Ω, and thus $f \in S(\Omega)$ by Hurwitz's theorem in higher dimensions (see e.g. [19, Theorem 6.1.17]). Also, since $(f_k)_{k\in\mathbb{N}} \subset S^*(\Omega)$, it follows that

$$\Re\Big\langle g_k(z), \frac{\partial h^2}{\partial \overline{z}}(z)\Big\rangle > 0, \quad \forall\, z \in \Omega \setminus \{0\},\ k \in \mathbb{N}, \tag{2.4}$$

where $g_k(z) = (Df_k(z))^{-1} f_k(z)$, $\forall\, k \in \mathbb{N}$, and h is the Minkowski function of Ω. In view of Proposition 1.14, we have that $f_k^{-1} \to f^{-1}$ locally uniformly on $f(\Omega)$ as $k \to \infty$. Taking into account the Weierstrass theorem for holomorphic mappings, we deduce that $(Df_k)^{-1} \to (Df)^{-1}$ locally uniformly on $f(\Omega)$ as $k \to \infty$, and thus $Df_k^{-1} \circ f_k \to Df^{-1} \circ f$ locally uniformly on Ω as $k \to \infty$. Consequently, $g_k \to g$ locally uniformly on Ω as $k \to \infty$, where $g(z) = (Df(z))^{-1} f(z)$, for all $z \in \Omega$. Finally, letting $k \to \infty$ in (2.4) and using the minimum principle for harmonic mappings, we deduce that

$$\Re\Big\langle g(z), \frac{\partial h^2}{\partial \overline{z}}(z)\Big\rangle > 0, \quad \forall\, z \in \Omega \setminus \{0\}.$$

Hence, $g \in \mathcal{M}(\Omega)$. In view of Proposition 1.12, we deduce that $f \in S^*(\Omega)$. Consequently, $S^*(\Omega)$ is a closed subset of $S(\Omega)$.

Thus $S^*(\Omega)$ is compact, as desired. This completes the proof. □

Now, we are able to prove that $\mathcal{M}(\Omega)$ is a compact subset of $H(\Omega)$ (see [14, Corollary 1.3], in the case $\Omega = \mathbb{B}^n$). Note that the proof of Theorem 2.6 provides an alternative proof of [14, Corollary 1.3] in the case $\Omega = \mathbb{B}^n$.

Theorem 2.6. *Let Ω be as in Lemma* 2.1. *Then $\mathcal{M}(\Omega)$ is compact in $H(\Omega)$.*

Proof. Let $(g_k)_{k\in\mathbb{N}}$ be an arbitrary sequence in $\mathcal{M}(\Omega)$. By Lemma 2.3, for every $k \in \mathbb{N}$, there exists $f_k \in S^*(\Omega)$ such that $g_k(z) = (Df_k(z))^{-1} f_k(z)$, for $z \in \Omega$. Since $S^*(\Omega)$ is a compact subset of $S(\Omega)$ by Theorem 2.5, we deduce that there exists a subsequence $(f_{k_m})_{m\in\mathbb{N}}$ of $(f_k)_{k\in\mathbb{N}}$ that converges locally uniformly on Ω to a mapping $f \in S^*(\Omega)$. Further, as in the proof of Theorem 2.5, we have that $f_{k_m}^{-1} \to f^{-1}$ locally uniformly on $f(\Omega)$ and $Df_{k_m}^{-1} \circ f_{k_m} \to Df^{-1} \circ f$ locally uniformly on Ω as $m \to \infty$. Consequently, $g_{k_m} \to g$ locally uniformly on Ω as $m \to \infty$, where $g(z) = (Df(z))^{-1} f(z)$, for all $z \in \Omega$. Hence $\mathcal{M}(\Omega)$ is a normal family. Finally, since $f \in S^*(\Omega)$, it follows that $g \in \mathcal{M}(\Omega)$, by Lemma 2.3. Hence $\mathcal{M}(\Omega)$ is also a closed subset of $H(\Omega)$.

Taking into account the above arguments, we deduce that $\mathcal{M}(\Omega)$ is a compact subset of $H(\Omega)$, as desired. This completes the proof. □

The following result provides coefficient bounds for mappings in $\mathcal{M}(\Omega)$ (see [14, Theorem 1.2], in the case $\Omega = \mathbb{B}^n$; cf. [38]).

Theorem 2.7. *Let Ω and h satisfy Assumption 1.8. Let $g \in \mathcal{M}(\Omega)$ and let $P_m(z) = \frac{1}{m!}D^m g(0)(z^m)$, $z \in \mathbb{C}^n$, $m \in \mathbb{N}$. Then the following relations hold:*

$$\left|\left\langle P_m(z), \frac{\partial h^2}{\partial \overline{z}}(z)\right\rangle\right| \leq 2, \quad \forall\, z \in \partial\Omega,\ m \geq 2. \tag{2.5}$$

These estimates are sharp if $\Omega = \mathbb{B}^n(p_1, \ldots, p_n)$, where $p_j > 1$, $j = 1, \ldots, n$.

Proof. Fix $z \in \partial\Omega$ and let $p : \mathbb{U} \to \mathbb{C}$ be given by $p(\zeta) = \frac{1}{\zeta}\langle g(\zeta z), \frac{\partial h^2}{\partial \overline{z}}(z)\rangle$ for $\zeta \in \mathbb{U} \setminus \{0\}$, and $p(0) = 1$. Then $p \in H(\mathbb{U})$ and

$$\Re p(\zeta) = \frac{1}{|\zeta|^2}\Re\langle g(\zeta z), \frac{\partial h^2}{\partial \overline{z}}(\zeta z)\rangle > 0 \ \text{ for } \ \zeta \in \mathbb{U} \setminus \{0\}.$$

Hence $p \in \mathcal{P}$, and thus $|q_m| \leq 2$, $m \geq 1$, where $q_m = \frac{1}{m!}p^{(m)}(0)$ (see e.g. [28]). Hence, from the relation $q_{m-1} = \langle P_m(z), \frac{\partial h^2}{\partial \overline{z}}(z)\rangle$, $m \geq 2$, we obtain that $|\langle P_m(z), \frac{\partial h^2}{\partial \overline{z}}(z)\rangle| \leq 2$, for all $m \geq 2$, as desired.

