SOLVABILITY OF CONTROL PROBLEM FOR A NONLOCAL NEUTRAL STOCHASTIC FRACTIONAL INTEGRO-DIFFERENTIAL INCLUSION WITH IMPULSES

ALKA CHADHA and SWAROOP NANDAN BORA

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This paper considers a class of neutral impulsive stochastic integro-differential inclusion with nonlocal conditions involving the Caputo fractional derivative of order $1 < \alpha < 2$ in a Hilbert space. A new set of sufficient conditions for the approximate controllability of semilinear fractional stochastic systems is derived utilizing solution operator, stochastic analysis, fractional calculus and fixed point theorem for a multivalued operator under the assumption that the corresponding linear system is approximately controllable. An example is provided to illustrate the derived theory at the end of the paper.

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1. INTRODUCTION

The investigation of stochastic differential equations has been picking up much importance and attention of researchers due to its wide applicability in science and engineering, for example, biology, mechanics, electrical engineering, physics, filtering of diffusion processes, optimal stochastic control and so on. By utilizing numerous techniques, the existence, uniqueness, stability and qualitative analysis of the mild solutions of stochastic differential equations have been studied by many authors, see [12, 13, 25, 29, 30, 35, 45]. The theory of fractional evolution equations or inclusions has been proven to be applicable to problems arising in mechanics, viscoelasticity, electrical engineering, medicine, electro-chemistry, control, biology, ecology, etc. Fractional evolution inclusions are a kind of important differential inclusions describing the processes behaving in much more complex ways on time which appear as a generalization of fractional evolution equations through the application of multivalued analysis. The fractional order models of real systems are always more adequate than the classical integer order models, since the description of some systems is more accurate when the derivative of fractional order is utilized. For more details, see the references [8–10, 18–20, 32, 38, 39, 42, 43].

The dynamics of many evolving processes are subject to sudden changes, such as harvesting, shocks and natural disasters. These phenomena involve short-term perturbations from continuous and smooth dynamics, whose duration is negligible in comparison with the duration of an entire evolution. In models that involve such perturbation, it is natural to assume that these perturbations act instantaneously or in the form of impulses. As a consequence, differential equations involving impulses have been developed in modeling impulsive systems appearing in physics, ecology, population dynamics, industrial robotics, optimal control, pharmacokinetics and so on. The change of the water level of artificial reservoirs in environment sciences can be modeled as an impulsive differential equation. For more study of impulsive equations, we refer to monographs [3, 23] and articles [2, 5, 8, 9, 11, 19, 26, 38, 39, 42]. On the other hand, controllability is a very important concept which plays an important role in both the deterministic and the stochastic control theories. Generally, controllability implies that it is possible to steer a dynamical control system from an arbitrary beginning state to an arbitrary last state utilizing the set of admissible controls. In the case of infinite dimensional systems, two essential concepts of controllability can be recognized, known as exact and approximate controllability. This is emphatically identified by the fact that there exist linear subspaces in infinite dimension, which are not closed. Exact controllability enables to steer the system to an arbitrary last state while approximate controllability implies that system can be directed to an arbitrary small neighborhood of final state. Especially, approximate controllability gives the possibility of steering the system to states which form a dense subspace in the state space. Clearly, exact controllability is basically a stronger notion than approximate controllability which means that exact controllability implies approximate controllability but generally, the converse statement is false. On the other hand, exact controllability appears rather exceptionally in case of infinite dimensional systems. However, it should be noticed that in the case of finite dimensional systems, notions of exact and approximate controllability coincide. The approximate controllability of nonlinear deterministic and stochastic systems is well studied by many authors in the literature, see [1, 2, 4, 14, 17, 18, 20–22, 24, 26–28, 34, 36, 37, 41, 44–46].

