Dedicated to the memory of Cabiria Andreian-Cazacu

ON SMOOTHNESS OF THE ELEMENTS OF SOME INTEGRABLE TEICHMÜLLER SPACES

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In this paper we focus on the integrable Teichmüller spaces $\mathbf{T}^p$ ($p > 0$) which are subspaces of the symmetric subspace of the universal Teichmüller space. We prove that every element of $\mathbf{T}^p$ for $0 < p \leq 1$, is a $\mathcal{C}^1$-diffeomorphism.

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1. INTRODUCTION

The universal Teichmüller space $\mathbf{T}$ is the space of *quasisymmetric* mappings of the unit circle $\mathbb{S}^1$ fixing 1, i, and -1. An orientation-preserving homeomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$ is said to be quasisymmetric if there exists $M > 0$ such that

$$\forall\theta \in \mathbb{R}, \forall t > 0,\ \frac{1}{M} \leq \left| \frac{f(e^{i(\theta+t)}) - f(e^{i\theta})}{f(e^{i\theta}) - f(e^{i(\theta-t)})} \right| \leq M.$$

Due to a well-known result by Ahlfors and Beurling [3] one can give an equivalent description of $\mathbf{T}$ by means of *quasiconformal mappings.* More precisely, the universal Teichmüller space can be defined as the set of *Teichmüller equivalence classes* of *quasiconformal mappings* of the unit disc $\mathbb{D}$ fixing 1, i, and -1 where two such mappings are Teichmüller equivalent if they coincide on $\mathbb{S}^1$. A mapping $F : D \to F(D)$, where $D \subset \mathbb{C}$ is a domain, is called quasiconformal (or q.c. for short) if it is an orientation-preserving homeomorphism and if its distributional derivatives $\partial_z F$ and $\partial_{\overline{z}} F$ can be represented by locally square integrable functions (also denoted by $\partial_z F$ and $\partial_{\overline{z}} F$) on D such that

$$\left\| \frac{\partial_{\overline{z}} F}{\partial_z F} \right\|_\infty = \operatorname*{ess.sup}_{z\in D} \left| \frac{\partial_{\overline{z}} F\,(z)}{\partial_z F\,(z)} \right| < 1.$$

We also recall that for $z = x + iy$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ and $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$. Furthermore, if F is a quasiconformal mapping, the function $\mu_F = \frac{\partial_{\bar{z}} F}{\partial_z F}$, defined a.e., is called the *Beltrami coefficient* associated with F. By the measurable Riemann mapping theorem, if a measurable function μ on D is such that $\|\mu\|_\infty < 1$, then the *Beltrami equation* $\partial_{\bar{z}} F = \mu \partial_z F$ has (up to normalization) an unique solution (which is by definition quasiconformal) we will denote here by F^μ. For further details we refer to [17].

Let us now introduce an important subspace of $\mathbf{T}$, namely, the *symmetric Teichmüller space* denoted here by $\mathbf{T}_{\mathrm{s}}$. Following a terminology introduced by Gardiner and Sullivan [14], it is the space of *symmetric* homeomorphism of $\mathbb{S}^1$ fixing 1, i, and -1. One recalls that $f : \mathbb{S}^1 \to \mathbb{S}^1$ is symmetric if it is an orientation-preserving homeomorphism of $\mathbb{S}^1$ such that

$$\frac{f(e^{i(\cdot+t)}) - f(e^{i\cdot})}{f(e^{i\cdot}) - f(e^{i(\cdot-t)})} \underset{t\to 0^+}{\longrightarrow} 1, \tag{1}$$

with respect to the uniform convergence on $\mathbb{R}$. As for the universal Teichmüller space one has an equivalent description of such a space that involves quasiconformal mappings. Indeed, Gardiner and Sullivan proved (see Theorem 2.1 in [14]) that $\mathbf{T}_{\mathrm{s}}$ corresponds to the space of Teichmüller equivalent classes of quasiconformal mappings of $\mathbb{D}$ fixing 1, i, and -1 admitting a representative which is *asymptotically conformal* on $\mathbb{S}^1$. Let us recall that a quasiconformal mapping $F : \mathbb{D} \to \mathbb{D}$ is said to be asymptotically conformal on $\mathbb{S}^1$ if for every $\epsilon > 0$, there exists a compact subset K_ϵ of $\mathbb{D}$ such that for any $z \in \mathbb{D} \setminus K_\epsilon$, $|\mu_F(z)| < \epsilon$.