To prove sharpness of (2.5) in the case $\Omega = \mathbb{B}^n(p_1, p_2, \ldots, p_n)$, let $g : \mathbb{B}^n(p_1, p_2, \ldots, p_n) \to \mathbb{C}^n$ be given by

$$g(z) = \left(z_1\frac{1+z_1}{1-z_1}, \ldots, z_n\frac{1+z_n}{1-z_n}\right), \forall\, z = (z_1, \ldots, z_n) \in \mathbb{B}^n(p_1, p_2, \ldots, p_n).$$

Then $g \in \mathcal{M}(\mathbb{B}^n(p_1, \ldots, p_n))$ by arguments similar to those in the proof of [20, Theorem 5]. Also, it is clear that $P_m(z) = 2(z_1^m, \ldots, z_n^m)$, for all $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $m \geq 2$. Since $p_j > 1$, $j = 1, \ldots, n$, it follows that $\mathbb{B}^n(p_1, \ldots, p_n)$ is a bounded balanced convex domain with C^1 plurisubharmonic defining functions (see e.g. [25]). Moreover, if $z_0 = (1, 0, \ldots, 0)$, then $z_0 \in \partial\mathbb{B}^n(p_1, \ldots, p_n)$ and $\langle z_0, \frac{\partial h^2}{\partial \overline{z}}(z_0)\rangle = 1$. Hence

$$\left|\left\langle P_m(z_0), \frac{\partial h^2}{\partial \overline{z}}(z_0)\right\rangle\right| = 2, \quad m \geq 2.$$

This completes the proof. □

Taking into account Theorem 2.7, it would be interesting to give an answer to the following question. In the case of the Euclidean unit ball $\mathbb{B}^n$ and the unit ball of $\mathbb{C}^n$ with respect to an arbitrary norm, see [14] and [18] (see also [6], [11] and [19], and the references therein).

Question 2.8. Let Ω be as in Lemma 2.1. Let $g \in \mathcal{M}(\Omega)$ and $P_m(z) = \frac{1}{m!}D^m g(0)(z^m)$, $z \in \mathbb{C}^n$, $m \in \mathbb{N}$. Find bounds for $\|P_m(z)\|$, $z \in \partial\Omega$, $m \geq 2$.

Remark 2.9. Let $n \geq 2$ and $f \in S^*(\Omega)$. Since $f(\Omega)$ is a starlike domain in $\mathbb{C}^n$, it follows that $f(\Omega)$ is a Runge domain, by a result due to El Kasimi

[12]. In view of the well known Andersén-Lempert result [1, Theorem 2.1], there is a sequence $(\psi_k)_{k\in\mathbb{N}} \subset \mathrm{Aut}(\mathbb{C}^n)$ that converges locally uniformly on Ω to f, where $\mathrm{Aut}\,(\mathbb{C}^n)$ is the family of biholomorphic automorphisms of $\mathbb{C}^n$.

Recent applications of the Andersén-Lempert theory [1] in the study of approximation properties by automorphisms of $\mathbb{C}^n$ of various compact subsets of $S(\mathbb{B}^n)$ $(n \geq 2)$ may be found in [22] and [23]; see also [35].

3. THE LOEWNER DIFFERENTIAL EQUATION AND THE FAMILY $S^0(\Omega)$

We begin this section with the following definitions (see [24], e.g. [4], [10]). Recall that $\Omega \subset \mathbb{C}^n$ and h satisfy Assumption 1.8.

Definition 3.1. A mapping $f = f(z,t) : \Omega \times [0,\infty) \to \mathbb{C}^n$ is called a Loewner chain if $e^{-t}f(\cdot,t) \in S(\Omega)$, for all $t \geq 0$, and $f_s(\Omega) \subseteq f_t(\Omega)$, $0 \leq s \leq t < \infty$, where $f_t(z) = f(z,t)$, for $z \in \Omega$, $t \geq 0$. The Loewner chain $f(z,t)$ is called a normal Loewner chain if $\{e^{-t}f_t\}_{t\geq 0}$ is a normal family on Ω.

Remark 3.2 (cf. [27], for $\Omega = \mathbb{B}^n$). Let $f = f(z,t) : \Omega \times [0,\infty) \to \mathbb{C}^n$ be a Loewner chain. Then there exists a unique univalent mapping $v(\cdot,s,t) : \Omega \to \Omega$, called the transition mapping associated with $f(z,t)$, such that $v(0,s,t) = 0$, $Dv(0,s,t) = e^{s-t}I_n$, and $f(\cdot,s) = f(v(\cdot,s,t),t)$, for $t \geq s \geq 0$.

The following semigroup property holds (see [19, Chpater 8] for $\Omega = \mathbb{B}^n$):

$$v(z,s,u) = v(v(z,s,t),t,u), \quad z \in \Omega, \quad 0 \leq s \leq t \leq u < \infty.$$

In view of Proposition 1.9 (viii) and the above equality, we obtain that $h(v(z,s,u)) \leq h(v(z,s,t))$ for $z \in \Omega$ and $u \geq t \geq s \geq 0$, and thus $h(v(z,s,\cdot))$ is a decreasing function on $[s,\infty)$ (see e.g. [19, Chpater 8] for $\Omega = \mathbb{B}^n$).

Definition 3.3. Let $g : \Omega \times [0,\infty) \to \mathbb{C}^n$ be a mapping. We say that g is a Herglotz vector field if the following conditions hold:

(i) $g(\cdot,t) \in \mathcal{M}(\Omega)$, for almost all $t \geq 0$.

