Dedicated to the memory of Professor Cabiria Andreian-Cazacu

ANALYTIC FUNCTIONAL CALCULUS IN QUATERNIONIC FRAMEWORK

FLORIAN-HORIA VASILESCU

Communicated by Aurelian Gheondea

Regarding quaternions as normal matrices, we first characterize the 2×2 matrix-valued functions, defined on subsets of quaternions, whose values are quaternions. Then we investigate the regularity of quaternionic-valued functions, defined by the analytic functional calculus, showing its equivalence with the slice hyperholomorphy.

AMS 2010 Subject Classification: 47A10; 30G35; 30A05; 47A60.

Key words: spectrum of a quaternion, quaternionic valued functions, analytic functional calculus for quaternions.

1. INTRODUCTION

Introduced in science by W. R. Hamilton as early as 1843, the quaternions form a unital non commutative division algebra, with numerous applications in mathematics and physics. In mathematics, the celebrated Frobenius theorem, proved in 1877, placed the algebra of quaternions among the only three finite dimensional division algebras over the real numbers, which is a remarkable feature shared with the real and complex fields.

Concerning physics, one can find a first suggestion of a quaternion quantum mechanics in a footnote of a paper by G. Birkhoff and J. von Neumann (see [1]), as mentioned in [6]. The work [6] itself presents a quaternionic quantum mechanics, using various entities assuming quaternionic values. The existence of serious physical hypotheses incited several mathematicians to develop a branch of analysis in the framework of quaternions.

One of the most important investigation in the quaternionic context has been to find a convenient manner to express the "analyticity" of functions depending on quaternions. Among the pioneer contributions in this direction one should mention the works [14] and [7].

More recently, a concept of *slice regularity (or hyperholomorphy)* for functions of one quaternionic variable has been introduced in [9], leading to a large development sythesized in [4] (which contains a large list of references), whose impact is still actual (see [10], [3], etc.).

Unlike in [9], the basic idea of the present paper is to investigate the regularity of quaternionic-valued functions, which are defined by an analytic functional calculus acting on quaternions. We have chosen to consider the algebra of quaternions not as an abstract object but as a real subalgebra of the complex algebra of 2×2 matrices with complex entries, which is a useful classica representation appearing in many works (see for instance [2]). Among the advantages of this representation is that we may view the quaternions as linear operators actually on complex spaces, commuting with the complex numbers. Another one is to regard each quaternion as a normal operator, having a spectrum which can be used to define various compatible functional calculi, including the analytic one.

One of the main results of this work is Theorem 1, giving a characterization of those matrix-valued functions, defined on some open sets in the complex plane, producing quaternions when applied, by functional calculus, to quaternions having spectra in their domain of definition. Such a function, temporarily called *skew conjugate symmetric*, corresponds to the more known concept of *stem function* (a notion going back to [7]), transposed in our framework (see Remarks 4 and 5).

Roughly speaking, and unlike in [9], a "quaternionic regular function" can and will be obtained by a pointwise application of the analytic functional calculus with stem functions on a conjugate symmetric open set U in the complex plane, to quaternions whose spectra are in U, via the matrix version of Cauchy's formula (10), with no need of slice derivatives. Moreover, unlike in [9], our Cauchy type kernel is partially commutative, making many arguments more transparent. In this way, we obtain a whole class of "regular functions" (in fact, quaternionic Cauchy transforms of stem functions; see Theorem 2). Of course, we can recapture, with our methods and in our terms, several properties of slice regular functions (see Remark 11 etc.), presented for example in [4].

The discussion concerning the "regularity" of the quaternionic-valued functions is ended with a comaparison between our concept of regularity with that of slice regularity (Theorem 6), showing that these concepts are equivalent on open sets called in this work *spectrally saturated*, which happen to be axially symmetric sets, introduced in [4] (see Proposition 1).

Finally, we should mention that this text is largely inspired by our homonym work [15], where the reader can find more details about some proofs, which are often presented here in a shorter form.

2. HAMILTON'S ALGEBRA

We start this discussion with some well known facts. Abstract Hamilton's algebra $\mathbb{H}_0$ is the four-dimensional $\mathbb{R}$-algebra with unit 1, generated by $\{\mathbf{j},\mathbf{k},\mathbf{l}\}$, where $\mathbf{j},\mathbf{k},\mathbf{l}$ satisfy

$$\mathbf{jk}=-\mathbf{kj}=\mathbf{l},\ \mathbf{kl}=-\mathbf{lk}=\mathbf{j},\ \mathbf{lj}=-\mathbf{jl}=\mathbf{k},\ \mathbf{jj}=\mathbf{kk}=\mathbf{ll}=-1.$$

In this work, *Hamiltonn's algebra* (or *algebra of quaternions*) will be identified with an $\mathbb{R}$-subalgebra of $\mathbb{M}_2$ of 2×2 matrices with complex entries. Specifically, and using a well-known idea, one considers the following 2×2-matrices with complex entries

$$\mathbf{I}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \mathbf{J}=\begin{pmatrix}i&0\\0&-i\end{pmatrix},\quad \mathbf{K}=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad \mathbf{L}=\begin{pmatrix}0&i\\i&0\end{pmatrix},$$

with $i^2=-1$. Because we have

$$\mathbf{J}^2=\mathbf{K}^2=\mathbf{L}^2=-\mathbf{I},$$

$$\mathbf{JK}=\mathbf{L}=-\mathbf{KJ},\ \mathbf{KL}=\mathbf{J}=-\mathbf{LK},\ \mathbf{LJ}=\mathbf{K}=-\mathbf{JL},$$

the assignment

$$\mathbb{H}_0\ni x_0+x_1\mathbf{j}+x_2\mathbf{k}+x_3\mathbf{l}\mapsto x_0\mathbf{I}+x_1\mathbf{J}+x_2\mathbf{K}+x_3\mathbf{L}\in\mathbb{M}_2 \tag{1}$$

is an injective unital $\mathbb{R}$-algebra morphism, which is also an isometry. For this reason, from now on, the algebra of quaternions, denoted by $\mathbb{H}$, is defined as the $\mathbb{R}$-subalgebra of the algebra $\mathbb{M}_2$, generated by the matrices $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$ and $\mathbf{L}$. Although this identification is not canonical, the realization of $\mathbb{H}_0$ as a matrix algebra $\mathbb{H}$ offers more freedom when acting with its elements, as we shall see in the sequel. In particular, the quaternions in this paper commute with complex scalars because they are, in fact, *matricial quaternions*.

We shall often use the notation

$$Q(\mathbf{z})=\begin{pmatrix}z_1&z_2\\-\bar{z}_2&\bar{z}_1\end{pmatrix}$$

for every $\mathbf{z}=(z_1,z_2)\in\mathbb{C}^2$, and noticing that

$$Q(\mathbf{z})=\Re z_1\mathbf{I}+\Im z_1\mathbf{J}+\Re z_2\mathbf{K}+\Im z_2\mathbf{L},$$

and $\mathbf{I}=Q((1,0))$, $\mathbf{J}=Q((i,0))$, $\mathbf{K}=Q((0,1))$, $\mathbf{L}=Q((0,i))$, we obtain that the map $\mathbb{C}^2\ni\mathbf{z}\mapsto Q(\mathbf{z})\in\mathbb{H}$ is $\mathbb{R}$-linear and bijective. In other words, the set $\mathbb{C}^2$ can be identified, as an $\mathbb{R}$-vector space, with the algebra $\mathbb{H}$. For technical reasons, we often represent a fixed element of $\mathbb{H}$ under the form $Q(\mathbf{z})$, for some $\mathbf{z}\in\mathbb{C}^2$ uniquely determined, via the assignment (1).

Regarding the elements of $\mathbb{M}_2$ as linear maps acting on the space $\mathbb{C}^2$, endowed with the natural scalar product $\langle \mathbf{z}, \mathbf{w} \rangle = z_1\bar{w}_1 + z_2\bar{w}_2$ and the associated norm $\|\mathbf{z}\|^2 = |z_1|^2 + |z_2|^2$, $\mathbf{z} = (z_1, z_2), \mathbf{w} = (w_1, w_2) \in \mathbb{C}^2$, we see that the algebra $\mathbb{H}$ also has a natural involution $Q(\mathbf{z}) \mapsto Q(\mathbf{z})^*$, $\mathbf{z} \in \mathbb{C}^2$, where $\mathbf{z} = (z_1, z_2)$,

$$Q(\mathbf{z})^* = \begin{pmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{pmatrix} = Q(\mathbf{z}^*),$$

with $\mathbf{z}^* = (\bar{z}_1, -z_2)$. In particular, $\mathbf{J}^* = -\mathbf{J}$, $\mathbf{K}^* = -\mathbf{K}$, $\mathbf{L}^* = -\mathbf{L}$.

It is easily seen that $Q(\mathbf{z})Q(\mathbf{z})^* = Q(\mathbf{z})^*Q(\mathbf{z}) = \|\mathbf{z}\|^2\mathbf{I}$ for all $\mathbf{z} \in \mathbb{C}^2$, and so $Q(\mathbf{z})$ is a normal matrix for each $\mathbf{z} \in \mathbb{C}^2$. Moreover, $\|Q(\mathbf{z})\| = \|\mathbf{z}\|$ for all $\mathbf{z} \in \mathbb{C}^2$, that is, the map $\mathbb{C}^2 \ni \mathbf{z} \mapsto Q(\mathbf{z}) \in \mathbb{H}$ is an isometry. In addition, $Q(\mathbf{z})^{-1} = \|\mathbf{z}\|^{-2}Q(\mathbf{z})^*$ for all $\mathbf{z} \in \mathbb{C}^2 \setminus \{0\}$, so every nonnull element of $\mathbb{H}$ is invertible.