Here we focus on some interesting infinite dimensional subspaces of $\mathbf{T}$, the *p-integrable Teichmüller spaces*, which we define for each $p > 0$ as the set

$$\mathbf{T}^p = \left\{ f \in \mathbf{T} \mid \exists F : \mathbb{D} \to \mathbb{D},\ \text{q.c. such that } F_{|_{\mathbb{S}^1}} = f \text{ and } \mu_F \in \mathrm{L}^p(\mathbb{D}, \sigma) \right\},$$

where σ is the hyperbolic measure on $\mathbb{D}$, that is, for any $z = x + iy \in \mathbb{D}$, $d\sigma(z) = (1 - |z|^2)^{-2}\mathrm{d}x\mathrm{d}y$. It is elementary to observe from such a definition that if $q > p > 0$, then $\mathbf{T}^p \subset \mathbf{T}^{\mathrm{q}}$. The spaces $\mathbf{T}^p$, $p \geq 2$, were first introduced by Guo [15] through an equivalent description involving univalent functions. At about the same time, Cui [9] studied the case $p = 2$ and gave a few important characterizations of the elements of $\mathbf{T}^2$. In particular, he proved that the Beltrami coefficient associated with the *Douady–Earle extension* (see [10]) of any element of $\mathbf{T}^2$ belongs to $\mathrm{L}^2(\mathbb{D}, \sigma)$. Later on, Takhtajan and Teo [22] introduced a Hilbert manifold structure on the universal Teichmüller space that makes the space $\mathbf{T}^2$ the connected component of the identity mapping $\mathrm{id}_{\mathbb{S}^1}$. With respect to such a structure, they proved that the so-called *Weil–Petersson* metric is a Riemannian metric on $\mathbf{T}$. Following Takhtajan and Teo's work, the space $\mathbf{T}^2$ is now reffered to as the *Weil–Petersson Teichmüller space*.

For further results on $\mathbf{T}^2$ we refer to [20]. Let us point out that one can obtain $\mathbf{T}^2 \subset \mathbf{T}_{\mathrm{s}}$ by combining [9, Theorem 2 and Lemma 2] and [12, Theorem 4], see [13, Section 3] for a more detailed explanation. One can also mention the paper [21] by Tang where in particular, Cui's result concerning the Douady–Earle extension is extended to all spaces $\mathbf{T}^p$ with $p \geq 2$. Recently, the second author of this paper proved in [6] that $\mathbf{T}^2 \subset \mathbf{T}_{\mathrm{s}}$ using an approach based on module techniques and the so-called *Teichmüller's Modulsatz* (see [23, §4]), and later on using a different method she proved that for any $p > 0$, $\mathbf{T}^p \subset \mathbf{T}_{\mathrm{s}}$ (see [5]).

In this paper we only deal with $\mathbf{T}^p$ for $0 < p \leq 1$ and we give a proof of the following result:

THEOREM 1. *Let $p \in \mathbb{R}$, $0 < p \leq 1$. Then, every element of $\mathbf{T}^p$ is a $\mathcal{C}^1$-diffeomorphism.*

The strategy of the proof takes advantage of an approach used by the second author of this paper and J. A. Jenkins [8], modified to the case of the unit disc. We first use the *Teichmüller–Wittich–Bellinskiĭ* theorem to show that every element of $\mathbf{T}^1$ has a non-vanishing derivative at each point of $\mathbb{S}^1$. Then, we use properties of the *reduced module* of a simply-connected domain to show that the derivatives of the elements of $\mathbf{T}^1$ are continuous. As mentioned above, since $\mathbf{T}^p \subset \mathbf{T}^1$ for $0 < p \leq 1$, it follows immediately that for $0 < p \leq 1$, any element in $\mathbf{T}^p$ is continuously differentiable with non-vanishing derivative.

2. BACKGROUND

In this section we recall some classic notions from geometric function theory. Such notions are most notably and thoroughly investigated in Teichmüller's *Habilitationsschrift* (Habilitation Thesis) [23].

2.1. Module of a doubly-connected domain

Let D be a (non-degenerate) doubly-connected domain of the extended complex plane, that is, the complement of D is an union of two disjoint simply-connected domains, each bounded by a Jordan curve. It is well known (see [17, 23]) that there exists a biholomorphic function that maps D onto an annulus of inner radius r_1 and outer radius r_2 for some $0 < r_2 < r_1 < \infty$. The *module* $\mathrm{Mod}(D)$ of D is $\ln\left(\frac{r_2}{r_2}\right)$. It is a *conformal invariant*, namely, if $\Psi : D \to \Psi(D)$ is a biholomorphic function, then $\mathrm{Mod}(D) = \mathrm{Mod}(\Psi(D))$.