(ii) $g(z,\cdot)$ is measurable on $[0,\infty)$, for all $z \in \Omega$.

Remark 3.4. In [24], a Herglotz vector field g should satisfy the following condition:

(iii) for each $T > 0$ and $r \in (0,1)$, there exists a constant $K(r,T)$ such that $\|g(z,t)\| \leq K(r,T)$ for $z \in \overline{\Omega}_r$ and $t \in [0,T]$.

We note that this condition is satisfied by the compactness of $\mathcal{M}(\Omega)$. So, the results in [24] hold without the assumption (iii).

Recall that a mapping $v \in H(\Omega)$ is called a Schwarz mapping if $h(v(z)) \le h(z)$, for all $z \in \Omega$.

The following result may be deduced by arguments similar to those in the proof of [14, Theorem 1.4] (cf. [24, Proposition 1.2 and Lemma 2.1], [27], [29]).

PROPOSITION 3.5. *Let $g = g(z,t) : \Omega \times [0,\infty) \to \mathbb{C}^n$ be a Herglotz vector field. Then for each $s \ge 0$ and $z \in \Omega$, the initial value problem*

$$\frac{\partial v}{\partial t} = -g(v,t), \qquad a.e.\ \ t \ge s, \quad v(z,s,s) = z, \tag{3.1}$$

has a unique solution $v = v(z,s,t)$ such that $v(z,s,\cdot)$ is locally absolutely continuous on $[s,\infty)$ locally uniformly with respect to $z \in \Omega$, and $v(\cdot,s,t)$ is a univalent Schwarz mapping with $Dv_{s,t}(0) = e^{s-t}I_n$, where $v_{s,t}(\cdot) = v(\cdot,s,t)$. Moreover, the limit

$$\lim_{t\to\infty} e^t v(\cdot,s,t) = f(\cdot,s) \tag{3.2}$$

exists locally uniformly on Ω, for $s \ge 0$. In addition, $f(z,s)$ is a normal Loewner chain which is locally Lipschitz continuous of $s \in [0,\infty)$ locally uniformly with respect to $z \in \Omega$ and which satisfies the generalized Loewner differential equation

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)g(z,t), \qquad a.e.\ \ t \in [0,\infty), \quad \forall\, z \in \Omega. \tag{3.3}$$

Proof. As in the proof of [19, Theorem 8.1.3], there exists a unique solution $v = v(z,s,t)$ to the initial value problem (3.1) such that $v(z,s,\cdot)$ is locally absolutely continuous on $[s,\infty)$ locally uniformly with respect to $z \in \Omega$, and $v(\cdot,s,t)$ is a univalent Schwarz mapping with $Dv_{s,t}(0) = e^{s-t}I_n$, where $v_{s,t}(\cdot) = v(\cdot,s,t)$. We give a proof of (3.2). We use arguments similar to those in the proof of [19, Theorem 8.1.5]. Fix $s \ge 0$ and let $\varphi(z,t) = e^t v(z,s,t)$ for $t \ge s$. Let $G(z,t) = g(z,t) - z$. Then, for each $r \in (0,1)$, there exist constants $K(r) > 0$ and $t(r) > s$ such that

$$h(G(e^{-t}\varphi(z,t),t)) \le K(r)e^{-2t}t^2, \quad h(z) \le r,\ t \ge t(r),$$

where h is the Minkowski function of Ω. In view of Proposition 1.9 (vii), there exists $c_1 > 0$ such that $h(z) \ge c_1\|z\|$ for $z \in \mathbb{C}^n$, and from the above relation, we have that

$$\|\varphi(z,t_2) - \varphi(z,t_1)\| \le \frac{K(r)}{c_1}\int_{t_1}^{t_2} t^2 e^{-t}dt$$

for $t_2 > t_1 > t(r)$ and $h(z) \le r$. Hence the limit (3.2) exists locally uniformly on Ω and by the Weierstrass theorem for holomorphic mappings, the limit is holomorphic on Ω. Moreover, $f(0,t)=0$ and $Df(0,t)=e^tI_n$, for $t\geq 0$, and in view of Hurwitz's theorem we deduce that $f(\cdot,t)$ is univalent on Ω, for $t\geq 0$.

As in the proof of [19, Theorem 8.1.5], we have that $f(z,s)=f(v(z,s,t),t)$, for all $z\in\Omega$ and $t\geq s\geq 0$. Hence, $f=f(z,t)$ is a Loewner chain.

In view of [24, Proposition 1.3], we obtain that

$$\frac{h(z)}{(1+h(z))^2}\leq h(e^{-t}f(z,t))\leq\frac{h(z)}{(1-h(z))^2},\quad\forall\, z\in\Omega,\ t\geq 0. \tag{3.4}$$

The above relation and Proposition 1.9 (vii) imply that $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is a locally uniformly bounded family on Ω, and thus $f(z,t)$ is a normal Loewner chain.

The fact that $f(z,t)$ is locally Lipschitz in t uniformly in z on each closed ball contained in Ω can be proved by an argument similar to that in the proof of [19, Theorem 8.1.5]. Indeed, since (see (3.4))

$$h(f(z,t))\leq e^t\frac{h(z)}{(1-h(z))^2},\quad z\in\Omega,\quad t\geq 0,$$

there exists a constant $C>0$ such that

$$\|f(z,t)\|\leq Ce^t\frac{h(z)}{(1-h(z))^2},\quad z\in\Omega,\quad t\geq 0.$$

Let $\overline{B}$ be a closed ball contained in Ω. Since $\overline{B}$ is a compact subset of Ω, there exists $r\in(0,1)$ such that $\overline{B}\subset\Omega_r=r\Omega$. There exists an open ball B_2 such that $\overline{B}\subset B_2$ and $\overline{B}_2\subset\Omega_r$. Therefore, for each $T>0$, there exists a constant $L(B,T)$ such that

$$\|f(z,t)-f(w,t)\|\leq L(B,T)\|z-w\|,\quad z,w\in B_2,\quad t\in[0,T].$$

Also, since $\mathcal{M}(\Omega)$ is compact, there exists a constant $M(B)$ such that

$$\|g(z)\|\leq M(B),\quad g\in\mathcal{M}(\Omega),\quad z\in\Omega_r.$$

Then, we have

$$\|v(z,s,t)-z\|=\left\|\int_s^t g(v(z,s,\tau),\tau)d\tau\right\|\leq M(B)(t-s),\ z\in\Omega_r, 0\leq s\leq t<\infty.$$