In this paper, we consider the following neutral stochastic integro-differential inclusions with nonlocal conditions in a separable Hilbert space $(\mathbb{U}; \|\cdot\|_{\mathbb{U}})$ with the inner product $(\cdot,\cdot)_{\mathbb{U}}$:

$$(1.1)\qquad {}^{c}D_t^{\alpha-1}\Big[y'(t) - F\Big(t, y(h_1(t)), \int_0^t a_1(t,\tau,y(h_2(\tau)))\mathrm{d}\tau\Big)\Big] \in$$

$$Ay(t) + Bu(t) + K(t, y(h_3(t))) + G(t, y(h_4(t)))\frac{\mathrm{d}w}{\mathrm{d}t}, \ t \in J = (0, T],$$

$$\Delta y(t_i) = I_i(y(t_i)), \quad \Delta y'(t_i) = J_i(y(t_i)), \ \ i = 1, \cdots, m, \tag{1.2}$$

$$y(0) + h(y) = y_0 \in \mathbb{U}, \ \ y'(0) + g(y) = y_1 \in \mathbb{U}, \tag{1.3}$$

where ${}^cD_t^\alpha$ means the Caputo fractional derivative of order $1 < \alpha < 2$, $0 < T < \infty$, A is a closed and densely defined linear operator on a common domain in a Hilbert space $\mathbb{U}$, the control function $u \in L^2_{\mathcal{F}}([0, T], H)$, a Hilbert space of admissible control functions and B is a bounded linear operator from a Banach space H to $\mathbb{U}$. The functions F, G, K, h_1, h_2, h_3, h_4 h are appropriate continuous functions to be specified later and $h_j \in C(J, J)$, $j = 1, 2, 3, 4$.

The rest of the paper is organized as follows. Section 2 presents some basic definitions of fractional calculus, notations, lemmas and theorems. In section 3, the existence of the mild solution to impulsive neutral stochastic control system (1.2)–(1.3) is shown by virtue of solution operator via fixed point technique, stochastic analysis and functional analysis. Then, a set of sufficient conditions proving approximate controllability of the system is formulated with the assumption that the associated linear system is approximately controllable. Section 4 provides an example illustrating the abstract results that are obtained.

2. PRELIMINARIES

Throughout the work, the notations $(\mathbb{U}, \|\cdot\|_{\mathbb{U}}, (\cdot,\cdot)_{\mathbb{U}})$ and $(\mathbb{V}, \|\cdot\|_{\mathbb{V}}, (\cdot,\cdot)_{\mathbb{V}})$ stand for the separable Hilbert spaces. The notation $C(J, \mathbb{U})$ stands for the Banach space of continuous functions from J to $\mathbb{U}$ with supremum norm, i.e., $\|y\|_J = \sup_{t\in J} \|y(t)\|$, $\forall\ y \in C(J, \mathbb{U})$ and $L^1(J, \mathbb{U})$ denotes the Banach space of functions $y : J \to \mathbb{U}$ which are Bochner integrable normed by $\|y\|_{L^1} = \int_0^T \|y(t)\|\mathrm{d}t$, for all $y \in L^1(J, \mathbb{U})$. A measurable function $y : J \to \mathbb{U}$ is Bochner integrable if and only if $\|y\|$ is Lebesgue integrable. The notation $\mathbb{B}(\mathbb{U})$ stands for the Banach space of all linear bounded operator from $\mathbb{U}$ into itself with norm

$$\|f\|_{\mathbb{B}(\mathbb{U})} = \sup\{\|f(y)\| : \ \|y\| \le 1\}, \ \ \forall \ f \in \mathbb{B}(\mathbb{U}). \tag{2.1}$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a normal filtration $\mathbf{F} = \mathcal{F}_t$, $t \in [0, T]$. A filtration $\mathbf{F}$ is a sequence of σ-algebra $\{\mathcal{F}_t\}_{t\ge 0}$ with $\mathcal{F}_t \subset \mathcal{F}$ for each t and $t_1 \le t_2 \Rightarrow \mathcal{F}_{t_1} \subset \mathcal{F}_{t_1}$. An $\mathbb{U}$-valued random variable is an $\mathcal{F}_t$-measurable function $y(t) : \Omega \to \mathbb{U}$ and the space $S = \{y(t, \omega) : \Omega \to \mathbb{U} : t \in [0, T]\}$ which contains all random variables is called a stochastic process. In addition, we use the notation $y(t)$ instead of $y(t, w)$ and $y(t) : J \to \mathbb{U} \in$