Let us clarify the position of the $\mathbb{R}$-subalgebra $\mathbb{H}$ into the $\mathbb{C}$-algebra $\mathbb{M}_2$.

Remark 1. On the algebra $\mathbb{M}_2$ we define what we will call a *skew complex conjugation*, setting

$$\mathbf{a}^\sim := \begin{pmatrix} \bar{a}_4 & -\bar{a}_3 \\ -\bar{a}_2 & \bar{a}_1 \end{pmatrix} \text{ for every } \mathbf{a} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathbb{M}_2.$$

The map $\mathbf{a} \mapsto \mathbf{a}^\sim$ is conjugate homogeneous and additive, in particular $\mathbb{R}$-linear, multiplicative, unital, $(\mathbf{a}^\sim)^\sim = \mathbf{a}$, and $(\mathbf{a}^*)^\sim = (\mathbf{a}^\sim)^*$. In addition, $\mathbf{a} = \mathbf{a}^\sim$ if and only if $\mathbf{a}$ is a quaternion.

Being a $*$-automorphism $\mathbb{R}$-linear, the map $\mathbf{a} \mapsto \mathbf{a}^\sim$ is actually an isometry.

Note that $\mathbf{a} = \frac{\mathbf{a}+\mathbf{a}^\sim}{2} + i\frac{\mathbf{a}-\mathbf{a}^\sim}{2i}$, $\mathbf{a} \in \mathbb{M}_2$, with $\mathbf{a} + \mathbf{a}^\sim$, $i(\mathbf{a} - \mathbf{a}^\sim) \in \mathbb{H}$. In other words, $\mathbb{M}_2 = \mathbb{H} + i\mathbb{H}$. We also have $\mathbb{H} \cap i\mathbb{H} = \{0\}$. Indeed, if $q = ir$ with $q, r \in \mathbb{H}$, we have $q^\sim = q = (ir)^\sim = -ir = -q$, whence $q = 0$, showing that the decomposition $\mathbb{M}_2 = \mathbb{H} + i\mathbb{H}$ is a direct sum.

A map similar to the skew complex conjugation is defined, under the name of *reflexion*, in C^*-algebras (see [13], Definition 2.6.). Nevertheless, a reflexion is an anti-automorphism.

3. A SPECTRAL APPROACH TO $\mathbb{H}$-VALUED FUNCTIONS

As before, the space $\mathbb{C}^2$ is endowed with its natural scalar product $\langle *, * \rangle$, and norm $\| * \|$. We also have the $\mathbb{R}$-linear the map $\mathbb{C}^2 \ni \mathbf{z} \mapsto Q(\mathbf{z}) \in \mathbb{H}$, which is a bijective isometry. In other words, giving a quaternion $q \in \mathbb{H}$, there is a unique point $\mathbf{z}_q \in \mathbb{C}^2$ such that $q = Q(\mathbf{z}_q)$. Moreover, the algebra $\mathbb{H}$ will be regarded as an $\mathbb{R}$-subalgebra of the $\mathbb{C}$-algebra $\mathbb{M}_2$. In this way, the elements of

$\mathbb{H}$ will be regarded as linear operators on $\mathbb{C}^2$, so every element $Q(\mathbf{z})$ is a normal operator on the Hilbert space $\mathbb{C}^2$, whose spectrum is denoted by $\sigma(Q(\mathbf{z}))$.

Occasionally, we use the notation $\Re q = \Re z_1$ and $\|q\| = \|Q(\mathbf{z})\|$ if $q = Q(\mathbf{z})$ and $\mathbf{z} = (z_1, z_2)$.

We start with an elementary result:

LEMMA 1. *Let $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$ be fixed. The spectrum $\sigma(Q(\mathbf{z})) = \{s_\pm(\mathbf{z})\}$ of the normal operator $Q(\mathbf{z})$ is given by*

$$s_\pm(\mathbf{z}) = \Re z_1 \pm i\sqrt{(\Im z_1)^2 + |z_2|^2},\ \mathbf{z} = (z_1, z_2) \in \mathbb{C}^2. \tag{2}$$

We have $s_+(\mathbf{z}) = \overline{s_-(\mathbf{z})}$, and the points $s_+(\mathbf{z}), s_-(\mathbf{z})$ are distinct if and only if $Q(\mathbf{z}) \notin \mathbb{R}\mathbf{I}$. Moreover:

(a) *if $z_2 \neq 0$, the elements*

$$\nu_\pm(\mathbf{z}) = (\sqrt{|z_2|^2 + |s_\pm(\mathbf{z}) - z_1|^2})^{-1}(z_2, s_\pm(\mathbf{z}) - z_1) \in \mathbb{C}^2,\ \mathbf{z} = (z_1, z_2) \in \mathbb{C}^2 \tag{3}$$

are eigenvectors corresponding to the eigenvalues $\{s_\pm(\mathbf{z})\}$ respectively, and they form an orthonormal basis of the Hilbert space $\mathbb{C}^2$;

(b) *if $z_2 = 0$ but $\Im z_1 \neq 0$, we have $\sigma(Q(\mathbf{z})) = \{s_\pm(\mathbf{z})\}$, with $s_+(\mathbf{z}) = z_1, s_-(\mathbf{z}) = \bar{z}_1$, and $\nu_+(\mathbf{z}) = (1, 0)$, $\nu_-(\mathbf{z}) = (0, 1)$ are eigenvectors corresponding the eigenvalues $z_1, \bar{z}_1$, respectively;*

(c) *if $z_2 = 0$ and $\mathbf{z} = (x, 0)$ with $x \in \mathbb{R}$, we have $\sigma(Q(\mathbf{z})) = \{x\}$, with $s_+(\mathbf{z}) = s_-(\mathbf{z}) = x$, and $\nu_+(\mathbf{z}) = (1, 0)$, $\nu_-(\mathbf{z}) = (0, 1)$ are eigenvectors corresponding to the eigenvalue x.*

Proof. The spectrum $\sigma(Q(\mathbf{z}))$ of $Q(\mathbf{z})$ is given by the roots of the equation

$$s^2 - 2s\Re z_1 + |z_1|^2 + |z_2|^2 = 0, \tag{4}$$

leading to the equality (2).

(a) Let $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$ be with $z_2 \neq 0$, so $Q(\mathbf{z}) \notin \mathbb{R}\mathbf{I}$. In this case, clearly $|z_2|^2 + |s_\pm(\mathbf{z}) - z_1|^2 > 0$. The vectors $(z_2, s_+(\mathbf{z}) - z_1)$, $(z_2, s_-(\mathbf{z}) - z_1)$ are orthogonal eigenvectors of $Q(\mathbf{z})$ in $\mathbb{C}^2$, corresponding to the eigenvalues $\nu_+(\mathbf{z}), \nu_-(\mathbf{z})$, via equation (4). Hence $\{\nu_+(\mathbf{z}), \nu_-(\mathbf{z})\}$ is an orthonormal basis of $\mathbb{C}^2$, with the stated properties.

The assertions (b), (c) are easily obtained and left to the reader. We only note that $z_2 = 0$ implies, in fact, $s_\pm(\mathbf{z}) = \Re z_1 \pm i|\Im z_1|$, leading to the eigenvectors $(\Re z_1 + i|\Im z_1|, 0)$ and $(0, \Re z_1 - i|\Im z_1|)$, corresponding to the eigenvalues $z_1, \bar{z}_1$, respectively. For the sake of simplicity, we take $s_+(\mathbf{z}) = z_1$, $s_-(\mathbf{z}) = \bar{z}_1$, and replace the eigenvectors from above by $(1, 0)$ and $(0, 1)$, respectively, with no loss of generality.

Example 1. Let $\mathbb{S} = \{\mathfrak{s} = x_1\mathbf{J} + x_2\mathbf{K} + x_3\mathbf{L}; x_1, x_2, x_3 \in \mathbb{R}, x_1^2 + x_2^2 + x_3^2 = 1\}$, that is, the unit sphere of purely imaginary quaternions. Every quaternion $q \in \mathbb{H} \setminus \mathbb{R}$ can be written as $q = x\mathbf{I} + y\mathfrak{s}$, for some $\mathfrak{s} \in \mathbb{S}$, where x, y are real numbers. We have $q = x\mathbf{I} + y\mathfrak{s}$, $x, y \in \mathbb{R}$, we have $\sigma(q) = \{x \pm iy\}$. Indeed, the quaternion $\mathfrak{s} \in \mathbb{S}$ can be written as a matrix under the form

$$\mathfrak{s} = \begin{pmatrix} ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & -ia_1 \end{pmatrix},$$

where a_1, a_2, a_3 are real numbers with $a_1^2 + a_2^2 + a_3^2 = 1$. The detrminant of the matrix $(\lambda - x)\mathbf{I} - y\mathfrak{s}$ is equal to $\lambda^2 - 2\lambda x + x^2 + y^2$, via the equality $a_1^2 + |a_2 + ia_3|^2 = 1$, whose roots are equal to $x \pm iy$, so the spectrum of q does not depend on $\mathfrak{s}$.

Definition 1. For a fixed point $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$, let $\{s_\pm(\mathbf{z})\}$ be the spectrum of the operator $Q(\mathbf{z})$, given by Lemma 1. The eigenvectors $\{\nu_\pm(\mathbf{z})\}$ of $Q(\mathbf{z})$ corresponding to the eigenvalues $\{s_\pm(\mathbf{z})\}$ respectively, again given by Lemma 1, will be called the *canonical eigenvectors* of $Q(\mathbf{z})$. When $q \in \mathbb{H}$ is arbitrary, we put $\sigma(q) = \sigma(Q(\mathbf{z})) = \{s_\pm(\mathbf{z})\}$, where $q = Q(\mathbf{z})$. We also use the notation $s_\pm(q) = s_\pm(\mathbf{z})$ and $\nu_\pm(q) = \nu_\pm(\mathbf{z})$.