Therefore, there exists a positive integer $k(B)$ such that $\frac{M(B)}{k(B)}<\operatorname{dist}(B,\partial B_2)$, which implies that

$$v(z,s,t)\in B_2,\ z\in\overline{B},\ 0\leq s\leq t<\infty,\ t-s<\frac{1}{k(B)}.$$

Thus, we may conclude that

$$\|f(z,s)-f(z,t)\| \;=\; \|f(v(z,s,t),t)-f(z,t)\|$$

$$\leq L(B,T)\|v(z,s,t)-z\|$$
$$\leq L(B,T)M(B)(t-s),$$

for all $z \in \overline{B}$ and $0 \leq s \leq t \leq T$ with $t-s<\frac{1}{k(B)}$.

For any fixed positive integer $T>0$ and for any s,t with $0 \leq s \leq t \leq T$, let $s=t_0<t_1<\ldots<t_{k(B)(T+1)+1}=t$ so that $t_{i+1}-t_i<\frac{1}{k(B)}$ for all i. Then

$$\|f(z,s)-f(z,t)\| \leq \sum_{i=0}^{k(B)(T+1)} \|f(z,t_{i+1})-f(z,t_i)\|$$

$$\leq \sum_{i=0}^{k(B)(T+1)} L(B,T)M(B)(t_{i+1}-t_i) = L(B,T)M(B)(t-s).$$

This implies that $f(z,t)$ is locally Lipschitz continuous on $[0,\infty)$ on $\overline{B}$, as claimed.

Finally, we prove (3.3). We have

$$\frac{f(z,t+\eta)-f(z,t)}{\eta} = A(z,t,\eta)\left(\frac{z-v(z,t,t+\eta)}{\eta}\right), \quad z \in \Omega,\ t \geq 0,\ \eta > 0,$$

where $A(z,t,\eta)$ is a real-linear operator which tends to the invertible complex linear operator $Df(z,t)$ as $\eta \to 0^+$. Since $f(z,t)$ is locally Lipschitz in t locally uniformly in $z \in \Omega$, in view of Vitali's theorem in several complex variables, by letting $\eta \to 0^+$ in the above equality, we obtain that

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)\widetilde{g}(z,t), \quad \text{a.e. } t \in [0,\infty), \quad \forall\, z \in \Omega, \tag{3.5}$$

where $\widetilde{g}(\cdot,t) \in H(\Omega)$. Since h is C^1 on $\mathbb{C}^n \setminus \{0\}$ and $h(v(z,s,t)) \leq h(z)$ for $z \in \Omega \setminus \{0\}$, $0 \leq s \leq t < \infty$, we obtain that $\widetilde{g}(\cdot,t) \in \mathcal{M}(\Omega)$, for a.e. $t \geq 0$. Indeed, fix $z \in \Omega \setminus \{0\}$ and $t \geq 0$ be such that $\exists \lim_{\eta \to 0^+} \frac{z-v(z,t,t+\eta)}{\eta}$. Then

$$0 \leq \lim_{\eta \to 0^+} \frac{h^2(z)-h^2(v(z,t,t+\eta))}{\eta} = \lim_{\eta \to 0^+} \frac{h^2(v(z,t,t))-h^2(v(z,t,t+\eta))}{\eta}$$
$$= 2\Re\left\langle \widetilde{g}(z,t), \frac{\partial h^2}{\partial \overline{z}}(z)\right\rangle.$$

Since $f(z,s)=f(v(z,s,t),t)$, for $z \in \Omega$ and $t \geq s \geq 0$, we obtain in view of (3.5) that

$$0 = \frac{\partial f}{\partial t}(z,s) = Df(v(z,s,t),t)\frac{\partial v}{\partial t}(z,s,t) + \frac{\partial f}{\partial t}(v(z,s,t),t)$$
$$= Df(v(z,s,t),t)\big(-g(v(z,s,t),t)+\widetilde{g}(v(z,s,t),t)\big), \text{ a.e. } t \geq s \geq 0,\ \forall\, z \in \Omega.$$

Since $Df(\cdot,t)$ has a bounded inverse, we deduce from the above relation that $\widetilde{g}(v,t)=g(v,t)$ for a.e. $t \geq 0$. This implies that $\widetilde{g}(\cdot,t)=g(\cdot,t)$, for a.e. $t \geq 0$ by the identity theorem for holomorphic mappings. Hence, the Loewner differential equation (3.3) holds, as desired. This completes the proof. □

Taking into account the above result, we introduce the following notions (see [10, Definition 2.7], in the case $\Omega = \mathbb{B}^n$).

Definition 3.6. Let $g : \Omega \times [0, \infty) \to \mathbb{C}^n$ be a Herglotz vector field.

(i) The Loewner chain given by (3.2) is called the canonical solution of the Loewner differential equation (3.3) associated with g.

(ii) A mapping $f = f(z, t) : \Omega \times [0, \infty) \to \mathbb{C}^n$ is called a standard solution of the Loewner differential equation (3.3) if $f(\cdot, t) \in H(\Omega)$, $f(0, t) = 0$, $Df(0, t) = e^t I_n$, $t \geq 0$, $f(z, \cdot)$ is locally absolutely continuous on $[0, \infty)$ locally uniformly with respect to $z \in \Omega$, and f satisfies the Loewner differential equation (3.3).