S. We assume that $\{w(t) : t \geq 0\}$ is a $\mathbb{V}$-valued Wiener process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbf{F})$ with covariance operator $(\mathcal{Q})$, where $\mathcal{Q}$ is a positive, self-adjoint, trace class operator on $\mathbb{V}$. Especially, $w(t)$ denotes an $\mathbb{V}$-valued $\mathcal{Q}$-Wiener process with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of K and $\{\lambda_n\}_{n=1}^{\infty}$ be a bounded sequence of nonnegative real numbers with $\mathcal{Q}e_i = \lambda_i e_i$. Further, we consider a sequence β_i of independent Brownian motions with

$$(w(t), e) = \sum_{n=1}^{\infty} \lambda_n (e_n, e)\beta_n(t), \quad e \in \mathbb{V}, \quad t \in [0, T]$$

and $\mathcal{F}_t = \mathcal{F}_t^w$ is the σ-algebra generated by $\{w(s) : 0 \leq s \leq t\}$. The symbol $L(\mathbb{V}, \mathbb{U})$ stands for the space of all bounded linear operators from $\mathbb{V}$ into $\mathbb{U}$ with the usual norm $\| \cdot \|_{L(\mathbb{V},\mathbb{U})}$ and $L(\mathbb{U})$ when $\mathbb{V} = \mathbb{U}$. Suppose that $\mathcal{F}_t = \sigma\{w(s) : 0 \leq s \leq t\}$ is the σ-algebra generated by w and $\mathcal{F}_T = \mathcal{F}$.

For defining stochastic integrals with respect to the $\mathcal{Q}$-Wiener process $w(t)$, we consider the subspace $\mathbb{V}_0 = \mathcal{Q}^{1/2}(\mathbb{V})$ of $\mathbb{V}$ with the inner product $(x, y)_{\mathbb{V}_0} = (\mathcal{Q}^{-1/2}x, \mathcal{Q}^{-1/2}y)_{\mathbb{V}}$. It is easy to verify that $\mathbb{V}_0$ is a Hilbert space. Let $L_0^2 = L^2(\mathbb{V}_0, \mathbb{U})$ be the space of all Hilbert-Schmidt operators from $\mathbb{V}_0$ to $\mathbb{U}$ with the following norm

$$\|\varphi\|_{L_0^2}^2 = Tr((\varphi\mathcal{Q}^{1/2})(\varphi\mathcal{Q}^{1/2})^*), \quad \text{for any} \quad \varphi \in L_0^2.$$

For any bounded operators $\varphi \in L(\mathbb{V}, \mathbb{U})$, it is clear that the above norm can be reduced to $\|\varphi\|_{L_0^2}^2 = Tr(\varphi\mathcal{Q}\varphi^*)$. The notation $L^2(\mathcal{F}_T, \mathbb{U})$ stands for the Banach space which contains all $\mathcal{F}_T$-measurable square integrable random variables with values in the Hilbert space $\mathbb{U}$. The notation $C([0, T]; L^2(\mathcal{F}, \mathbb{U}))$ represents the Banach space of continuous maps from $[0, T]$ into $L^2(\mathcal{F}, \mathbb{U})$ that satisfies the condition $\sup_{t\in[0,T]} \mathbb{E}\|y(t)\|^2 < \infty$, where $\mathbb{E}$ denotes the integration with respect to a probability measure $\mathbb{P}$ *i.e.*, $\mathbb{E}y = \int_\Omega y d\mathbb{P}$.

Let $\mathcal{C} = \mathcal{C}([0, T], \mathbb{U})$ be the closed subspace of $C([0, T], L^2(\mathcal{F}, \mathbb{U}))$ consisting of measurable and $\mathcal{F}_t$-adapted $\mathbb{U}$-valued stochastic processes $y \in C([0, T], L^2(\mathcal{F}, \mathbb{U}))$ endowed with the norm $\|y\|_{\mathcal{C}} = (\sup_{t\in[0,T]} E\|y(t)\|_{\mathbb{U}}^2)^{1/2}$. It is easy to verify that $(\mathcal{C}, \| \cdot \|_{\mathcal{C}})$ is a Banach space. For a basic study on stochastic differential equation, we refer to the book [13].

To set the structure for our primary existence results, we give the following definitions.

Definition 2.1. The Riemann-Liouville fractional integral operator $\mathbb{J}$ of order $\beta > 0$ is defined by

$$^{RL}\mathbb{J}_t^{\beta} F(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} F(s) \mathrm{d}s, \tag{2.2}$$

where $F \in L^1((0,T),\mathbb{U})$.