LEMMA 2. *Let* $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$, *and let* $\nu_\pm(\mathbf{z}) = (\nu_{\pm 1}(\mathbf{z}), \nu_{\pm 2}(\mathbf{z})) \in \mathbb{C}^2$ *be the canonical eigenvectors of* $Q(\mathbf{z})$. *Then we have*

$$|\nu_{-1}(\mathbf{z})|^2 = |\nu_{+2}(\mathbf{z})|^2,\ |\nu_{-2}(\mathbf{z})|^2 = |\nu_{+1}(\mathbf{z})|^2,$$
$$\nu_{-1}(\mathbf{z})\overline{\nu_{-2}}(\mathbf{z}) + \nu_{+1}(\mathbf{z})\overline{\nu_{+2}}(\mathbf{z}) = 0 \qquad (*)$$

Sketch of proof. We set $\nu_\pm := \nu_\pm(\mathbf{z})$ and $\nu_\pm = (\nu_{\pm 1}, \nu_{\pm 2}) \in \mathbb{C}^2$. If $z_2 = 0$, we have $\nu_{+1} = \nu_{-2} = 1$, $\nu_{-1} = \nu_{+2} = 0$, so relations $(*)$ are trivial.

Assume $z_2 \neq 0$. Because $|s_+ - z_1| = |s_- - \bar{z}_1|$, and relation (4) is equivalent to $(s_\pm - z_1)(s_\pm - \bar{z}_1) + |z_2|^2 = 0$, we have $|s_- - z_1|^2 |s_+ - z_1|^2 = |z_2|^4$. Therefore $|\nu_{-1}|^2 = |\nu_{+2}|^2$. A similar argument shows that $|\nu_{+1}|^2 = |\nu_{-2}|^2$.

We also have $\nu_{-1}\overline{\nu_{-2}} = -\nu_{+1}\overline{\nu_{+2}}$, via equation (4). Consequently, equalities $(*)$ hold true. When $z_2 = 0$, the assertion is obvious.

We note that equalities $(*)$ do not follow, in general, from the orthogonality of $\nu_+(\mathbf{z})$ and $\nu_-(\mathbf{z})$.

Remark 2. Given a complex number ζ, we can determine all quaternions q with $\sigma(q) = \{\zeta, \bar{\zeta}\}$. Assuming, with no loss of generality, that $\Im\zeta \geq 0$, we look for the points $\mathbf{z} = (z_1, z_2) \in \mathbb{C}$ satisfying the equation $\zeta = s_+(\mathbf{z}) = \Re z_1 + i\sqrt{(\Im z_1)^2 + |z_2|^2}$, so $\bar{\zeta} = s_-(\mathbf{z}) = \Re z_1 - i\sqrt{(\Im z_1)^2 + |z_2|^2}$. Setting $u = z_2$ as a parameter, we obtain $\Re z_1 = \Re\zeta$, and $(\Im z_1)^2 = (\Im\zeta)^2 - |u|^2$, provided $|u|^2 \leq (\Im\zeta)^2$. The solutions are given by the set

$$\{\mathbf{z} = (\Re\zeta \pm i\sqrt{(\Im\zeta)^2 - |u|^2}, u) \in \mathbb{C}^2,\ |u| \leq \Im\zeta\},$$

so we have, for every such a $\mathbf{z}$, $\sigma(Q(\mathbf{z})) = \{\zeta, \bar{\zeta}\}$, via Lemma 1.

Remark 3. (1) A subset $U \subset \mathbb{C}$ is said to be *conjugate symmetric* if $\zeta \in U$ if and only if $\bar{\zeta} \in U$. For an arbitrary conjugate symmetric subset $U \subset \mathbb{C}$ we put $U_{\mathbb{H}} = \{q \in \mathbb{H}; \sigma(q) \subset U\}$. Note that, for every $\zeta \in U$ and $u \in \mathbb{C}$ with $|u| \le |\Im\zeta|$, setting $q_\zeta^{\pm}(u) := \Re\zeta \pm i\sqrt{(\Im\zeta)^2 - |u|^2}, u) \in \mathbb{C}^2$, $|u| \le |\Im\zeta|$, we have

$$U_{\mathbb{H}} = \{Q(q_\zeta^{\pm}(u)); \zeta \in U, u \in \mathbb{C}, |u| \le |\Im\zeta|\},$$

via Remark 2.

If $U \subset \mathbb{C}$ is open and conjugate symmetric, the set $U_{\mathbb{H}}$ is also open via the upper semi-continuity of the spectrum (see [5], Lemma VII.6.3.).

(2) A subset $A \subset \mathbb{H}$ is said to be *spectrally saturated* if whenever $\sigma(r) = \sigma(q)$ for some $r \in \mathbb{H}$ and $q \in A$, we also have $r \in A$.

For an arbitrary $A \subset \mathbb{H}$, we put $\mathfrak{S}(A) = \{\zeta \in \mathbb{C}; \exists q \in A : \zeta \in \sigma(q)\}$. As above, we also put $S_{\mathbb{H}} = \{q \in \mathbb{H}; \sigma(q) \subset S\}$ for an arbitrary subset $S \subset \mathbb{C}$.

(3) A subset $A \subset \mathbb{H}$ is spectrally saturated if and only if there exists a conjugate symmetric subset $S \subset \mathbb{C}$ such that $A = S_{\mathbb{H}}$. In this case, $S = \mathfrak{S}(A)$.

(4) If $\Omega \subset \mathbb{H}$ is an open spectraly saturated set, then $\mathfrak{S}(\Omega) \subset \mathbb{C}$ is open.

An important particular case is when $U = \mathbb{D}_r := \{\zeta \in \mathbb{C}; |\zeta| < r\}$, for some $r > 0$. Because the norm of the normal operator induced by q on $\mathbb{C}^2$ is equal to its spectral radius, we must have $U_{\mathbb{H}} = \{q \in \mathbb{H}; \|q\| < r\}$.

(5) We finally note that, for a given conjugate symmetric subset $U \subset \mathbb{C}$, the set $U_{\mathbb{H}}$ is precisely the *circularization* of U, via Proposition 1, so it is *axially symmetric* (see [10], Section 1.1 and [4], Definition 4.3.1). Nevertheless, we continue to call such a set spectrally saturated, a name which better reflects our spectral approach.

Remark 4. Let $U \subset \mathbb{C}$ be conjugate symmetric, and let $F : U \mapsto \mathbb{M}_2$. We write

$$F(\zeta) = \begin{pmatrix} f_{11}(\zeta) & f_{12}(\zeta) \\ f_{21}(\zeta) & f_{22}(\zeta) \end{pmatrix}, \ \zeta \in U,$$

with $f_{mn} : U \mapsto \mathbb{C}$, $m, n \in \{1, 2\}$, and set $F^{\sim}(\zeta) = (F(\zeta))^{\sim}$ for all $\zeta \in U$, where "$\sim$" designates the skew complex conjugation (see Remark 1).

We temporarily say that F is *skew conjugate symmetric* if $F(\bar{\zeta}) = F^{\sim}(\zeta)$, $\zeta \in U$. In fact, the function F is skew conjugate symmetric if and only if F has the form

$$F(\zeta) = \begin{pmatrix} f_1(\zeta) & f_2(\zeta) \\ -\overline{f_2(\bar{\zeta})} & \overline{f_1(\bar{\zeta})} \end{pmatrix}, \ \zeta \in U, \tag{5}$$

for some functions $f_1, f_2 : U \mapsto \mathbb{C}$.

Remark 5. It is interesting to compare the stem functions (see for instance [10], Section 1.1), with the skew symmetric conjugate functions. To discuss this question in our context, let us remark that the tensor product $\mathbb{H}\otimes_{\mathbb{R}}\mathbb{C}$ may be identified with $\mathbb{M}_2=\mathbb{H}+i\mathbb{H}$, which is a direct sum, via the isomorphism induced by the decomposition $\mathbf{a}=\frac{\mathbf{a}+\mathbf{a}^\sim}{2}+i\frac{\mathbf{a}-\mathbf{a}^\sim}{2i}$, $\mathbf{a}\in\mathbb{M}_2$, with $\mathbf{a}+\mathbf{a}^\sim$, $i(\mathbf{a}-\mathbf{a}^\sim)\in\mathbb{H}$ (see Remark 1). The corresponding conjugation of $\mathbb{M}_2$ is in this case $\mathbf{a}=\mathbf{b}+i\mathbf{c}\mapsto\bar{\mathbf{a}}=\mathbf{b}-i\mathbf{c}$, where $\mathbf{b},\mathbf{c}\in\mathbb{H}$ are uniquely determined by a given $\mathbf{a}\in\mathbb{M}_2$.

With this identification, a stem function is a map $F:U\mapsto\mathbb{M}_2$, where $U\subset\mathbb{C}$ is conjugate symmetric, with the property $F(\bar\zeta)=\overline{F(\zeta)}$ for all $\zeta\in U$. This leads to the equality

$$F(\zeta)=\begin{pmatrix} f_1(\zeta) & f_2(\zeta)\\ -\overline{f_2(\bar\zeta)} & \overline{f_1(\bar\zeta)}\end{pmatrix},\ \zeta\in U,$$

showing that every stem function is skew symmetric conjugate, via (5).