It is of interest to study properties of standard solutions of (3.3) and connections with the canonical solution of (3.3). In the case of the unit ball $\mathbb{B}^n$, see [10] (see [2], [4], [37]).

The following result yields that the standard solutions $f = f(z, t)$ of the Loewner differential equation (3.3), such that $\{e^{-t} f(\cdot, t)\}_{t\geq 0}$ is a normal family on Ω, provide Loewner chains (cf. [27, Theorem 2.2] and [14, Lemma 1.6], in the case $\Omega = \mathbb{B}^n$).

THEOREM 3.7. *Let $g : \Omega \times [0, \infty) \to \mathbb{C}^n$ be a Herglotz vector field. Assume that $f : \Omega \times [0, \infty) \to \mathbb{C}^n$ is a standard solution of the Loewner differential equation* (3.3) *associated to g. Also, assume that there exists an increasing sequence $\{t_m\}_{m\in\mathbb{N}}$ such that $0 < t_m \to \infty$ and $\lim_{m\to\infty} e^{-t_m} f(\cdot, t_m) = F$ locally uniformly on Ω, where $F \in H(\Omega)$. Then $f = f(z, t)$ is a Loewner chain such that*

$$f(\cdot, s) = \lim_{t\to\infty} e^t v(\cdot, s, t)$$

locally uniformly on Ω, for each $s \geq 0$, where $v = v(z, s, t)$ is the unique locally absolutely continuous solution on $[s, \infty)$ of the initial value problem (3.1). *Hence $f = f(z, t)$ is the canonical solution of* (3.3).

Proof. As in the proof of [19, Theorem 8.1.6], we obtain that $f(z, s) = f(v(z, s, t), t)$, for $z \in \Omega$, $t \geq s \geq 0$.

Next, we prove that

$$f(\cdot, s) = \lim_{t\to\infty} e^t v(\cdot, s, t)$$

locally uniformly on Ω, for all $s \geq 0$. We use an argument similar to that in the proof of [19, Theorem 8.1.6]. Since $\lim_{m\to\infty} e^{-t_m} f(\cdot, t_m) = F$ locally uniformly on Ω, we deduce that for each $\rho \in (0, 1)$ there exists $K_0(\rho) > 0$ such that $h(e^{-t_m} f(z, t_m) - z) \leq K_0(\rho)$ for $h(z) \leq \rho$. By Proposition 1.10, we have

$$h\big(e^{-t_m} f(z, t_m) - z\big) \leq K_0(\rho) \frac{h^2(z)}{\rho^2}, \quad h(z) \leq \rho, \quad m \in \mathbb{N}.$$

Since $f(z,s)=f(v(z,s,t),t)$, for $z\in\Omega$, $t\geq s\geq 0$, we obtain from the above relations and [24, Proposition 1.3] that

$$h\big(f(z,s)-e^{t_m}v(z,s,t_m)\big)\leq \frac{K_0(\rho)}{(1-\rho)^4}e^{2s-t_m},\quad m\in\mathbb{N},\quad h(z)\leq\rho.$$

This implies that $\lim_{m\to\infty}e^{t_m}v(\cdot,s,t_m)=f(\cdot,s)$ locally uniformly on Ω. Since the limit $\lim_{t\to\infty}e^t v(\cdot,s,t)$ exists locally uniformly on Ω, for $s\geq 0$, by Proposition 3.5, we obtain that

$$f(\cdot,s)=\lim_{t\to\infty}e^t v(\cdot,s,t)$$

locally uniformly on Ω. This completes the proof. □

Next, we recall the notion of parametric representation on bounded balanced pseudoconvex domains with C^1 plurisubharmonic defining functions (cf. [24], [14] and [19] and the references therein; see also [29], [30] for applications of this notion on the unit polydisc in $\mathbb{C}^n$).

Definition 3.8. A normalized mapping $f\in H(\Omega)$ has parametric representation (denoted by $f\in S^0(\Omega)$) if there exists a Herglotz vector field $g:\Omega\times[0,\infty)\to\mathbb{C}^n$ such that $f=\lim\limits_{t\to\infty}e^t v(\cdot,t)$ locally uniformly on Ω, where $v(z,t)=v(z,0,t)$ and $v(z,s,\cdot)$ is the unique locally absolutely continuous solution on $[s,\infty)$ of the initial value problem (3.1), for all $z\in\Omega$ and $s\geq 0$.

Taking into account Proposition 3.5 (and the arguments used in its proof) and Theorem 3.7, we obtain the following characterization of parametric representation on Ω, which is a generalization of the corresponding result in the case on the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ (see [14] and [19, Chapter 8]).

Note that if $f=f(z,t):\mathbb{B}^n\times[0,\infty)\to\mathbb{C}^n$ is a Loewner chain, then $f(z,\cdot)$ is locally Lipschitz continuous on $[0,\infty)$ locally uniformly with respect to $z\in\mathbb{B}^n$ (see [14] and [19, Theorem 8.1.8]).

THEOREM 3.9. *Let $f:\Omega\to\mathbb{C}^n$ be a normalized holomorphic mapping. Then $f\in S^0(\Omega)$ if and only if there exists a normal Loewner chain $(f_t)_{t\geq 0}$ which is locally Lipschitz continuous on $[0,\infty)$ locally uniformly with respect to $z\in\Omega$, such that $f=f_0$.*

Example 3.10. Let $f:\Omega\to\mathbb{C}^n$ be a normalized holomorphic mapping. Then $f\in S^*(\Omega)$ if and only if $f(z,t)=e^t f(z)$ is a Loewner chain (see [36] in the case $\Omega=\mathbb{B}^n$; cf. [24]).

The following growth result for $S^0(\Omega)$ was obtained in [24, Theorem 2.4].