Definition 2.2. The Riemann-Liouville fractional derivative is given as

$$ {}^{RL}D_t^{\beta}F(t) = D_t^m \mathbb{J}_t^{m-\beta}F(t),\ m-1<\beta<m,\ m\in\mathbb{N}, \tag{2.3}$$

where $D_t^m = \frac{d^m}{dt^m}$, $F \in L^1((0,T),\mathbb{U})$, ${}^{RL}\mathbb{J}_t^{m-\beta}F \in W^{m,1}((0,T),\mathbb{U})$. Here the notation $W^{m,1}((0,T),\mathbb{U})$ stands for the Sobolev space defined by

$$\begin{aligned} W^{m,1}((0,T),\mathbb{U}) &= \{y\in\mathbb{U} : \exists z\in L^1((0,T),\mathbb{U}) : y(t) = \sum_{k=0}^{m-1} d_k \frac{t^k}{k!} \\ &\quad + \frac{t^{m-1}}{(m-1)!} * z(t),\ \ t\in(0,T)\}. \end{aligned} \tag{2.4}$$

Note that $z(t) = y^m(t)$, $d_k = y^k(0)$.

Definition 2.3. The Caputo fractional derivative is given as

$$ {}^{c}D_t^{\beta}F(t) = \frac{1}{\Gamma(m-\beta)}\int_0^t (t-s)^{m-\beta-1}F^m(s)\mathrm{d}s,\ \ m-1<\beta<m, \tag{2.5}$$

where $F \in C^{m-1}((0,T),\mathbb{U}) \cap L^1((0,T),\mathbb{U})$.

Definition 2.4. Let $A : D(A) \subset \mathbb{U} \to \mathbb{U}$ be a closed linear operator. The operator A is said to be sectorial operator of type (M,θ,α,ω) if there exist constants $\omega\in\mathbb{R}$, $0<\theta<\pi/2$, $M>0$ such that

(i) The α-resolvent of A exists outside the sector $\omega + S_\theta = \{\omega+\lambda^\alpha : \lambda \in \mathbb{C},\ |Arg(-\lambda^\alpha)|<\theta\}$,

(ii) $\|R(\lambda^\alpha, A)\| = \|(\lambda^\alpha - A)^{-1}\| \le \frac{M}{|\lambda^\alpha-\omega|}$, $\lambda\notin\omega+S_\theta$.

If A is a sectorial operator of type (M,θ,α,ω), then it is easy to see that A generates an α-resolvent family $\{\mathcal{R}_\alpha(t) : t\ge 0\}$ in a Banach space, where $\mathcal{R}_\alpha(t) = \frac{1}{2\pi i}\int_C e^{\lambda t}R(\lambda^\alpha,A)\mathrm{d}\lambda$. For more details, see [39].

Lemma 2.1 ([39]). *Suppose that A is a sectorial operator of type (M,θ,α,ω) and f satisfies the uniform Hölder condition with the exponent $\beta\in(0,1]$. Then, the unique solution of the Cauchy problem*

$$ D^\alpha y(t) = Ay(t) + f(t),\ \ t\in[0,T],\ 1<\alpha<2, \tag{2.6}$$

$$ y(0) = y_0\in\mathbb{U},\ \ y'(0) = y_1 \in\mathbb{U}, \tag{2.7}$$

is given by

$$ y(t) = \mathcal{T}_\alpha(t)y_0 + \mathcal{K}_\alpha(t)y_1 + \int_0^t \mathcal{R}_\alpha(t-s)f(s)\mathrm{d}s,\ t\in[0,T]. \tag{2.8}$$

The operators $\mathcal{T}_\alpha(t)$, $\mathcal{K}_\alpha(t)$ and $\mathcal{R}_\alpha(t)$ are defined as

$$\mathcal{T}_\alpha(t) = \frac{1}{2\pi i}\int_C e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha,A)\mathrm{d}\lambda,$$

$$\begin{aligned} \mathcal{K}_\alpha(t) &= \frac{1}{2\pi i}\int_C e^{\lambda t}\lambda^{\alpha-2}R(\lambda^\alpha, A)\mathrm{d}\lambda, \\ \mathcal{R}_\alpha(t) &= \frac{1}{2\pi i}\int_C e^{\lambda t}R(\lambda^\alpha, A)\mathrm{d}\lambda, \end{aligned} \tag{2.9}$$

where C is a suitable path such that $\lambda^\alpha \notin \omega + S_\theta,\ \lambda \in \mathbb{C}$.