Conversely, when F is given by (5), so it is skew symmetric conjugate, we have

$$F^\sim(\zeta)=\begin{pmatrix} \overline{f_1(\bar\zeta)} & \overline{f_2(\bar\zeta)}\\ -f_2(\zeta) & f_1(\zeta)\end{pmatrix},\ \zeta\in U.$$

Setting $F_1(\zeta)=(1/2)(F(\zeta)+F^\sim(\zeta))$ and $F_2(\zeta)=(1/2i)(F(\zeta)-F^\sim(\zeta))$, which are clearly $\mathbb{H}$-valued functions, and $F(\zeta)=F_1(\zeta)+iF_2(\zeta)$, a direct computation shows that $F(\bar\zeta)=F_1(\zeta)-iF_2(\zeta)$, implying that F is a stem function.

As the term "stem function“ is currently used in literature, from now on we shall designate a skew symmetric function as a stem function. Nevertheless, we shall use the definition of the skew symmetric function rather than that equivalent of stem function, which is more appropriate to our framework.

Finally, note that a stem function is not necessarily $\mathbb{H}$-valued. Using the notation from above, the stem function F is $\mathbb{H}$-valued if and only if $f_1(\bar\zeta)=\overline{f_1(\zeta)}$ and $f_2(\bar\zeta)=-\overline{f_2(\zeta)}$ for all $\zeta\in U$.

Remark 6. With the notation from Definition 1, and because for each $\mathbf{z}\in\mathbb{C}^2$ the operator $Q(\mathbf{z})$ is normal on the Hilbert space $\mathbb{C}^2$, we have a direct sum decomposition $\mathbb{C}^2=N_+(\mathbf{z})\oplus N_-(\mathbf{z})$, where $N_\pm(\mathbf{z})=\{\mathbf{w}\in\mathbb{C}^2;Q(\mathbf{z})\mathbf{w}=s_\pm(\mathbf{z})\mathbf{w}\}$. The projections $E_\pm(\mathbf{z})$ of $\mathbb{C}^2$ onto $N_\pm(\mathbf{z})$ are given by $E_\pm(\mathbf{z})\mathbf{w}=\langle\mathbf{w},\nu_\pm(\mathbf{z})\rangle\nu_\pm(\mathbf{z})$, $\mathbf{w}\in\mathbb{C}^2$.

For every function $f:\sigma(Q(\mathbf{z}))\mapsto\mathbb{C}$ we may define the operator

$$(6)\qquad f(Q(\mathbf{z}))\mathbf{w}=f(s_+(\mathbf{z}))\langle\mathbf{w},\nu_+(\mathbf{z})\rangle\nu_+(\mathbf{z})+f(s_-(\mathbf{z}))\langle\mathbf{w},\nu_-(\mathbf{z})\rangle\nu_-(\mathbf{z}),$$

where $\mathbf{w}\in\mathbb{C}^2$ is arbitrary, with a slight but traditional abuse of notation.

We note that formula (6) is a particular case of the functional calculus given by the spectral theorem for compact normal operators (see for instance [11], Chapter 3; see also [8] for other connexions).

In particular, for $f(\zeta)=(\lambda-\zeta)^{-1}$, $\zeta\in\sigma(Q(\mathbf{z}))$, $\lambda\neq\zeta$, we have

$$(\lambda\mathbf{I}-Q(\mathbf{z}))^{-1}=(\lambda-s_+(\mathbf{z}))^{-1}E_+(\mathbf{z})+(\lambda-s_-(\mathbf{z}))^{-1}E_-(\mathbf{z}),\ \lambda\notin\sigma(Q(\mathbf{z})),$$

a formula to be later used.

More generally, for a fixed $\mathbf{z}=(z_1,z_2)\in\mathbb{C}^2$ and a function $F:\sigma(Q(\mathbf{z}))\mapsto\mathbb{M}_2$, we may define the operator (in fact a matrix with respect to the canonical basis of $\mathbb{C}^2$) by the formula

$$F(Q(\mathbf{z}))\mathbf{w}=F(s_+(\mathbf{z}))\langle\mathbf{w},\nu_+(\mathbf{z})\rangle\nu_+(\mathbf{z})+F(s_-(\mathbf{z}))\langle\mathbf{w},\nu_-(\mathbf{z})\rangle\nu_-(\mathbf{z}), \tag{7}$$

where $\mathbf{w}\in\mathbb{C}^2$ is arbitrary, as an extension of (6). Formula (7) can be also written as

$$F(q)\mathbf{w}=F(s_+(q))\langle\mathbf{w},\nu_+(q)\rangle\nu_+(q)+F(s_-(q))\langle\mathbf{w},\nu_-(q)\rangle\nu_-(q),\ \mathbf{w}\in\mathbb{C}^2,$$

for each $F:\sigma(q)\mapsto\mathbb{M}_2$. In addition, if $q=Q(\mathbf{z})$, one can use the notation $N_\pm(q)=N_\pm(\mathbf{z})$ and $E_\pm(q)=E_\pm(\mathbf{z})$. In fact, if $U\subset\mathbb{C}$ is conjugate symmetric, the formula from above leads to a function $F:U_\mathbb{H}\mapsto\mathbb{M}_2$. Finally, when $q=s\mathbf{I}$, $s\in\mathbb{R}$, then $F(q)=F(s)\mathbf{I}$, via Lemma 1(c).

THEOREM 1. *Let $U\subset\mathbb{C}$ be a conjugate symmetric subset, and let $F:U\mapsto\mathbb{M}_2$. The matrix $F(q)$ is a quaternion for all $q\in U_\mathbb{H}$ if and only if F is a stem function.*

Sketch of proof We fix a point $\zeta\in U$. As $\bar\zeta\in U$, we may suppose, with no loss of generality, that $\Im\zeta\geq 0$.

We assume first that $\Im\zeta>0$, and choose a quaternion $q\in U_\mathbb{H}$ with $\sigma(q)=\{\zeta,\bar\zeta\}$. Writing $q=Q(\mathbf{z})$ with $\mathbf{z}=(z_1,z_2)\in\mathbb{C}^2$, because $\Im\zeta>0$, we may assume $z_2\neq 0$, via Remark 2. Let $\nu_\pm(\mathbf{z})$ be the canonical eigenvectors of $Q(\mathbf{z})$, given by Definition 1. We also have $s_+(\mathbf{z})=\zeta$, $s_-(\mathbf{z})=\bar\zeta$.

We show first that $F(Q(\mathbf{z}))\in\mathbb{H}$ if and only if

$$F(s_+(\mathbf{z}))\nu_+(\mathbf{z})=F^\sim(s_-(\mathbf{z}))\nu_+(\mathbf{z}). \tag{8}$$

To simplify the computation, we set $s_\pm=s_\pm(\mathbf{z}),F_\pm=F(q_\pm(\mathbf{z})),\nu_\pm(\mathbf{z})=(\nu_{\pm1},\nu_{\pm2})$, and fix a $\mathbf{w}=(w_1,w_2)\in\mathbb{C}^2$. Using this notation, we obtain that the matrix $F(Q(\mathbf{z}))$ is a quaternion if and only if

$$(f_{11}^+-\overline{f_{22}^-})|\nu_{+1}|^2+(f_{11}^--\overline{f_{22}^+})|\nu_{+2}|^2+(f_{12}^+-f_{12}^--\overline{f_{21}^+}+\overline{f_{21}^-})\overline{\nu_{+1}}\nu_{+2}=0$$

(9)

$$(f_{12}^-+\overline{f_{21}^+})|\nu_{+1}|^2+(f_{12}^++\overline{f_{21}^-})|\nu_{+2}|^2+(f_{11}^+-f_{11}^-+\overline{f_{22}^+}-\overline{f_{22}^-})\nu_{+1}\overline{\nu_{+2}}=0.$$

To have a notation even simpler, we set $x := f_{11}^+ - \overline{f_{22}^-}$, $y := f_{11}^- - \overline{f_{22}^+}$, $u := f_{12}^+ + \overline{f_{21}^-}$, $v = f_{12}^- + \overline{f_{21}^+}$. Equations (9) lead to

$$\begin{pmatrix} f_{11}^+ & f_{12}^+ \\ f_{21}^+ & f_{22}^+ \end{pmatrix}\begin{pmatrix} z_2 \\ s_+(\mathbf{z}) - z_1 \end{pmatrix} = \begin{pmatrix} \overline{f_{22}^-} & -\overline{f_{21}^-} \\ -\overline{f_{12}^-} & \overline{f_{11}^-} \end{pmatrix}\begin{pmatrix} z_2 \\ s_+(\mathbf{z}) - z_1 \end{pmatrix},$$

which is (8), modulo a multiplicative constant. In other words, $F(Q(\mathbf{z})) \in \mathbb{H}$ if and only if (8) holds.

Next we apply Remark 2. Finding two linearly independent vectors $\nu_+(\mathbf{z}_{u_1})$, $\nu_+(\mathbf{z}_{u_2})$, with the hypothesis $F(Q(\mathbf{z}_{u_1})), F(Q(\mathbf{z}_{u_2})) \in \mathbb{H}$, we deduce that the equality $F(\zeta) = F^\sim(\bar\zeta)$ holds.

If $\Im\zeta < 0$, a similar argument shows that $F(\zeta) = F^\sim(\bar\zeta)$. In particular, if $F(Q(\mathbf{z})) \in \mathbb{H}$ for $Q(\mathbf{z}) \in U_{\mathbb{H}}$, the equality $F(\zeta) = F^\sim(\bar\zeta)$ is true for all $\zeta \in U$ with $\Im\zeta \neq 0$.

The remaining situations can be proved via direct calculatuins.

Corollary 1. *Let $U \subset \mathbb{C}$ be a conjugate symmetric subset, and let $f : U \mapsto \mathbb{C}$. We have $f(q) \in \mathbb{H}$ for all $q \in U_{\mathbb{H}}$ if and only if $f(\zeta) = \overline{f(\bar\zeta)}$ for all $\zeta \in U$.*

Proof. We apply Theorem 1 to the function $F = f\mathbf{I} : U \mapsto \mathbb{M}_2$. This function is a stem one if and only if $f(\zeta) = \overline{f(\bar\zeta)}$ for all $\zeta \in U$.