It is also well known (see [17, 23]) that the module is *superadditive*. More precisely, if D_1 and D_2 are two disjoint doubly-connected subdomains of a doubly-connected domain D_3, where each separates some $z_0 \in \mathbb{C}$ from ∞, then

$$\mathrm{Mod}(D_1) + \mathrm{Mod}(D_2) \leq \mathrm{Mod}(D_3)\,. \tag{2}$$

In saying that a doubly-connected domain separates z_0 from ∞, we mean that one component of its complement contains z_0 in its interior while the other component contains ∞.

Let us now recall two inequalities that will be used in the proof of the main result. For $0 < r_2 < r_1$ and $\zeta \in \mathbb{C}$ we set $A_{\zeta,r_2,r_1} = \{z \mid r_2 < |z-\zeta| < r_1\}$. Let $F : A_{\zeta,r_2,r_1} \to F\left(A_{\zeta,r_2,r_1}\right)$ be a quasiconformal mapping. Then setting $z = \zeta + re^{i\theta}, r_2 < r < r_1$ we have

$$\mathrm{Mod}(F\left(A_{\zeta,r_2,r_1}\right)) \leq \frac{1}{2\pi} \iint_{A_{\zeta,r_2,r_1}} \frac{1+|\mu_F(z)|}{1-|\mu_F(z)|} \cdot \frac{\mathrm{d}x\mathrm{d}y}{|z-\zeta|^2}, \tag{3}$$

and

$$2\pi \int_{r_2}^{r_1} \frac{1}{\int_0^{2\pi} \frac{1+|\mu_F(z)|}{1-|\mu_F(z)|}\mathrm{d}\theta} \cdot \frac{dr}{r} \leq \mathrm{Mod}(F\left(A_{\zeta,r_2,r_1}\right))\,. \tag{4}$$

These estimates could be obtained following Teichmüller's approach based on the *length-area method* in [23, §6.3], where he arrived at weaker versions of (3) and (4). Estimates equivalent to (3) and (4) – some proved under more general assumptions and different methods – can be found in [18, 16, 4], and others.

2.2. **Reduced module of a simply-connected domain**

Let Ω be a simply-connected domain of the complex plane different from $\mathbb{C}$. Let $\zeta \in \Omega$. For $r > 0$, let $D(\zeta, r)$ denote the disc of radius r centered at ζ and let $0 < r_2 < r_1$ be small enough so that $D(\zeta, r_1) \subset \Omega$. From (2) follows

$$\mathrm{Mod}(\Omega \setminus D(\zeta, r_1)) + \ln\left(\frac{r_1}{r_2}\right) \leq \mathrm{Mod}(\Omega \setminus D(\zeta, r_2))\,,$$

and therefore

$$\mathrm{Mod}(\Omega \setminus D(\zeta, r_1)) + \ln(r_1) \leq \mathrm{Mod}(\Omega \setminus D(\zeta, r_2)) + \ln(r_2)\,.$$

One defines the reduced module $\mathrm{M}^{\mathrm{red}}(\Omega, \zeta)$ of Ω at ζ as

$$\lim_{r\to 0^+} \left(\mathrm{Mod}(\Omega \setminus D(\zeta, r)) + \ln(r)\right).$$

Using, for example, *Koebe distortion theorem* one can show that this limit is finite and $\mathrm{M}^{\mathrm{red}}(\Omega, \zeta) = \ln(|\Psi'(0)|)$, where $\Psi : \mathbb{D} \to \Omega$ is a biholomorphic function mapping 0 onto ζ. A detailed proof can be found in [23, §1.6]. From here it follows directly that $\zeta \mapsto \mathrm{M}^{\mathrm{red}}(\Omega, \zeta)$ is continuous on Ω.

Before concluding this subsection let us add one more property of the reduced module that we will use later.