We use the symbol $\mathcal{P}(\mathbb{U})$ for the family of all subsets of $\mathbb{U}$ and denote

$$\begin{aligned} \mathcal{P}_{cl}(\mathbb{U}) &= \{Z \in \mathcal{P}(\mathbb{U}) : Z \text{ is closed}\}, \\ \mathcal{P}_{bd}(\mathbb{U}) &= \{Z \in \mathcal{P}(\mathbb{U}) : Z \text{ is bounded}\}, \\ \mathcal{P}_{cv}(\mathbb{U}) &= \{Z \in \mathcal{P}(\mathbb{U}) : Z \text{ is convex}\}, \\ \mathcal{P}_{cp}(\mathbb{U}) &= \{Z \in \mathcal{P}(\mathbb{U}) : Z \text{ is compact}\}. \end{aligned} \tag{2.10}$$

We now present a few facts on multi-valued operators.

The multi-valued operator $\Upsilon : \mathbb{U} \to \mathcal{P}(\mathbb{U})$ is called convex (closed) valued if $\Upsilon(x)$ is convex (closed) for every $x \in \mathbb{U}$. If $\Upsilon(F) = \cup_{z\in F}\Upsilon(z)$ is bounded in $\mathbb{U}$ for all $F \in \mathcal{P}_{bd}(\mathbb{U})$, *i.e.*, $\sup_{z\in F}\{\sup\{\|y\| : y \in \Upsilon(z)\}\} < \infty$, then map Υ is bounded on bounded sets. A multi-valued map $\Upsilon : \mathbb{U} \to \mathcal{P}(\mathbb{U})$ is called upper semicontinuous (u.s.c.) if, for any $u \in \mathbb{U}$, the set $\Upsilon(u)$ is a nonempty closed subset of $\mathbb{U}$ and if for each open set $\mathcal{G}$ of $\mathbb{U}$ which is contained $\Upsilon(u)$ and there exists an open neighborhood $\mathcal{N}$ of u such that $\Upsilon(\mathcal{N}) \subset \mathcal{G}$. The map Υ is called completely continuous if $\Upsilon(\mathcal{G})$ is relatively compact for every bounded subset of $\mathcal{G} \subseteq \mathbb{U}$. If the multi-valued map Υ is completely continuous with nonempty compact values, then Υ is u.s.c. if and only if Υ has a closed graph, *i.e.*, $z_n \to z$, $y_n \to y$, $y_n \in \Upsilon(z_n) \Rightarrow y \in \Upsilon(z)$. The map Υ is called completely continuous if $\Upsilon(F)$ is relatively compact, for every bounded subset $F \subseteq \mathbb{U}$. A multi-valued function $\Upsilon : [0, T] \to \mathcal{P}_{cl}$ is said to be measurable if, for each $y \in \mathbb{U}$, the function $\mathbb{Y} : [0, T] \to \mathbb{R}^+$ given by $\mathbb{Y}(t) = d(y, \Upsilon(t)) = \inf\{d(y, z) : z \in \Upsilon(t)\}$ is measurable.

Let us consider $\mathbb{U}_d : \mathcal{P}(\mathbb{U}) \times \mathcal{P}(\mathbb{U}) \to \mathbb{R}^+ \cup \{\infty\}$ defined by

$$\mathbb{U}_d(\widetilde{G}, \widetilde{H}) = \max\{\sup_{\widetilde{g}\in\widetilde{G}} d(\widetilde{g}, \widetilde{H}), \sup_{\widetilde{h}\in\widetilde{H}} d(\widetilde{G}, \widetilde{h})\},$$

where $d(\widetilde{G}, \widetilde{h}) = \inf_{\widetilde{g}\in\widetilde{G}} d(\widetilde{g}, \widetilde{h})$ and $d(\widetilde{g}, \widetilde{U}) = \inf_{\widetilde{h}\in\widetilde{H}} d(\widetilde{g}, \widetilde{h})$. Note that $(\mathcal{P}_{bd,cl}(\mathbb{U}), \mathbb{U}_d)$ is a metric space and $(\mathcal{P}_{cl}(\mathbb{U}, \mathbb{U}_d)$ is a generalized metric space. The map Υ has a fixed point if there exists a $y \in \mathbb{U}$ with $y \in \Upsilon(y)$. For more study on multi-valued maps, we refer to the book [15].