Corollary 2. *Let $U \subset \mathbb{C}$ be an open conjugate symmetric subset, and let $F : U \mapsto \mathbb{H}$. Then we have $F(q) \in \mathbb{H}$ for all $q \in U_{\mathbb{H}}$ if and only if $F(\zeta) = F(\bar\zeta)$ for all $\zeta \in U$.*

Proof. The property $F : U \mapsto \mathbb{H}$ implies that $F^\sim = F$. Therefore, F is a stem function if and only if $F(\zeta) = F(\bar\zeta)$ for all $\zeta \in U$.

Remark 7. Let $U \subset \mathbb{C}$ be a conjugate symmetric set, and let $F : U \mapsto \mathbb{M}_2$ be a stem function. The formula

$$F(q)\mathbf{w} = F(s_+(q))\langle \mathbf{w}, \nu_+(q)\rangle \nu_+(q) + F(s_-(q))\langle \mathbf{w}, \nu_-(q)\rangle \nu_-(q),$$

where $q \in U_{\mathbb{H}}$ and $\mathbf{w} \in \mathbb{C}^2$ are arbitrary, is an "extension" of the function F to $U_{\mathbb{H}}$, in a sense to be specified. Note that we have an embedding $U \ni \zeta \mapsto q_\zeta := Q((\zeta, 0)) \in \mathbb{H}$, which is the restriction of an $\mathbb{R}$-linear isometry. In fact, writing $\zeta = x + iy$, with $x, y \in \mathbb{R}$ unique, we have $q_\zeta = x\mathbf{I} + y\mathbf{J}$, allowing us to identify the set U with the set $U_{\mathbf{J}} := \{q_\zeta; \zeta \in U\} = \{x\mathbf{I} + y\mathbf{J}; x + iy \in U\} \subset \mathbb{H}$.

As we have $\sigma(q_\zeta) = \{\zeta, \bar\zeta\}$, it follows from Lemma 1 and formula (5) that $s_+(q_\zeta) = \zeta$, $s_-(q_\zeta) = \bar\zeta$, $\nu_+(q_\zeta) = (1, 0)$, $\nu_-(q_\zeta) = (0, 1)$, and hence

$$F(q_\zeta) = \begin{pmatrix} f_1(\zeta) & f_2(\bar\zeta) \\ -\overline{f_2(\bar\zeta)} & \overline{f_1(\zeta)} \end{pmatrix}, \ \zeta \in U.$$

Let $\mathcal{F}_s(U,\mathbb{M}_2) = \{F : U \mapsto \mathbb{M}_2; F(\bar{\zeta}) = F^{\sim}(\zeta), \zeta \in U\}$, which is an $\mathbb{R}$-algebra of $\mathbb{M}_2$-valued functions on U, consisting of all stem functions on U. Let also $\mathcal{F}(U,\mathbb{H}) = \{G : U \mapsto \mathbb{H}\}$, which an $\mathbb{R}$-algebra of $\mathbb{H}$-valued functions on U. Setting $\kappa(\zeta) = q_\zeta, \zeta \in U$, we have an injective unital morphism of $\mathbb{R}$-algebras given by $\mathcal{F}_s(U,\mathbb{M}_2) \ni F \mapsto F \circ \kappa \in \mathcal{F}(U,\mathbb{H})$. Therefore, the map $U_{\mathbb{H}} \ni q \mapsto F(q) \in \mathbb{H}$ given by (7), which extends the map $q_\zeta \mapsto F(q_\zeta)$, may be also regarded as an "extension" of $F \in \mathcal{F}_s(U,\mathbb{M}_2)$ (modulo the map κ). Note also that the function $U_{\mathbb{H}} \ni q \mapsto F(q) \in \mathbb{H}$ is uniquely determined by the function $U \ni \zeta \mapsto F(\zeta) \in \mathbb{M}_2$, when the latter is a stem function.

In particular, if $f \in \mathcal{F}_s(U) := \{g : U \mapsto \mathbb{C}; g(\bar{\zeta}) = \overline{g(\zeta)}, \zeta \in U\}$, then $f(Q(\zeta,0)) = Q((f(\zeta),0)),\ \zeta \in U$.

4. ANALYTIC FUNCTIONAL CALCULUS FOR QUATERNIONS

Regarding, as before, the quaternions as normal operators, we now investigate some consequences of their analytic functional calculus, in the classical sense (see [5], Section VII.3, for details). Let $U \subset \mathbb{C}$ be open. We recall tat an open subset $\Delta \subset U$ is said to be a *Cauchy domain* (in U) if $\Delta \subset \bar{\Delta} \subset U$ and the boundary $\partial\Delta$ of Δ consists of a finite family of closed curves, piecewise smooth, positively oriented. Note that a Cauchy domain is bounded but not necessarily connected.

Lemma 3. *Let $U \subset \mathbb{C}$ be a conjugate symmetric open set and let $F : U \mapsto \mathbb{M}_2$ be an analytic function. For every $q \in U_{\mathbb{H}}$ we set*

$$F_{\mathbb{H}}(q) = \frac{1}{2\pi i}\int_\Gamma F(\zeta)(\zeta\mathbf{I} - q)^{-1}d\zeta, \tag{10}$$

where Γ is the boundary of a Cauchy domain in U containing the spectrum $\sigma(q)$. Then we have $F_{\mathbb{H}}(q) \in \mathbb{H}$ for all $q \in U_{\mathbb{H}}$ if and only if F is a stem function.

Proof. We first assume that $q \notin \mathbb{R}\mathbf{I}$. If $\sigma(q) = \{s_+, s_-\}$ with $s_\pm = s_\pm(q)$, the points s_+, s_- are distinct and not real, by Lemma 1. We fix an $r > 0$ sufficiently small such that, setting $D_\pm := \{\zeta \in U; |\zeta - s_\pm| \le r\}$, we have $D_\pm \subset U$ and $D_+ \cap D_- = \emptyset$. Then

$$F_{\mathbb{H}}(q) = \frac{1}{2\pi i}\int_{\Gamma_+} F(\zeta)(\zeta\mathbf{I} - q)^{-1}d\zeta + \frac{1}{2\pi i}\int_{\Gamma_-} F(\zeta)(\zeta\mathbf{I} - q)^{-1}d\zeta,$$

where $\Gamma_\pm$ is the boundary of $D_\pm$. We may write $F(\zeta) = \sum_{k\ge0}(\zeta - s_+)^k A_k$ with $\zeta \in D_+$, $A_k \in \mathbb{M}_2$ for all $k \ge 0$, as a uniformly convergent series. Similarly, $F(\zeta) = \sum_{k\ge0}(\zeta - s_-)^k B_k$ with $\zeta \in D_-$, $B_k \in \mathbb{M}_2$ for all $k \ge 0$, as a uniformly convergent series.

Note that

$$\frac{1}{2\pi i}\int_{\Gamma_+} F(\zeta)(\zeta\mathbf{I}-q)^{-1}d\zeta = \sum_{k\geq 0}\left(A_k\frac{1}{2\pi i}\int_{\Gamma_+}(\zeta-s_+)^k(\zeta\mathbf{I}-q)^{-1}d\zeta\right) = A_0E_+,$$

because the integral $\frac{1}{2\pi i}\int_{\Gamma_+}(\zeta-s_+)^k(\zeta\mathbf{I}-q)^{-1}d\zeta$ is equal to

$$E_+ := \frac{1}{2\pi i}\int_{\Gamma_+}(\zeta\mathbf{I}-q)^{-1}d\zeta$$

when $k=0$, which is the projection of $\mathbb{C}^2$ onto the space $N_+ := \{\mathbf{v}; q\mathbf{v} = s_+\mathbf{v}\}$, and it is equal to 0 when $k\geq 1$, because $(\zeta\mathbf{I}-q)^{-1}E_+ = (\zeta-s_+)^{-1}E_+$ (see Remark 6).

Note also that $E_+\mathbf{w} = \langle\mathbf{w},\nu_+(q)\rangle\nu_+(q)$, and $E_-\mathbf{w} = \langle\mathbf{w},\nu_-(q)\rangle\nu_-(q)$, all $\mathbf{w}\in\mathbb{C}^2$. Consequently, $F_{\mathbb{H}}(q) = F(s_+)E_+ + F(s_-)E_-$, and the right hand side of this equality coincides with formula (7).

When $\sigma(q) = \{s\}$, where $s := s_+ = s_- \in \mathbb{R}$, we obtain $F_{\mathbb{H}}(q) = A_0 = F(s)\mathbf{I}$.

In both situations, the matrix $F_{\mathbb{H}}(q)$ is equal to the right hand side of formula (7). Therefore, we must have $F_{\mathbb{H}}(q)\in\mathbb{H}$ if and only if $F(s_+) = F^{\sim}(s_-)$, via Theorem 1. In other words, $F_{\mathbb{H}}(q)\in\mathbb{H}$ for all $q\in U_{\mathbb{H}}$ if and only if $F: U\mapsto\mathbb{M}_2$ is a stem function.

Remark 8. It follows from the proof of the previous lemma that the element $F_{\mathbb{H}}(q)$, given by formula (10), coincides with the element $F(q)$ given by (7). Nevertheless, we keep the notation $F_{\mathbb{H}}(q)$ whenever we want to emphasize that it is defined via (10).