If $F : \mathbb{C} \to \mathbb{C}$ is a homeomorphism then, for any $r > 0$, the function $\zeta \mapsto \mathrm{M}^{\mathrm{red}}(F(D(\zeta, r)), F(\zeta))$ is continuous. Indeed, if $\zeta_n \underset{n\to\infty}{\longrightarrow} \zeta$, then by applying the sequence of biholomorphic functions $z \mapsto F(z) - F(\zeta_n) + F(\zeta), z \in D(\zeta, r)$, one obtains a sequence of domains D_n, which contain $F(\zeta)$. Since $F(z)$ is a homeomorphism it follows that $D_n \underset{n\to\infty}{\longrightarrow} F(D(\zeta, r))$ (with respect to the topology induced by the Hausdorff distance on the set of subsets of $\mathbb{C}$). Consider the sequence of biholomorphic functions $\Psi_n : \mathbb{D} \to D_n$, mapping 0 onto $F(\zeta)$, normalized by $\Psi_n'(0) > 0$. Then for any n, $\ln(\Psi_n'(0)) = \mathrm{M}^{\mathrm{red}}(D_n, F(\zeta)) = \mathrm{M}^{\mathrm{red}}(F(D(\zeta_n, r)), F(\zeta_n))$ since a translation does not change the reduced module. Furthermore, the sequence of functions Ψ_n forms a normal family and thus, up to a subsequence, Ψ_n converges uniformly (on any compact subset of $\mathbb{D}$) to a biholomorphic function $\Psi_\infty : \mathbb{D} \to F(D(\zeta, r))$ mapping 0 onto ζ. This implies

$$\begin{aligned}
\mathrm{M}^{\mathrm{red}}(F(D(\zeta, r)), F(\zeta)) &= \ln\left(\Psi_\infty'(0)\right) \\
&= \lim_{n\to\infty} \ln\left(\Psi_n'(0)\right) \\
&= \lim_{n\to\infty} \mathrm{M}^{\mathrm{red}}(F(D(\zeta_n, r)), F(\zeta_n)),
\end{aligned}$$

and thus we have continuity.

2.3. The Teichmüller–Wittich–Bellinskiĭ theorem

First, let us recall that a mapping $F : \mathbb{C} \to \mathbb{C}$ is said to be *conformal* at z_0 if $\lim_{z\to z_0} \frac{F(z)-F(z_0)}{z-z_0}$ exists and is not equal to 0. Following [17, Chapter V, Theorem 6.1], the well-known Teichmüller–Wittich–Bellinskiĭ theorem can be stated as follows:

THEOREM 2. *Let D be a domain of the complex plane and let $z_0 \in D$. Let $F : D \to F(D)$ be a quasiconformal mapping. If there exists a neighborhood $\mathcal{U}$ of z_0 contained in D such that*

$$\iint_{\mathcal{U}} \frac{|\mu_F(z)|}{|z - z_0|^2} \mathrm{d}x\mathrm{d}y < \infty;$$

then F is conformal at $z = z_0$.

The history of this theorem and its extensions is rather long and we may refer the curious reader to some of the following papers [2, 18, 11, 7, 16, 4, 19] and to [1].

3. PROOF OF THEOREM 1

Let $f \in \mathbf{T}^1$. By definition, there exists a quasiconformal mapping $F : \mathbb{D} \to \mathbb{D}$, such that $F_{|_{\mathbb{S}^1}} = f$ and

$$\iint_{\mathbb{D}} |\mu_F(z)| \, \mathrm{d}\sigma(z) < \infty. \tag{5}$$

Let $\widetilde{\mu}$ be a function defined on the extended complex plane which coincides with μ_F on $\mathbb{D}$ and which is identically 0 outside the disc. Let $F^{\widetilde{\mu}}$ be the unique quasiconformal mapping of the (extended) complex plane with Beltrami coefficient $\widetilde{\mu}$ that fixes 1, i, and $-i$. Since solutions to the Beltrami equation are unique up to a normalization, we have $F^{\widetilde{\mu}}_{|_{\mathbb{D}}} = F$. Since $F(\mathbb{D}) = \mathbb{D}$, we also have that $F^{\widetilde{\mu}}(\mathbb{D}) = \mathbb{D}$, and $F^{\widetilde{\mu}}_{|_{\mathbb{S}^1}} = f$.

Claim 1. The quasiconformal mapping $F^{\widetilde{\mu}}$ is conformal at any point of $\mathbb{S}^1$. Therefore, f is a diffeomorphism of $\mathbb{S}^1$.