Definition 2.5. Let $\Upsilon : \mathbb{U} \to \mathcal{P}_{bd,cl}(\mathbb{U})$ be a multivalued mapping. Then, Υ is called a multivalued contraction if there exists a constant $\mu \in (0, 1)$ such that

$$\mathbb{U}_d(\Upsilon(x) - \Upsilon(y)) \leq \mu\|x - y\|_{\mathbb{U}}, \tag{2.11}$$

for each $x, y \in \mathbb{U}$. The constant μ is called a contraction constant of Υ.

Definition 2.6. The multi-valued map $G : [0,T] \times \mathbb{U} \to \mathcal{P}_{bd,cl,cv}(L(\mathbb{V},\mathbb{U}))$ is called L^2-Carathéodory if

(a) The map $t \mapsto G(t,y)$ is measurable for each $y \in \mathbb{U}$;

(b) The map $y \mapsto G(t,y)$ is u.s.c. for almost all $t \in J$;

(c) There are a continuous function $W_G \in L^1([0,T];\mathbb{R}_+)$ and a continuous nondecreasing function $\Theta_G : [0,\infty) \to (0,\infty)$ such that

$$\|G(t,z)\|_{\mathbb{U}}^2 = \sup_{g \in G(t,z)} \mathbb{E}\|g\|_{\mathbb{U}}^2 \leq W_G(t)\Theta_G(\|z\|_{\mathbb{U}}^2), \text{and for a.e. } t \in J. \tag{2.12}$$

Thus, we have the following result stated as

LEMMA 2.2 ([15]). *Let $\mathbb{U}$ be a Hilbert space and I be a compact interval. If G is a L^2-Carathéodory multi-valued map with $\mathcal{N}_{G,y} \neq 0$ and Υ is a linear continuous mapping from $L^2(I,\mathbb{U})$ to $C(I,\mathbb{U})$, then the map*

$$\Upsilon \circ \mathcal{N}_G : C(I,\mathbb{U}) \to \mathcal{P}_{cp,cv}(\mathbb{U}), \quad u \mapsto (\Upsilon \circ \mathcal{N}_G)(y) = \Upsilon(\mathcal{N}_{G,y}), \tag{2.13}$$

is a closed graph operator in $C(I,\mathbb{U}) \times C(I,\mathbb{U})$, where $\mathcal{N}_{G,y}$ denotes the selectors set from G defined as

$$g \in \mathcal{N}_{G,y} = \{g \in L^2(I, L(\mathbb{V},\mathbb{U})) : g(t) \in G(t, y(h_4(t))) \textit{ for a.e. } t \in [0,T]\}. \tag{2.14}$$

Next, we present the definition of the mild solution to the problem (1.2)–(1.3) based on the papers [19] and [39].

Definition 2.7. An $\mathcal{F}_t$-adapted stochastic process $y \in \mathcal{C}$ is called a mild solution of the problem (1.2)–(1.3) if, for each control $L^2_{\mathcal{F}}([0,T],H)$,

(i) $y_0, h \in L_0^p(\Omega,\mathcal{C})$;

(ii) $y(0) + h(y) = y_0$, $y'(0) + g(y) = y_1$;

(iii) $y(t) \in \mathbb{U}$ has càdlàg paths on $t \in [0,T]$ a.s., and there is a function $g \in \mathcal{N}_{G,y(h_3(t))}$ such that

$$\begin{aligned} y(t) \in\ & \mathcal{T}_\alpha(t)[y_0 - h(y)] + \mathcal{K}_\alpha(t)[y_1 - g(y) - F(0, y(h_1(0)), 0)] \\ & + \int_0^t \mathcal{T}_\alpha(t-s) F(s, y(h_1(s)), \int_0^s a_1(s,\tau,y(h_2(\tau)))\mathrm{d}\tau)\mathrm{d}s \\ & + \int_0^t \mathcal{R}_\alpha(t-s)Bu(s)\mathrm{d}s + \int_0^t \mathcal{R}_\alpha(t-s)K(s, y(h_3(s)))\mathrm{d}s \\ & + \int_0^t \mathcal{R}_\alpha(t-s)g(s)\mathrm{d}w(s) + \sum_{0<t_i<t} \mathcal{T}_\alpha(t-t_i)I_i(y(t_i)) \\ & + \sum_{0<t_i<t} \mathcal{K}_\alpha(t-t_i)J_i(y(t_i)), \qquad t \in [0,T]. \end{aligned} \tag{2.15}$$

The symbol $y(t;u)$ denotes the state value of the system (1.2)–(1.3) at time t corresponding to the control $u \in L^2_{\mathcal{F}}([0,T],H)$. Particularly, the state of system (1.2)–(1.3) at $t=T$, $y(T;u)$ is called the terminal state with control u. We consider the space

$$\mathcal{RS}(T) = \{y(T;u) : u(\cdot) \in L^2_{\mathcal{F}}([0,T],H)\}$$

which is known as the reachable set of (1.2) at the terminal time T. Here $L^2_{\mathcal{F}}([0,T],H)$ is the closed subspace of $L^2_{\mathcal{F}}([0,T]\times\Omega,H)$ which consists of all $\mathcal{F}_t$-adapted, H-valued stochastic processes.