COROLLARY 3. *Let $U\subset\mathbb{C}$ be a conjugate symmetric open set and let $f: U\mapsto\mathbb{C}$ be an analytic function. For every $q\in U_{\mathbb{H}}$ we set*

$$f_{\mathbb{H}}(q) = \frac{1}{2\pi i}\int_{\Gamma} f(\zeta)(\zeta\mathbf{I}-q)^{-1}d\zeta, \tag{11}$$

where Γ is the boundary of a Cauchy domain in U containing the spectrum $\sigma(q)$. Then we have $f_{\mathbb{H}}(q)\in\mathbb{H}$ if and only if $f(s_+(q)) = \overline{f(s_-(q))}$ for all $q\in U_{\mathbb{H}}$.

Proof. The assertion is a direct consequence of Lemma 3, applied to the the function $f\mathbf{I}$.

Remark 9. Let $U\subset\mathbb{C}$ be open and conjugate symmetric. As already seen, for every point $\zeta\in U$ the quaternion $Q((\zeta,0))$ is an element of $U_{\mathbb{H}}$ because its spectrum equals the set $\{\zeta,\bar\zeta\}$. According to Corollary 3, an analytic function $f: U\mapsto\mathbb{C}$ has the property $f_{\mathbb{H}}(q)\in\mathbb{H}$ for all $q\in U_{\mathbb{H}}$ if and only if $f(\bar\zeta) = \overline{f(\zeta)}$ for all $\zeta\in U$. This shows, in particular, that the set $\mathcal{O}_s(U)$, consisting of all analytic functions $f : U \mapsto \mathbb{C}$ with the property $f(\bar{\zeta}) = \overline{f(\zeta)}$ for all $\zeta \in U$, is compatible with the analytic functional calculus of the quaternions. Clearly, $\mathcal{O}_s(U)$ is a unital $\mathbb{R}$-subalgebra of the $\mathbb{C}$-algebra $\mathcal{O}(U)$ of all analytic functions in U.

More generally, if $F \in \mathcal{O}(U, \mathbb{M}_2)$, where $\mathcal{O}(U, \mathbb{M}_2)$ is the $\mathbb{C}$-algebra of all $\mathbb{M}_2$-valued analytic functions in U, we have the property $F_{\mathbb{H}}(q) \in \mathbb{H}$ for all $q \in U_{\mathbb{H}}$ if and only if F is a stem function, by Lemma 3, whose matrix representation is given by formula (5).

Let us denote by $\mathcal{O}_s(U, \mathbb{M}_2)$ the set of all analytic stem functions $F \in \mathcal{O}(U, \mathbb{M}_2)$, so $\mathcal{O}_s(U, \mathbb{M}_2) \subset \mathcal{F}_s(U, \mathbb{M}_2)$, where the latter is introduced in Remark 7. It is easily seen that $\mathcal{O}_s(U, \mathbb{M}_2)$ is a unital $\mathbb{R}$-subalgebra of the $\mathbb{C}$-algebra $\mathcal{O}(U, \mathbb{M}_2)$, and it is also an $\mathcal{O}_s(U)$-module.

As an example, if $\Delta \subset \mathbb{C}$ is an open disk centered at 0, each function $F \in \mathcal{O}_s(\Delta, \mathbb{M}_2)$ can be represented as a convergent series $F(\zeta) = \sum_{k \geq 0} a_k \zeta^k$, $\zeta \in \Delta$, with $a_k \in \mathbb{H}$ for all $k \geq 0$.

Definition 2. Let $\Omega \subset \mathbb{H}$ be a spectrally saturated open set, and let $U = \mathfrak{S}(\Omega) \subset \mathbb{C}$ (which is also open by Remark 3(4)). Then we put

$$\mathcal{R}(\Omega) = \{f_{\mathbb{H}}; f \in \mathcal{O}_s(U)\} \text{ and } \mathcal{R}(\Omega, \mathbb{H}) = \{F_{\mathbb{H}}; F \in \mathcal{O}_s(U, \mathbb{M}_2)\}.$$

In fact, these are $\mathbb{R}$-linear spaces, having some useful properties.

THEOREM 2. *Let $\Omega \subset \mathbb{H}$ be a spectrally saturated open set, and let $U = \mathfrak{S}(\Omega) \subset \mathbb{C}$. The space $\mathcal{R}(\Omega)$ is a unital commutative $\mathbb{R}$-algebra, the space $\mathcal{R}(\Omega, \mathbb{H})$ is a right $\mathcal{R}(\Omega)$-module, and the map $\mathcal{O}_s(U, \mathbb{M}_2) \ni F \mapsto F_{\mathbb{H}} \in \mathcal{R}(\Omega, \mathbb{H})$ is a right module isomorphism. Moreover, for every polynomial*

$$P(\zeta) = \sum_{n=0}^{m} a_n \zeta^n, \; \zeta \in \mathbb{C},$$

with $a_n \in \mathbb{H}$ for all $n = 0, 1, \ldots, m$, we have $P_{\mathbb{H}}(q) = \sum_{n=0}^{m} a_n q^n \in \mathbb{H}$ for all $q \in \mathbb{H}$.

Sketch of proof. The $\mathbb{R}$-linearity of the maps

$$\mathcal{O}_s(U, \mathbb{M}_2) \ni F \mapsto F_{\mathbb{H}} \in \mathcal{R}(\Omega, \mathbb{H}), \; \mathcal{O}_s(U) \ni f \mapsto f_{\mathbb{H}} \in \mathcal{R}(\Omega),$$

is clear. That the second one is also multiplicative follows from the multiplicativity of the analytic functional calculus at any point $q \in \mathbb{H}$. In fact, we have a more general property, specifically $(Ff)_{\mathbb{H}}(q) = F_{\mathbb{H}}(q) f_{\mathbb{H}}(q)$ for all $F \in \mathcal{O}_s(U, \mathbb{M}_2)$, $f \in \mathcal{O}_s(U)$, and $q \in \Omega$, which follows from the fact that $f(\zeta)$ commutes with $(\eta \mathbf{I} - q)^{-1}$ for all ζ, η where the functions are defined.

In particular, for every polynomial $P(\zeta) = \sum_{n=0}^{m} a_n \zeta^n$ with $a_n \in \mathbb{H}$ for all $n = 0, 1, \ldots, m$, we have $P_{\mathbb{H}}(q) = \sum_{n=0}^{m} a_n q^n \in \mathbb{H}$ for all $q \in \mathbb{H}$.

Another stated property is the injectivity of the map $\mathcal{O}_s(U,\mathbb{M}_2)\ni F\mapsto F_{\mathbb{H}}\in\mathcal{R}(\Omega,\mathbb{H})$. Indeed, if the function $F_{\mathbb{H}}$ is null, the function $U\ni\zeta\mapsto F_{\mathbb{H}}(q_\zeta)\in\mathbb{H}$ is null too, so $F\in\mathcal{O}_s(U,\mathbb{M}_2)$ should be null as well (see Remark 7).

COROLLARY 4. *Let $\Omega\subset\mathbb{H}$ be a spectrally saturated open set, and let $U=\mathfrak{S}(\Omega)\subset\mathbb{C}$. The map $\mathcal{O}_s(U)\ni f\mapsto f_{\mathbb{H}}\in\mathcal{R}(\Omega)$ is a unital $\mathbb{R}$-algebra isomorphism. Moreover,*

(a) *for every polynomial $p(\zeta)=\sum_{n=0}^m a_n\zeta^n$ with a_n real for all $n=0,1,\ldots,$*
m, we have $p_{\mathbb{H}}(q)=\sum_{n=0}^m a_nq^n\in\mathbb{H}$ for all $q\in\Omega$;

(b) *if $f\in\mathcal{O}_s(U)$ has no zero in U, we have $(f_{\mathbb{H}}(q))^{-1}=f_{\mathbb{H}}^{-1}(q)$ for all $q\in\Omega$.*

The assertions are direct consequences of the previous proof.

COROLLARY 5. *Let $r>0$ and let $U\supset\{\zeta\in\mathbb{C};|\zeta|\le r\}$ be a conjugate symmetric open set. Then for every $F\in\mathcal{O}_s(U,\mathbb{M}_2)$ one has*

$$F_{\mathbb{H}}(q)=\sum_{n\ge0}\frac{F^{(n)}(0)}{n!}q^n,\ \ \|q\|<r,$$

where the series is absolutely convergent.

The assertion follows as in the classical case.

Remark 10. For every function $F\in\mathcal{O}_s(U,\mathbb{M}_2)$, the derivatives $F^{(n)}$ also belong to $\mathcal{O}_s(U,\mathbb{M}_2)$, where $U\subset\mathbb{C}$ is a conjugate symmetric open set.

Fixing $F\in\mathcal{O}_s(U,\mathbb{M}_2)$, we may define its *extended derivatives* with respect to the quaternionic variable via the formula

$$F_{\mathbb{H}}^{(n)}(q)=\frac{1}{2\pi i}\int_\Gamma F^{(n)}(\zeta)(\zeta\mathbf{I}-q)^{-1}d\zeta, \tag{12}$$

for the boundary Γ of a Cauchy domain $\Delta\subset U$, $n\ge0$ an arbitrary integer, and $\sigma(q)\subset\Delta$.

In particular, if $F(\zeta)=\sum_{k\ge0}a_k\zeta^k$ with coefficients in $\mathbb{H}$, then (12) gives the equality $F'_{\mathbb{H}}(q)=\sum_{k\ge1}ka_kq^{k-1}$, which looks like a (formal) derivative of the function $F_{\mathbb{H}}(q)=\sum_{k\ge0}a_kq^k$.