Proof of Claim 1. We apply Theorem 2 to derive the conformality of $F^{\widetilde{\mu}}$. Let $\zeta_0 \in \mathbb{S}^1$. Because of (5), one can find a compact subset K of $\mathbb{D}$ such that

$$\iint_{\mathbb{D}\setminus K} |\mu_F(z)| \, \mathrm{d}\sigma(z) < 1. \tag{6}$$

Let $r > 0$ be such that $\mathbb{D} \cap D(\zeta_0, r) \subset \mathbb{D} \setminus K$. One first observes that

$$\begin{aligned} \forall z \in D(\zeta_0, r) \cap \mathbb{D}, \ \left(1 - |z|^2\right)^2 &= (1 - |z|)^2 \cdot (1 + |z|)^2 \\ &\leq |\zeta_0 - z|^2 \cdot (1 + |z|)^2 \\ &< 4 \cdot |\zeta_0 - z|^2 , \end{aligned}$$

and therefore

$$\forall z \in D(\zeta_0, r) \cap \mathbb{D}, \ \frac{1}{|z - \zeta_0|^2} < 4 \cdot \frac{1}{\left(1 - |z|^2\right)^2}. \tag{7}$$

It follows

$$\begin{aligned} \iint_{D(\zeta_0, r)} \frac{|\widetilde{\mu}(z)|}{|z - \zeta_0|^2} \mathrm{d}x\mathrm{d}y &= \iint_{D(\zeta_0, r)\cap\mathbb{D}} \frac{|\mu_F(z)|}{|z - \zeta_0|^2} \mathrm{d}x\mathrm{d}y \\ &\leq 4 \iint_{D(\zeta_0, r)\cap\mathbb{D}} |\mu_F(z)| \, \mathrm{d}\sigma(z) \\ &\leq 4. \end{aligned}$$

We deduce, by Theorem 2, that $F^{\widetilde{\mu}}$ is conformal at $z = \zeta_0$ which proves that f is differentiable at ζ_0 and $|f'(\zeta_0)| > 0$. Since this is true for any $\zeta_0 \in \mathbb{S}^1$, we deduce that f is a diffeomorphism of $\mathbb{S}^1$. □

The following two additional results will be needed in the proof of the continuity of f' on $\mathbb{S}^1$.

Claim 2. Let $\epsilon > 0$. Then, there exists $r_\epsilon > 0$ such that

$$\forall \zeta \in \mathbb{S}^1, \forall 0 < \rho_2 < \rho_1 \leq r_\epsilon, \ \left| \mathrm{Mod}\Big(F^{\widetilde{\mu}}\left(A_{\zeta,\rho_2,\rho_1}\right)\Big) - \ln\left(\frac{\rho_1}{\rho_2}\right) \right| < \epsilon.$$

Claim 3. Let $\zeta \in \mathbb{S}^1$ and $r > 0$. Then,

$$\lim_{\rho\to 0} \mathrm{Mod}\Big(F^{\widetilde{\mu}}\left(A_{\zeta,\rho,r}\right)\Big) + \ln\left(\left|f'(\zeta)\right|\rho\right) = \mathrm{M}^{\mathrm{red}}\Big(F^{\widetilde{\mu}}\left(D\left(\zeta,r\right)\right), f(\zeta)\Big).$$

Proof of Claim 2. Let $\zeta \in \mathbb{S}^1$ and $0 < \rho_2 < \rho_1$. One the one hand, by applying (3) one gets

$$\begin{aligned}
\mathrm{Mod}\Big(F^{\widetilde{\mu}}\left(A_{\zeta,\rho_2,\rho_1}\right)\Big) - \ln\left(\frac{\rho_1}{\rho_2}\right) &\leq \frac{1}{2\pi} \iint_{A_{\zeta,\rho_2,\rho_1}} \frac{1+|\widetilde{\mu}(z)|}{1-|\widetilde{\mu}(z)|} \cdot \frac{\mathrm{d}x\mathrm{d}y}{|z-\zeta|^2} - \ln\left(\frac{\rho_1}{\rho_2}\right) \\
&= \frac{1}{2\pi} \iint_{A_{\zeta,\rho_2,\rho_1}} \left(\frac{1+|\widetilde{\mu}(z)|}{1-|\widetilde{\mu}(z)|} - 1 \right) \cdot \frac{\mathrm{d}x\mathrm{d}y}{|z-\zeta|^2} \\
&\leq \frac{1}{\pi\left(1-\|\mu_F\|_\infty\right)} \iint_{A_{\zeta,\rho_2,\rho_1}\cap\mathbb{D}} |\mu_F(z)| \cdot \frac{\mathrm{d}x\mathrm{d}y}{|z-\zeta|^2}. \qquad (8)
\end{aligned}$$