Definition 2.8. The system (1.2) is said to be approximately controllable on the interval $[0,T]$ if $\overline{\mathcal{RS}}(T) = L^2(\mathcal{F}_T,\mathbb{U})$, where $\overline{\mathcal{RS}}(T)$ denotes the closure of the reachable set.

Now, we consider the following notations

$$\begin{aligned}
\Gamma^T_\zeta &= \int_\zeta^t \mathcal{R}_\alpha(T-s)BB^*\mathcal{R}^*_\alpha(T-s)\mathrm{d}s, \quad \zeta\in[0,T),\\
\Gamma^T_0 &= \int_0^t \mathcal{R}_\alpha(T-s)BB^*\mathcal{R}^*_\alpha(T-s)\mathrm{d}s,\\
R(\kappa,\Gamma^T_0) &= (\kappa I+\Gamma^T_0)^{-1}, \qquad (2.16)
\end{aligned}$$

where B^* denotes the adjoint of B and $\mathcal{R}^*_\alpha(t)$ is the adjoint of $\mathcal{R}_\alpha(t)$. It is simple to show that the operator Γ^T_0 is a linear bounded operator.

Lemma 2.3 ([27]). *The linear integro-differential Cauchy problem corresponding to system (1.2)–(1.3) is approximately controllable on $[0,T]$ if and only if $\kappa R(\kappa,\Gamma^T_\zeta)\to 0$, $0\le\zeta<s\le T$ as $\kappa\to 0^+$ in the strong operator topology.*

Lemma 2.4 ([20]). *For any $\hat{y}_T\in L^2(\mathcal{F}_T,\mathbb{U})$, there exists $\hat{\phi}\in L^2_{\mathcal{F}}(\Omega;L^2([0,T],L^0_2))$ with $\hat{y}_T=\mathbb{E}\hat{y}_T+\int_0^T\hat{\phi}(s)\mathrm{d}w(s)$.*

Now, for any $\kappa>0$ and $\hat{y}_T\in L^2(\mathcal{F}_T,\mathbb{U})$, we define the control function

$$\begin{aligned}
u^\kappa(t) = {}& B^*\mathcal{R}^*_\alpha(T-t)(\kappa I+\Gamma^T_0)^{-1}\Bigg[E\hat{y}_T+\int_0^T\hat{\phi}(s)\mathrm{d}w(s)\\
&-\mathcal{T}_\alpha(T)(y_0-h(y))-\mathcal{K}_\alpha(T)\Big(y_1-g(y)-F(0,y(h_1(0)),0)\Big)\Bigg]\\
&-B^*\mathcal{R}^*_\alpha(T-t)\int_0^T(\kappa I+\Gamma^T_s)^{-1}\mathcal{T}_\alpha(T-s)\\
&\times F(s,y(h_1(s)),\int_0^s a_1(s,\tau,y(h_2(\tau)))\mathrm{d}\tau)\mathrm{d}s-B^*\mathcal{R}^*_\alpha(T-t)
\end{aligned} \qquad (2.17)$$

$$
\begin{aligned}
&\times \int_0^T (\kappa I + \Gamma_s^T)^{-1}\mathcal{R}_\alpha(T-s)K(s, y(h_3(s)))\mathrm{d}s - B^*\mathcal{R}_\alpha^*(T-t) \\
&\times \int_0^T (\kappa I + \Gamma_s^T)^{-1}\mathcal{R}_\alpha(T-s)g(s)\mathrm{d}w(s) \\
&- B^*\mathcal{R}_\alpha^*(T-t) \sum_{0<t_i<t} (\kappa I + \Gamma_s^T)^{-1}\mathcal{T}_\alpha(T-t_i)I_i(y(t_i)) \\
&- B^*\mathcal{R}_\alpha^*(T-t) \sum_{0<t_i<t} (\kappa I + \Gamma_s^T)^{-1}\mathcal{K}_\alpha(T-t_i)J_i(y(t_i)), \;\; t \in [0, T].
\end{aligned}
$$

Now, we present the following fixed point theorem (Nonlinear alternative of Leray-Schauder type for multivalued maps due to D. O'Regan) which will be used to prove existence results.