Theorem 3.11. *If $f \in S^0(\Omega)$, then*

$$\frac{h(z)}{(1+h(z))^2} \leq h(f(z)) \leq \frac{h(z)}{(1-h(z))^2}, \quad \forall\, z \in \Omega.$$

This result is sharp in the case $\Omega = \mathbb{B}^n(p_1, \dots, p_n)$, where $p_j > 1$, $j = 1, \dots, n$.

In view of Theorems 3.9, 3.11 and the fact that Lipschitz constants for $f(z,s)$ in Proposition 3.5 do not depend on the chain $f(z,s)$, we obtain the following compactness result, by using arguments similar to those in the proof of [19, Corollary 8.3.11]. This result is important in the study of extremal problems associated with the family $S^0(\Omega)$ (see [7], [15], [16], [17], [21], [35], in the case $\Omega = \mathbb{B}^n$).

Theorem 3.12. *The family $S^0(\Omega)$ is a compact subset of $H(\Omega)$.*

Using arguments similar to those in the proof of [10, Theorem 3.1] (see also [19, Chapter 8]), we deduce the following covering result related to the Loewner range of a normal Loewner chain on Ω.

Proposition 3.13. *Let $(f_t)_{t\geq 0}$ be a normal Loewner chain on Ω, which is locally Lipschitz continuous on $[0,\infty)$ locally uniformly with respect to $z \in \Omega$, and let $\mathcal{R}(f_t) := \bigcup_{t\geq 0} f_t(\Omega)$ be the Loewner range of $(f_t)_{t\geq 0}$. Then $\mathcal{R}(f_t) = \mathbb{C}^n$.*

It would be interesting to give an answer to the following question. Note that if $\Omega = \mathbb{B}^n$, the answer to Question 3.14 is positive (see [35]).

Question 3.14. Let $(f_t)_{t\geq 0}$ be a normal Loewner chain on Ω. Then is it true that $\mathcal{R}(f_t) = \mathbb{C}^n$?

4. QUESTIONS AND CONJECTURES

Taking into account Proposition 3.5 and Theorem 3.7, we propose the following conjecture, which is true in the case $\Omega = \mathbb{B}^n$ (see e.g. [3], [14], [19, Chapter 8] and the references therein, in the case $\Omega = \mathbb{B}^n$).

Conjecture 4.1. *Let $f : \Omega \times [0,\infty) \to \mathbb{C}^n$ be a Loewner chain and let $v = v(z,s,t)$ be its transition mapping. Then the following statements hold:*

(i) *The mapping $v(z,s,\cdot)$ is locally Lipschitz continuous on $[s,\infty)$ locally uniformly with respect to $z \in \Omega$, for $s \geq 0$.*

(ii) *The mapping $f(z,\cdot)$ is locally Lipschitz continuous on $[0,\infty)$ locally uniformly with respect to $z \in \Omega$, and there exists a Herglotz vector field $g : \Omega \times [0,\infty) \to \mathbb{C}^n$ such that*

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)g(z,t), \ \ a.e.\ \ t \in [0,\infty), \quad \forall\, z \in \Omega.$$

Let $\mathrm{Aut}\,(\mathbb{C}^n)$ be the family of biholomorphic automorphisms of $\mathbb{C}^n$ onto itself. Also, let $S^1(\Omega)$ be the family of all mappings $f \in S(\Omega)$ for which there exists a Loewner chain $(f_t)_{t\geq 0}$ on Ω with $\mathcal{R}(f_t) = \mathbb{C}^n$, such that $f_0 = f$.

Taking into account Theorem 3.9 and Proposition 3.13, we deduce that $S^0(\Omega) \subseteq S^1(\Omega)$. If $\Omega = \mathbb{B}^n$, $n \geq 2$, we have that $S^0(\mathbb{B}^n) \subsetneq S^1(\mathbb{B}^n)$ (see [14]).

Next, we consider the following conjecture, which is true in the case $\Omega = \mathbb{B}^n$ (see [10]; see also [3]).

Conjecture 4.2. *Let $(f_t)_{t\geq 0}$ be a Loewner chain on Ω. Then there exist a normal Loewner chain $(g_t)_{t\geq 0}$ on Ω and an entire normalized biholomorphic mapping $\Psi : \mathbb{C}^n \to \mathbb{C}^n$ such that $f_t = \Psi \circ g_t$, for all $t \geq 0$. In particular, if $f \in S^1(\Omega)$, then there exist $\Phi \in \mathrm{Aut}(\mathbb{C}^n)$ and $g \in S^0(\Omega)$ such that $f = \Phi \circ g$.*

Taking into account Lemma 2.3, it would be interesting to give an answer to the following question. In the case of the Euclidean unit ball $\mathbb{B}^n$, see [14] (see also [6], and [19], and the references therein).

Question 4.3. Let $f \in S^0(\Omega)$ and let $P_m(z) = \frac{1}{m!}D^m f(0)(z^m)$, for $z \in \mathbb{C}^n$, $m \geq 2$. Find coefficient bounds for $\|P_m(z)\|$, $z \in \partial\Omega$, $m \geq 2$. In particular, find coefficient bounds for $\|P_m(z)\|$, $m \geq 2$, in the case that $f \in S^*(\Omega)$.

It is of interest to give an answer to the following question regarding the distortion result for the family $S^0(\Omega)$.

Question 4.4. Find sharp estimates for $\|Df(z)\|$ and $|\det Df(z)|$, $z \in \Omega$, in the case that $f \in S^0(\Omega)$ (in particular, for $f \in S^*(\Omega)$).

Next, we introduce the following families of bounded univalent mappings on Ω. These families were intensively investigated in the case $\Omega = \mathbb{B}^n$ (see [7], [15], [16], [21]).