On the other hand since

$$\int_0^{2\pi} \frac{1+|\widetilde{\mu}\left(z\right)|}{1-|\widetilde{\mu}\left(z\right)|} \mathrm{d}\theta \geq 2\pi,$$

by means of (4) one obtains

$$\begin{aligned}
\mathrm{Mod}\Big(F^{\widetilde{\mu}}\left(A_{\zeta,\rho_2,\rho_1}\right)\Big) - \ln\left(\frac{\rho_1}{\rho_2}\right) &\geq 2\pi \int_{\rho_2}^{\rho_1} \frac{1}{\int_0^{2\pi} \frac{1+|\widetilde{\mu}(z)|}{1-|\widetilde{\mu}(z)|} d\theta} \cdot \frac{\mathrm{d}r}{r} - \ln\left(\frac{\rho_1}{\rho_2}\right) \\
&= \int_{\rho_2}^{\rho_1} \frac{2\pi - \int_0^{2\pi} \frac{1+|\widetilde{\mu}(z)|}{1-|\widetilde{\mu}(z)|} \mathrm{d}\theta}{\int_0^{2\pi} \frac{1+|\widetilde{\mu}(z)|}{1-|\widetilde{\mu}(z)|} \mathrm{d}\theta} \cdot \frac{\mathrm{d}r}{r} \\
&= \int_{\rho_2}^{\rho_1} \frac{\int_0^{2\pi} \frac{-2|\widetilde{\mu}(z)|}{1-|\widetilde{\mu}(z)|} \mathrm{d}\theta}{\int_0^{2\pi} \frac{1+|\widetilde{\mu}(z)|}{1-|\widetilde{\mu}(z)|} \mathrm{d}\theta} \cdot \frac{\mathrm{d}r}{r} \\
&\geq \frac{-1}{\pi} \iint_{A_{\zeta,\rho_2,\rho_1}\cap\mathbb{D}} \frac{|\mu_F(z)|}{1-|\mu_F\left(z\right)|} \cdot \frac{\mathrm{d}x\mathrm{d}y}{|z-\zeta|^2} \\
&\geq -\frac{1}{\pi\left(1-\|\mu_F\|_\infty\right)} \iint_{A_{\zeta,\rho_2,\rho_1}\cap\mathbb{D}} |\mu_F(z)| \cdot \frac{\mathrm{d}x\mathrm{d}y}{|z-\zeta|^2}. \qquad (9)
\end{aligned}$$

Let $\epsilon > 0$. Still because of (5) there exists a compact set K_ϵ of $\mathbb{D}$ such that

$$\iint_{\mathbb{D}\setminus K_\epsilon} |\mu_F(z)| \, \mathrm{d}\sigma(z) < \frac{\pi(1-\|\mu_F\|_\infty)}{4}\epsilon. \qquad (10)$$

Let $r_\epsilon > 0$ be the distance between $\mathbb{S}^1$ and K_ϵ. Thus, for any $0 < \rho_2 < \rho_1 \leq r_\epsilon$ one obtains by combining (8), (9), (7), and (10)

$$\forall \zeta \in \mathbb{S}^1,\ -\epsilon < \mathrm{Mod}\Big(F^{\widetilde{\mu}}\left(A_{\zeta,\rho_2,\rho_1}\right)\Big) - \ln\left(\frac{\rho_1}{\rho_2}\right) < \epsilon,$$

and therefore Claim 2 follows. $\square$

Proof of Claim 3. Let $\zeta \in \mathbb{S}^1$ and let $r > 0$. For any ρ small enough, $0 < \rho < r$, let

$$m(\rho) = \min_{|z-\zeta|=\rho} \left|F^{\widetilde{\mu}}(z) - f(\zeta)\right| \text{ and } M(\rho) = \max_{|z-\zeta|=\rho} \left|F^{\widetilde{\mu}}(z) - f(\zeta)\right|.$$

Since $F^{\widetilde{\mu}}$ is conformal at ζ one has

$$\lim_{\rho\to 0} \frac{|f'(\zeta)|\,\rho}{M(\rho)} = \lim_{\rho\to 0} \frac{|f'(\zeta)|\,\rho}{m(\rho)} = 1. \tag{11}$$