Remark 11. (1) Let $U\subset\mathbb{C}$ be a conjugate symmetric open set and let $F\in\mathcal{O}_s(U,\mathbb{M}_2)$ be arbitrary. We can easily describe the zeros of $F_{\mathbb{H}}$. Indeed, as we have $F_{\mathbb{H}}(q)=F(s_+(q))\nu_+(q)+F(s_-(q))\nu_-(q)$, we have $F_{\mathbb{H}}(q)=0$ if and only if $F(s_\pm(q))=0$. Therefore, setting $\mathcal{Z}(F):=\{\lambda\in U;F(\lambda)=0\}$, and, similarly, $\mathcal{Z}(F_{\mathbb{H}}) := \{q \in U_{\mathbb{H}}; F_{\mathbb{H}}(q) = 0\}$, we must have $\mathcal{Z}(F_{\mathbb{H}}) = \{q \in U_{\mathbb{H}}; \sigma(q) \subset \mathcal{Z}(F)\}$.

In particular, if U is connected and $\mathcal{Z}(F)$ has an accumulation point in U, then $F_{\mathbb{H}} = 0$.

(2) Let us observe that if $F \in \mathcal{O}_s(U, \mathbb{M}_2)$ has the property that $F_{\mathbb{H}}(x + y\mathbf{J}) = 0$ for all $x + iy \in U$, then $F = 0$, and so $F_{\mathbb{H}} = 0$. Indeed, if $\zeta = x + iy$, this follows from Remark 7.

Remark 12. Theorem 2 and its consequences suggest a definition for $\mathbb{H}$-valued "analytic functions" as elements of the set $\mathcal{R}(\Omega, \mathbb{H})$, where Ω is a spectrally saturated open subset of $\mathbb{H}$. Because the expression "analytic function" seems to be quite improper in this context, the elements of $\mathcal{R}(\Omega, \mathbb{H})$ will be called *Q-regular functions* on Ω. In fact, the functions from $\mathcal{R}(\Omega, \mathbb{H})$ may be also regarded as *Cauchy transforms* of the (stem) functions from $\mathcal{O}_s(U, \mathbb{M}_2)$, with $U = \mathfrak{S}(\Omega)$.

We recall that there exists a large literature dedicated to a concept of "slice regularity (or hyperholomorphy)", which is a form of holomorphy in the context of quaternions (see for instance [4] and works quoted within). Till the end of this section we shall try to clarify the connection between these concepts, showing that they coincide on spectrally saturated open sets.

For $\mathbb{M}_2$-valued functions defined on subsets of $\mathbb{H}$, the concept of slice regularity (see [4]) is defined as follows.

Let $\mathbb{S}$ be the unit sphere of purely imaginary quaternions (see Example 1). Let also $\Omega \in \mathbb{H}$ be an open set, and let $F : \Omega \mapsto \mathbb{M}_2$ be a differentiable function. In the spirit of [4], we say that F is *(right) slice regular (or hyperholomorphic)* in Ω if for all $\mathfrak{s} \in \mathbb{S}$,

$$\bar{\partial}_{\mathfrak{s}} F(x\mathbf{I} + y\mathfrak{s}) := \frac{1}{2}\left(\frac{\partial}{\partial x} + R_{\mathfrak{s}} \frac{\partial}{\partial y}\right) F(x + y\mathfrak{s}) = 0,$$

on the set $\Omega \cap (\mathbb{R}\mathbf{I} + \mathbb{R}\mathfrak{s})$, where $R_{\mathfrak{s}}$ is the right multiplication of the elements of $\mathbb{M}_2$ by $\mathfrak{s}$.

Note that, unlike in [9], we use the right slice regularity rather than the left one because of our regard to $\mathbb{H}$ as an algebra of operators on $\mathbb{C}^2$.

Of course, we are mainly interested by slice regularity of $\mathbb{H}$-valued functions, but the concept is valid for $\mathbb{M}_2$-valued functions and plays an important role in our discussion.

Example 2. (1) The convergent series of the form $\sum_{k \geq 0} a_k q^k$, on a set $\{q \in \mathbb{H}; \|q\| < r\}$, with $a_k \in \mathbb{H}$ for all $k \geq 0$, are $\mathbb{H}$-valued slice regular on their domain of definition. In fact, if $a_k \in \mathbb{M}_2$, such functions are actually $\mathbb{M}_2$-valued right slice regular on their domain of definition.

(2) The matrix Cauchy kernel on the open set $\Omega \subset \mathbb{H}$, defined by $\Omega \ni q \mapsto (\zeta\mathbf{I} - q)^{-1} \in \mathbb{M}_2$, is slice regular on $\Omega \subset \mathbb{H}$, whenever $\zeta \notin \mathfrak{S}(\Omega)$. Indeed, for $q = x + y\mathfrak{s} \in \Omega \cap (\mathbb{R}\mathbf{I} + \mathbb{R}\mathfrak{s})$, we can write

$$\frac{\partial}{\partial x}((\zeta - x)\mathbf{I} - y\mathfrak{s})^{-1} = ((\zeta - x)\mathbf{I} - y\mathfrak{s})^{-2},$$

$$R_{\mathfrak{s}}\frac{\partial}{\partial y}((\zeta - x)\mathbf{I} - y\mathfrak{s})^{-1} = -((\zeta - x)\mathbf{I} - y\mathfrak{s})^{-2},$$

because $\zeta\mathbf{I}$, $\mathfrak{s}$ and $((\zeta - x)\mathbf{I} - y\mathfrak{s})^{-1}$ commute in $\mathbb{M}_2$, and $\mathfrak{s}^2 = -\mathbf{I}$. Therefore,

$$\bar{\partial}_{\mathfrak{s}}((\zeta\mathbf{I} - q)^{-1}) = \bar{\partial}_{\mathfrak{s}}(((\zeta - x)\mathbf{I} - y\mathfrak{s})^{-1}) = 0.$$

Lemma 4. *Let $\Omega \subset \mathbb{H}$ be a spectrally saturated open set. Then every function $F \in \mathcal{R}(\Omega, \mathbb{H})$ is right slice regular on Ω.*

Proof. We fix a function $F \in \mathcal{R}(\Omega, \mathbb{H})$, and let $U = \mathfrak{S}(\Omega)$. Using the integral representation of $F_{\mathbb{H}}(q)$ on a Cauchy domain $\Delta \subset \bar{\Delta} \subset U$, whose boundary is denoted by Γ, such that $\sigma(q) \subset \Delta$, and that $\bar{\partial}_{\mathfrak{s}}((\zeta\mathbf{I} - q)^{-1}) = 0$ by Example 2(2), we infer that

$$\bar{\partial}_{\mathfrak{s}}(F_{\mathbb{H}}(q)) = \frac{1}{2\pi i}\int_{\Gamma} F(\zeta)\bar{\partial}_{\mathfrak{s}}((\zeta\mathbf{I} - q)^{-1})d\zeta = 0,$$

which implies the assertion.

Lemma 5. *Let $U \subset \mathbb{C}$ be a conjugate symmetric open set, let $U_{\mathbf{J}} = \{x\mathbf{I} + y\mathbf{J} \in \mathbb{H}; x+iy \in U, x, y \in \mathbb{R}\}$, and let $f : U_{\mathbf{J}} \mapsto \mathbb{H}$ be such that $\bar{\partial}_{\mathbf{J}} f(x\mathbf{I}+y\mathbf{J}) = 0$. Then there are two functions $g, h : U_{\mathbf{J}} \mapsto \mathbb{C}_{\mathbf{J}}$ such that $\bar{\partial}_{\mathbf{J}} g = 0$, $\bar{\partial}_{\mathbf{J}} h = 0$ in $U_{\mathbf{J}}$, and $f = g + \mathbf{L}h$, where $\mathbb{C}_{\mathbf{J}} = \{x\mathbf{I} + y\mathbf{J} \in \mathbb{H}; x + iy \in U, x, y \in \mathbb{R}\}$.*

Proof. We proceed as in Lemma 4.1.7 in [4]. We write $f = f_0\mathbf{I} + f_1\mathbf{J} + f_2\mathbf{K} + f_3\mathbf{L}$, where f_0, f_1, f_2, f_3 are $\mathbb{R}$-valued functions. Then

$$\bar{\partial}_{\mathbf{J}} f(x\mathbf{I} + y\mathbf{J}) = \bar{\partial}_{\mathbf{J}}(f_0\mathbf{I} + f_1\mathbf{J}) + 2\mathbf{L}(\bar{\partial}_{\mathbf{J}}(f_3\mathbf{I} + f_2\mathbf{J}) = 0.$$

Therefore, we may take $g = f_0\mathbf{I} + f_1\mathbf{J}$ and $h = f_3\mathbf{I} + f_2\mathbf{J}$.

As mentioned in Remark 3(5), a fixed conjugate symmetric open set $U \subset \mathbb{C}$ can be associated with an axially symmetric set (see Definition 4.3.1 and Lemma 4.3.8 from [4]), given by the formula

$$\tilde{U} := \{x\mathbf{I} + y\mathfrak{s}; x + iy \in U, \mathfrak{s} \in \mathbb{S}\},$$

which is the *circularization of* U (as in [10], Section 1.1).

Proposition 1. *For every conjugate symmetric open set $U \subset \mathbb{C}$ we have the equality $U_{\mathbb{H}} = \tilde{U}$.*

Proof. If $q \in U_{\mathbb{H}}$, we can write $q = Q(\mathbf{z}^+(u))$ or $q = Q(\mathbf{z}^-(u))$, where $\mathbf{z}^{\pm}(u) = (x \pm i\sqrt{y^2 - |u|^2}, u) \in \mathbb{C}^2$, and $x \pm iy \in U$, for some complex number u with $|u| \le |y|$, by Remark 3. As we have

$$Q(\mathbf{z}^{\pm}(u)) = x\mathbf{I} \pm \sqrt{y^2 - |u|^2}\mathbf{J} + u_1\mathbf{K} + u_2\mathbf{L},$$

with $u = u_1 + iu_2$, $u_1, u_2 \in \mathbb{R}$, and for $y \neq 0$, we have

$$\mathfrak{s}_{\pm} := y^{-1}(\pm\sqrt{y^2 - |u|^2}\mathbf{J} + u_1\mathbf{K} + u_2\mathbf{L}) \in \mathbb{S},$$

it follows that $q \in \tilde{U}$.