The following notions were introduced in [15] in the case $\Omega = \mathbb{B}^n$ (cf. [34]):

Definition 4.5. Let $E \subseteq [0, \infty)$ be an interval and let $\Delta \subseteq H(\Omega)$ be a normal family. A mapping $g = g(z,t) : \Omega \times E \to \mathbb{C}^n$ is called a Carathéodory mapping on E with values in Δ if the following conditions hold:

(i) $g(\cdot, t) \in \Delta$ for a.e. $t \in E$.

(ii) $g(z, \cdot)$ is a measurable mapping on E, for all $z \in \Omega$.

We denote by $\mathcal{C}(E, \Delta)$ the family of all Carathéodory mappings on E with values in Δ.

Motivated by Proposition 3.5 and Theorem 2.6, we consider the following definition (see [15] for $\Omega = \mathbb{B}^n$).

Definition 4.6. Let $T \in (0, \infty)$, let $\Delta \subseteq \mathcal{M}(\Omega)$, $g \in \mathcal{C}([0,T], \Delta)$, and let $v = v(z,t;g)$ be the unique Lipschitz continuous solution on $[0,T]$ of the initial value problem

$$\frac{\partial v}{\partial t}(z,t) = -g(v(z,t),t) \text{ a.e. } t \in [0,T], \quad v(z,0) = z, \tag{4.1}$$

for $z \in \Omega$, such that $v(\cdot, t; g)$ is a univalent Schwarz mapping and $Dv(0,t;g) = e^{-t} I_n$ for $t \in [0,T]$. Also let

$$\widetilde{\mathcal{R}}_T(\mathrm{id}_\Omega, \Delta) := \Big\{ e^T v(\cdot, T; g) : g \in \mathcal{C}([0,T], \Delta) \Big\},$$

where $\mathrm{id}_\Omega(z) = z$, $\forall\, z \in \Omega$. The family $\widetilde{\mathcal{R}}_T(\mathrm{id}_\Omega, \Delta)$ is called the normalized time-T-reachable family of (4.1) (see [15] in the case $\Omega = \mathbb{B}^n$; cf. [34]).

For $M \in (1, \infty)$, let

$$S^0(M, \Omega) := \big\{ f \in S^0(\Omega) : h(f(z)) < M,\ z \in \Omega \big\}.$$

Remark 4.7. If $n = 1$, then $S^0(M, \mathbb{U}) = S(M)$, where $S(M)$ is the family of all functions in S which are uniformly bounded on $\mathbb{U}$ by the constant M. However, if $n \geq 2$, then $\widetilde{\mathcal{R}}_{\log M}(\mathrm{id}_{\mathbb{B}^n}, \mathcal{M}(\mathbb{B}^n)) \subsetneq S^0(M, \mathbb{B}^n)$ (see [7]).

Question 4.8. Is it true that $\widetilde{\mathcal{R}}_{\log M}(\mathrm{id}_\Omega, \mathcal{M}(\Omega)) \subsetneq S^0(M, \Omega)$, for $n \geq 2$?

Question 4.9. Find sharp estimates for $h(f(z))$ and $\|Df(z)\|$, $z \in \Omega$, in the case that $f \in \widetilde{\mathcal{R}}_{\log M}(\mathrm{id}_\Omega, \mathcal{M}(\Omega))$.

In view of Theorem 3.12, we deduce that the family $S^0(M, \Omega)$ is a compact subset of $S^0(\Omega)$. Regarding the family $\widetilde{\mathcal{R}}_{\log M}(\mathrm{id}_\Omega, \mathcal{M}(\Omega))$, we propose the following conjecture, which is true in the case $\Omega = \mathbb{B}^n$ (see [16, Corollary 4.7]).

CONJECTURE 4.10. *$\widetilde{\mathcal{R}}_{\log M}(\mathrm{id}_\Omega, \mathcal{M}(\Omega))$ is a compact subset of $S^0(\Omega)$.*

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REFERENCES

[1] E. Andersén and L. Lempert, *On the group of holomorphic automorphisms of* $\mathbb{C}^n$. Invent. Math. **110** (1992), 371–388.

[2] L. Arosio, *Resonances in Loewner equations.* Adv. Math. **227** (2011), 1413–1435.

[3] L. Arosio, F. Bracci, H. Hamada, and G. Kohr, *An abstract approach to Loewner chains.* J. Anal. Math. **119** (2013), 89–114.

[4] L. Arosio, F. Bracci, and E.F. Wold, *Solving the Loewner PDE in complete hyperbolic starlike domains of* $\mathbb{C}^n$. Adv. Math. **242** (2013), 209–216.

[5] T.J. Barth, *The Kobayashi indicatrix at the center of a circular domain.* Proc. Amer. Math. Soc. **88** (1983), 527–530.

[6] F. Bracci, *Shearing process and an example of a bounded support function in $S^0(\mathbb{B}^2)$.* Comput. Methods Funct. Theory. **15** (2015), 151–157.

[7] F. Bracci, I. Graham, H. Hamada, and G. Kohr, *Variation of Loewner chains, extreme and support points in the class S^0 in higher dimensions.* Constructive Approx. **43** (2016), 231–251.

[8] S. Dineen, *The Schwarz Lemma.* Oxford Mathematical Monographs, Clarendon Press, 1989.

[9] P.G. Dixon and J. Esterle, *Michael's problem and the Poincare-Fatou-Bieberbach phenomenon.* Bull. Amer. Math. Soc. New. Ser. **15** (1986), 127–187.

[10] P. Duren, I. Graham, H. Hamada, and G. Kohr, *Solutions for the generalized Loewner differential equation in several complex variables.* Math. Ann. **347** (2010), 411–435.

[11] M. Elin, S. Reich, and D. Shoikhet, *Numerical Range of Holomorphic Mappings and Applications.* Birkhäuser/Springer, Cham, 2019.

[12] A. El Kasimi, *Approximation polynômiale dans les domaines étoilés de $\mathbb{C}^n$.* Complex Var. Theory Appl. **10** (1988), 179–182.