Furthermore, it is evident that

$$\begin{aligned}\mathrm{Mod}\Big(F^{\widetilde{\mu}}\left(D(f(\zeta), r)\right) \setminus D(f(\zeta), M(\rho))\Big) &\leq \mathrm{Mod}\Big(F^{\widetilde{\mu}}\left(A_{\zeta,\rho,r}\right)\Big) \\ &\leq \mathrm{Mod}\Big(F^{\widetilde{\mu}}\left(D(\zeta, r)\right) \setminus D(f(\zeta), m(\rho))\Big).\end{aligned}$$

Therefore, by adding $\ln\left(|f'(\zeta)|\,\rho\right)$, using (11), and letting $\rho \to 0$ it follows that

$$\lim_{\rho\to 0} \mathrm{Mod}\Big(F^{\widetilde{\mu}}\left(A_{\zeta,\rho,r}\right)\Big) + \ln\left(\left|f'(\zeta)\right|\rho\right) = \mathrm{M}^{\mathrm{red}}\Big(F^{\widetilde{\mu}}\left(D\left(\zeta, r\right)\right), f(\zeta)\Big),$$

which proves Claim 3. $\square$

We have now all the ingredients necessary to complete the proof of Theorem 1.

Let $\zeta_0 \in \mathbb{S}^1$. Let $\epsilon > 0$. Let $r_{\frac{\epsilon}{5}} > 0$ be as in Claim 2. By the continuity of the reduced module discussed earlier one can find a $\delta_{\frac{\epsilon}{5}} > 0$ such that if $\zeta \in \mathbb{S}^1$ and $|\zeta - \zeta_0| < \delta_{\frac{\epsilon}{5}}$ then

$$\left|\mathrm{M}^{\mathrm{red}}\Big(F^{\widetilde{\mu}}\left(D(\zeta, r_{\frac{\epsilon}{5}})\right), f(\zeta)\Big) - \mathrm{M}^{\mathrm{red}}\Big(F^{\widetilde{\mu}}\left(D(\zeta_0, r_{\frac{\epsilon}{5}})\right), f(\zeta_0)\Big)\right| < \frac{\epsilon}{5}. \tag{12}$$

Let $\zeta \in \mathbb{S}^1$ be such that $|\zeta - \zeta_0| < \delta_{\frac{\epsilon}{5}}$. By Claim 3 there exist $r_{\zeta_0,1}, r_{\zeta,1} < r_{\frac{\epsilon}{5}}$ such that for any $\rho \leq r_{\zeta_0,1}$

(13)
$$\left|\mathrm{Mod}\Big(F^{\widetilde{\mu}}\left(A_{\zeta_0,\rho,r_{\frac{\epsilon}{5}}}\right)\Big) + \ln\left(\left|f'(\zeta_0)\right|\rho\right) - \mathrm{M}^{\mathrm{red}}\Big(F^{\widetilde{\mu}}\left(D\left(\zeta_0, r_{\frac{\epsilon}{5}}\right)\right), f(\zeta_0)\Big)\right| < \frac{\epsilon}{5},$$

and for any $\rho \leq r_{\zeta,1}$

(14)
$$\left|\mathrm{Mod}\Big(F^{\widetilde{\mu}}\left(A_{\zeta,\rho,r_{\frac{\epsilon}{5}}}\right)\Big) + \ln\left(\left|f'(\zeta)\right|\rho\right) - \mathrm{M}^{\mathrm{red}}\Big(F^{\widetilde{\mu}}\left(D\left(\zeta, r_{\frac{\epsilon}{5}}\right)\right), f(\zeta)\Big)\right| < \frac{\epsilon}{5}.$$

Thus, from the triangle inequality, Claim 2, and Inequalities (12), (13), and (14) we obtain