When $y = 0$, then $x \in U \subset \tilde{U}$.

Conversely, let $q \in \tilde{U}$, so $q = x\mathbf{I} + y\mathfrak{s}$ for some $\mathfrak{s} \in \mathbb{S}$, and $x + iy \in U$. Of course, $\mathfrak{s} = a_1\mathbf{J} + a_2\mathbf{K} + a_3\mathbf{L}; a_1, a_2, a_3 \in \mathbb{R}, a_1^2 + a_2^2 + a_3^2 = 1$.

To have $q \in U_{\mathbb{H}}$, we must solve the equation, $q = Q((x_1 \pm i\sqrt{y_1^2 - |u|^2}, u)$, for some $x_1 + iy_1 \in U$, and $|u|^2 \le y_1^2$, whose spectrum is $\{x_1 \pm iy_1\}$. On the other hand, according to Example 1, the spectrum of q is the set $\{x \pm iy\}$. Hence, we have the necessary conditions $x_1 = x$ and $y_1 = \pm y$. Note that we must have $u = a_2y + ia_3y$, and $a_1y = \pm|a_1y|$, which leads to a solution of the given equation for a suitable choice from $\{\pm y\}$. Consequently, $q \in U_{\mathbb{H}}$.

LEMMA 6. *Let $U \subset \mathbb{H}$ be a conjugate symmetric open set, and let $\Phi_{\mathbf{J}} : U_{\mathbf{J}} \mapsto \mathbb{H}$ be such that $\bar{\partial}_{\mathbf{J}}\Phi_{\mathbf{J}} = 0$. Then there exists a function $\Phi \in \mathcal{R}(U_{\mathbb{H}}, \mathbb{H})$ with $\Phi_{\mathbf{J}} = \Phi|U_{\mathbf{J}}$.*

Proof. According to Lemma 5, we can write $\Phi_{\mathbf{J}} = F_{\mathbf{J}} + \mathbf{L}G_{\mathbf{J}}$, with $F_{\mathbf{J}}, G_{\mathbf{J}} : U_{\mathbf{J}} \mapsto \mathbb{C}_{\mathbf{J}}$, and $\bar{\partial}_{\mathbf{J}}F_{\mathbf{J}} = 0$, $\bar{\partial}_{\mathbf{J}}G_{\mathbf{J}} = 0$ in $U_{\mathbf{J}}$. Note that we can write

$$\mathbb{C}_{\mathbf{J}} = \left\{ \begin{pmatrix} x + iy & 0 \\ 0 & x - iy \end{pmatrix}; x, y \in \mathbb{R} \right\}, \quad \bar{\partial}_{\mathbf{J}} = \frac{1}{2}\begin{pmatrix} \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}, \end{pmatrix}.$$

Because $\bar{\partial}_{\mathbf{J}}F_{\mathbf{J}} = 0$, showing that the function $F_{\mathbf{J}}$ is analytic in $U_{\mathbf{J}}$, we have a local convergent series representation of this function under the form $F_{\mathbf{J}}(x\mathbf{I} + y\mathbf{J}) = \sum_{k \ge 0} A_k((x - x_0) + (y - y_0)\mathbf{J})^k$ in a neighborhood of each fixed point $x_0 + y_0\mathbf{J} \in U_{\mathbf{J}}$, where $A_k \in \mathbb{C}_{\mathbf{J}}$ for all $k \ge 0$. Using this local representation written in a matricial form, we derive the existence of an analytic function $f_{\mathbf{J}} \in \mathcal{O}(U)$ such that

$$F_{\mathbf{J}}(x\mathbf{I} + y\mathbf{J}) = \begin{pmatrix} f_{\mathbf{J}}(x + iy) & 0 \\ 0 & \overline{f_{\mathbf{J}}(x + iy)} \end{pmatrix}, \; x\mathbf{I} + y\mathbf{J} \in U_{\mathbf{J}},$$

and a similar formula for $G_{\mathbf{J}}(x\mathbf{I} + y\mathbf{J})$, with $f_{\mathbf{J}}$ replaced by $g_{\mathbf{J}} \in \mathcal{O}(U)$. If

$$F(\zeta) = \begin{pmatrix} f_{\mathbf{J}}(\zeta) & 0 \\ 0 & \overline{f_{\mathbf{J}}(\bar{\zeta})} \end{pmatrix}, \; G(\zeta) = \begin{pmatrix} g_{\mathbf{J}}(\zeta) & 0 \\ 0 & \overline{g_{\mathbf{J}}(\bar{\zeta})} \end{pmatrix}, \; \zeta \in U,$$

we have $F, G \in \mathcal{O}_s(U, \mathbb{M}_2)$, and $F_{\mathbb{H}}(q_\zeta) = f_{\mathbf{J}}(x\mathbf{I} + y\mathbf{J})$, $G_{\mathbb{H}}(q_\zeta) = g_{\mathbf{J}}(x\mathbf{I} + y\mathbf{J})$, with $\zeta = x + iy$, via Remark 7. Moreover, $F + \mathbf{L}G \in \mathcal{O}_s(U, \mathbb{M}_2)$, so setting $\Phi(q) = F_{\mathbb{H}}(q) + \mathbf{L}G_{\mathbb{H}}(q)$ for all $q \in U_{\mathbb{H}}$, we have $\Phi \in \mathcal{R}(U_{\mathbb{H}}, \mathbb{H})$ via Theorem 2.

THEOREM 3. *Let $\Omega \subset \mathbb{H}$ be a spectrally saturated open set, and let $\Phi : \Omega \mapsto \mathbb{H}$. The following conditions are equivalent:*

(i) Φ is a slice regular function;

(ii) $\Phi \in \mathcal{R}(\Omega, \mathbb{H})$, that is, Φ is Q-regular.

Proof. If $\Phi \in \mathcal{R}(\Omega, \mathbb{H})$, then Φ is slice regular, by Lemma 4, so $(ii) \Rightarrow (i)$.

Conversely, let Φ be slice regular in Ω. Then we have $\bar{\partial}_{\mathbf{J}}\Phi_{\mathbf{J}} = 0$, where $\Phi_{\mathbf{J}} = \Phi|U_{\mathbf{J}}$. It follows from Lemma 6 that there exists $\Psi \in \mathcal{R}(U_{\mathbb{H}}, \mathbb{H})$ with $\Psi_{\mathbf{J}} = \Phi_{\mathbf{J}}$. This implies that $\Phi = \Psi$, because both Φ, Ψ are uniquely determined by $\Phi_{\mathbf{J}}, \Psi_{\mathbf{J}}$, respectively, the former by (the right side version of) Lemma 4.3.8 in[4], and the latter by Remark 11(2). Consequently, we also have $(i) \Rightarrow (ii)$.

REFERENCES

[1] G. Birkhoff and J. von Neumann, *The logic of quantum mechanics*. Ann. Math. **37** (1936), 823 – 843.

[2] J. L. Brenner, *Matrices of Quaternions*. Pacific J. Math. **1** (1951), 329 – 335.

[3] F. Colombo, J. Gantner, and D. P. Kimsey, *Spectral Theory on the S-Spectrum for Quaternionic Operators*. Birkhäuser, 2018.

[4] F. Colombo, I. Sabadini, and D. C. Struppa, *Noncommutative Functional Calculus, Theory and Applications of Slice Hyperholomorphic Functions*. Progress in Mathematics, Vol. **28** Birkhäuser/Springer Basel AG, Basel, 2011.

[5] N. Dunford and J. T. Schwartz, *Linear Operators, Part I: General Theory*. Interscience Publishers, New York, London, 1958.

[6] D. Finkelstein, J. M. Jauch, S. Schiminovich, and D. Speiser, *Foundations of Quaternion Quantum Mechanics*. Journal of Mathematical Physics **3** (1962), 207 – 220.

[7] R. Fueter, *Über eine Hartogs'schen Satz*. Comment. Math. Helv. **12** (1939/40), 75 – 80.

[8] G. Gentili, C. Stoppato, and D. C. Struppa, *Regular functions of a quaternionic variable*. Springer Monographs in Mathematics, Springer/Heidelberg, 2013.

[9] G. Gentili and D. C. Struppa, *A new theory of regular functions of a quaternionic variable*. Advances in Mathematics **216** (2007), 279 – 301.

[10] R. Ghiloni, V. Moretti, and A. Perotti, *Continuous slice functional calculus in quaternionic Hilbert spaces*. Rev. Math. Phys. **25** (2013), *4*, 1350006.

[11] C. S. Kubrusly, *Spectral Theory of Operators on Hilbert space*. Birkhäuser/Springer, New York, 2012.

[12] S. H. Kulkarni, *Representations of a Class of Real B^*-Algebras as Algebras of Quaternion-Valued Functions*. Proceedings of the American Mathematical Society **116** (1992), *1* , 61 – 66.

[13] T. A. Loring and P. W. Sorensen, *Almost commuting self-adjoint matrices: The real and self-dual cases*. Reviews in Mathematical Physics **28**:7 (2016), 1650017.

[14] G. Moisil and N. Theodorescu, *Fonctions holomorphes dans l'espace*. Mathematica (Cluj) **5** (1931), 142–159.

[15] F.-H. Vasilescu, *Analytic Functional Calculus in Quaternionic Framework*. Preprint arXiv:1902.03850v1.

University of Lille
Department of Mathematics
Campus Cité Scientifique
59655 Villeneuve d'Ascq, France
florian.vasilescu@univ-lille.fr