[13] S. Gong, *Convex and Starlike Mappings in Several Complex Variables.* Kluwer Acad. Publ. Dordrecht, 1998.

[14] I. Graham, H. Hamada, and G. Kohr, *Parametric representation of univalent mappings in several complex variables.* Canad. J. Math. **54** (2002), 324–351.

[15] I. Graham, H. Hamada, G. Kohr, and M. Kohr, *Extreme points, support points and the Loewner variation in several complex variables.* Sci. China Math. **55** (2012), 1353–1366.

[16] I. Graham, H. Hamada, G. Kohr, and M. Kohr, *Extremal properties associated with univalent subordination chains in $\mathbb{C}^n$.* Math. Ann. **359** (2014), 61–99.

[17] I. Graham, H. Hamada, G. Kohr, and M. Kohr, *Support points and extreme points for mappings with A-parametric representation in $\mathbb{C}^n$.* J. Geom. Anal. **26** (2016), 1560–1595.

[18] I. Graham H. Hamada, T. Honda, G. Kohr, and K.H. Shon, *Growth, distortion and coefficient bounds for Carathéodory families in $\mathbb{C}^n$ and complex Banach spaces.* J. Math. Anal. Appl. **416** (2014), 449–469.

[19] I. Graham and G. Kohr, *Geometric Function Theory in One and Higher Dimensions.* Marcel Dekker Inc., New York, 2003.

[20] H. Hamada, *Starlike mappings on bounded balanced domains with C^1-plurisubharmonic defining functions.* Pacif. J. Math. **194** (2000), 359–371.

[21] H. Hamada, M. Iancu, and G. Kohr, *Extremal problems for mappings with generalized parametric representation in $\mathbb{C}^n$.* Complex Anal. Oper. Theory. **10** (2016), 1045–1080.

[22] H. Hamada, M. Iancu, and G. Kohr, *Approximation of univalent mappings by automorphisms and quasiconformal diffeomorphisms in $\mathbb{C}^n$.* J. Approx. Theory. **240** (2019), 129–144.

[23] H. Hamada, M. Iancu, G. Kohr, and S. Schleissinger, *Approximation properties of univalent mappings on the unit ball in $\mathbb{C}^n$.* J. Approx. Theory. **226** (2018), 14–33.

[24] H. Hamada and G. Kohr, *Subordination chains and univalence of holomorphic mappings on bounded balanced pseudoconvex domains.* Ann. Univ. Mariae Curie Sklodowska. Sect. A **55** (2001), 61–80.

[25] M. Jarnicki and P. Pflug, *Invariant distances and metrics in complex analysis*. De Gruyter, 2nd Extended Edition. Berlin, 2013.

[26] H. Liu, Z. Zhang, and K. Lu, *The parametric representation for spiral-like mappings of type α on bounded balanced pseudoconvex domains*. Acta Math. Sci. **26B** (2006), 421–430.

[27] J.A. Pfaltzgraff, *Subordination chains and univalence of holomorphic mappings in $\mathbb{C}^n$*. Math. Ann. **210** (1974), 55–68.

[28] C. Pommerenke, *Univalent Functions*. Vandenhoeck & Ruprecht. Göttingen, 1975.

[29] T. Poreda, *On the univalent holomorphic maps of the unit polydisc in $\mathbb{C}^n$ which have the parametric representation, I-the geometrical properties*. Ann. Univ. Mariae Curie Skl. Sect. A **41** (1987), 105–113.

[30] T. Poreda, *On the univalent holomorphic maps of the unit polydisc in $\mathbb{C}^n$ which have the parametric representation, II-the necessary conditions and the sufficient conditions*. Ann. Univ. Mariae Curie Skl. Sect. A. **41** (1987), 115–121.

[31] T. Poreda, *On generalized differential equations in Banach spaces*. Dissertationes Mathematicae **310** (1991), 1–50.

[32] M. Range, *Holomorphic Functions and Integral Representations in Several Complex Variables*. Springer, New York, 1986.

[33] K. Roper and T.J. Suffridge, *Convexity properties of holomorphic mappings in $\mathbb{C}^n$*. Trans. Amer. Math. Soc. **351** (1999), 1803–1833.

[34] O. Roth, *Control theory in $\mathcal{H}(\mathbb{D})$*. Diss. Bayerischen Univ. Wuerzburg, 1998.

[35] S. Schleissinger, *On support points of the class $S^0(\mathbb{B}^n)$*. Proc. Amer. Math. Soc. **142** (2014), 3881–3887.

[36] T.J. Suffridge, *Starlikeness, convexity and other geometric properties of holomorphic maps in higher dimensions*. In: Lecture Notes Math. Springer, Berlin, **599** (1977), 146–159.

[37] M. Voda, *Solution of a Loewner chain equation in several complex variables*. J. Math. Anal. Appl. **375** (2011), 58–74.

[38] Q.H. Xu and T. Liu, *Biholomorphic mappings on bounded starlike circular domains*. J. Math. Anal. Appl. **366** (2010), 153–163.

[39] Q.H. Xu and T. Liu, *Subordination chains and biholomorphic mappings on bounded balanced pseudoconvex domains*. Comput. Math. Appl. **61** (2011), 1830–1836.

Kyushu Sangyo University
Faculty of Science and Engineering
3-1 Matsukadai, 2-Chome, Higashi-ku
Fukuoka 813-8503, Japan
h.hamada@ip.kyusan-u.ac.jp

Babeş-Bolyai University
Faculty of Mathematics and Computer Science
1 M. Kogălniceanu Str., 400084 Cluj-Napoca, Romania
miancu@math.ubbcluj.ro

Babeş-Bolyai University
Faculty of Mathematics and Computer Science
1 M. Kogălniceanu Str., 400084 Cluj-Napoca, Romania
gkohr@math.ubbcluj.ro