$$\begin{aligned}
&\left|\ln\left(\left|f'(\zeta)\right|\right)-\ln\left(\left|f'(\zeta_0)\right|\right)\right|\\
&=\left|\ln\left(\left|f'(\zeta)\right|r_{\zeta,1}\right)-\ln\left(r_{\zeta,1}\right)-\ln\left(\left|f'(\zeta_0)\right|r_{\zeta_0,1}\right)+\ln\left(r_{\zeta_0,1}\right)\right|\\
&\leq\left|\ln\left(\left|f'(\zeta)\right|r_{\zeta,1}\right)+\mathrm{Mod}\Big(F^{\widetilde{\mu}}\left(A_{\zeta,r_{\zeta,1},r_{\frac{\epsilon}{5}}}\right)\Big)-\mathrm{M}^{\mathrm{red}}\Big(F^{\widetilde{\mu}}\left(D\left(\zeta,r_{\frac{\epsilon}{5}}\right)\right),f(\zeta)\Big)\right|+\\
&\left|\ln\left(\left|f'(\zeta_0)\right|r_{\zeta_0,1}\right)+\mathrm{Mod}\Big(F^{\widetilde{\mu}}\left(A_{\zeta_0,r_{\zeta_0,1},r_{\frac{\epsilon}{5}}}\right)\Big)-\mathrm{M}^{\mathrm{red}}\Big(F^{\widetilde{\mu}}\left(D\left(\zeta_0,r_{\frac{\epsilon}{5}}\right)\right),f(\zeta_0)\Big)\right|\\
&+\left|\mathrm{M}^{\mathrm{red}}\Big(F^{\widetilde{\mu}}\left(D\left(\zeta,r_{\frac{\epsilon}{5}}\right)\right),f(\zeta)\Big)-\mathrm{M}^{\mathrm{red}}\Big(F^{\widetilde{\mu}}\left(D\left(\zeta_0,r_{\frac{\epsilon}{5}}\right)\right),f(\zeta_0)\Big)\right|\\
&+\left|-\mathrm{Mod}\Big(F^{\widetilde{\mu}}\left(A_{\zeta,r_{\zeta,1},r_{\frac{\epsilon}{5}}}\right)\Big)-\ln\left(r_{\zeta,1}\right)+\mathrm{Mod}\Big(F^{\widetilde{\mu}}\left(A_{\zeta_0,r_{\zeta_0,1},r_{\frac{\epsilon}{5}}}\right)\Big)+\ln\left(r_{\zeta_0,1}\right)\right|\\
&\leq 3\frac{\epsilon}{5}+\left|-\mathrm{Mod}\Big(F^{\widetilde{\mu}}\left(A_{\zeta,r_{\zeta,1},r_{\frac{\epsilon}{5}}}\right)\Big)+\ln\left(\frac{r_{\frac{\epsilon}{5}}}{r_{\zeta,1}}\right)\right|\\
&\qquad+\left|\mathrm{Mod}\Big(F^{\widetilde{\mu}}\left(A_{\zeta_0,r_{\zeta_0,1},r_{\frac{\epsilon}{5}}}\right)\Big)-\ln\left(\frac{r_{\frac{\epsilon}{5}}}{r_{\zeta_0,1}}\right)\right|\leq\epsilon.
\end{aligned}$$

This shows the continuity of $|f'|$ at any $\zeta_0 \in \mathbb{S}^1$, thus f' is continuously differentiable on $\mathbb{S}^1$ and since the derivative is never 0, any element $f \in \mathbf{T}^1$ is a $\mathcal{C}^1$-diffeomorphism on $\mathbb{S}^1$. Since $\mathbf{T}^p \subset \mathbf{T}^1$ $(p \leq 1)$ we have shown that Theorem 1 holds.

Since every differentiable quasisymmetric function f on $\mathbb{S}^1$ is symmetric in the sense of (1), the following already known property follows from Theorem 1.

Corollary 1. *Let* $0 < p \leq 1$. *Then,* $\mathbf{T}^p \subset \mathbf{T}_{\mathrm{s}}$.

Let us point out that although $\mathbf{T}^1 \subset \mathbf{T}_{\mathrm{s}}$, the quasiconformal extension F of f we were working with may not necessarily be asymptotically conformal on $\mathbb{S}^1$ and Claim 2 is not obvious. However, for $p \geq 2$, if one specifically employs the Douady–Earle extension, then Claim 2 holds. It seems natural to ask:

Question 1. Let $f \in \mathbf{T}^p$ (with $0 < p \leq 2$). Is there a quasinconformal extension F of f to the closed unit disc which is asymptotically conformal on $\mathbb{S}^1$ and such that $\mu_F \in \mathrm{L}^p(\mathbb{D}, \sigma)$?

Furthermore, since we obtain smoothness properties for the elements of $\mathbf{T}^p$ (for $p \leq 1$), we suggest that one can show higher and higher order of smoothness for $p < 1$, as p gets smaller and smaller. If this is the case we would like to find sharp results on how the order of smoothness depends on p, a question that seems to be similar to finding a characterization of $\mathbf{T}^p$ using Sobolev spaces for $p \geq 2$. In addition, we pose the following question:

Question 2. What is $\bigcap_{p>0} \mathbf{T}^p$?

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