LEMMA 2.5 ([33]). *Let $\mathbb{U}$ be a Hilbert space with V an open, convex subset of $\mathbb{U}$ and $y_0 \in V$. Assume*

(1) $\Psi : \overline{V} \to \mathcal{P}_{cd}(\mathbb{U})$ *has closed graph, and*

(2) $\Psi : \overline{V} \to \mathcal{P}_{cd}(\mathbb{U})$ *is a condensing map with $\Psi(\overline{V})$ a subset of a bounded set in $\mathbb{U}$ hold. Then either*

(i) Ψ *has a fixed point in $\overline{V}$, or*

(ii) *There exist $y \in \partial V$ and $\lambda \in (0, 1)$ with $y \in \lambda\Psi(y) + (1-\lambda)\{y_0\}$.*

3. MAIN RESULTS

Before expressing and demonstrating the main result, we assume the following assumptions to establish the required result.

(A1) The operators $\mathcal{T}_\alpha(t), \mathcal{K}_\alpha(t), \mathcal{R}_\alpha(t), t \geq 0$ generated by A are compact in $\overline{D(A)}$ and

$$
\sup_{t\in[0,T]} \|\mathcal{T}_\alpha(t)\| \leq \widetilde{M}, \quad \sup_{t\in[0,T]} \|\mathcal{K}_\alpha(t)\| \leq \widetilde{M}, \quad \sup_{t\in[0,T]} \|\mathcal{R}_\alpha\| \leq \widetilde{M}.
$$

(A2) (i) The function $F : [0, T] \times \mathbb{U} \times \mathbb{U} \to \mathbb{U}$ is a continuous function and there exists a positive constant $L_F > 0$ such that

$$
\|F(t, u_1, v_1) - F(t, u_2, v_2)\|^2 \leq L_F[\|u_1 - v_1\|_{\mathbb{U}}^2 + \|u_2 - v_2\|_{\mathbb{U}}^2],
$$

for all $u_1, v_1, u_2, v_2 \in \mathbb{U}$ and $t \in [0, T]$ and $\mathbb{D}_1 = \sup_{t\in[0,T]} \|F(t, 0, 0)\|_{\mathbb{U}}^2$ with $\mathbb{E}\|u_i\|^2 < \infty$, $\mathbb{E}\|v_i\|^2 < \infty$ for $i = 1, 2$.

(ii) The map $a_1 : D_1 \times \mathbb{U} \to \mathbb{U}$ is a continuous mapping and there exists a positive constant L_{a_1} such that

$$
\| \int_0^t [a_1(t, s, z_1) - a_1(t, s, z_2)]\mathrm{d}s\|_{\mathbb{U}}^2 \leq L_{a_1}\|z_1 - z_2\|_{\mathbb{U}}^2,
$$

for all $z_1, z_2 \in \mathbb{U}$ and $t \in [0,T]$ with $\mathcal{L}^1_{a_1} = T\sup_{(t,s)\in D_1} \|a_1(t,s,0)\|^2_{\mathbb{U}}$ and $\widetilde{M}^2 L_F[1+(1+2L_{a_1})] < 1$, where $D_1 = \{(t,s) \in [0,T]\times[0,T] : t \geq s\}$ and $\mathbb{E}\|z_i\|^2 < \infty$.

(A3) The multivalued map $G : J \times \mathbb{U} \to \mathcal{P}_{bd,cl,cv}(L(\mathbb{V},\mathbb{U}))$ and the map $K : J \times \mathbb{U} \to \mathbb{U}$ are L^2-Carathéodory functions such that
(i) The map $G(t,\cdot) : \mathbb{U} \to \mathcal{P}_{bd,cl,cv}(L(\mathbb{V},\mathbb{U}))$ is u.s.c. for each $t \in [0,T]$ and $G(\cdot,y)$ is measurable for each $y \in \mathbb{U}$. Then, the set

$$\mathcal{N}_{G,z} = \{\varrho \in L^2([0,T], L(\mathbb{V},\mathbb{U})) : \varrho(t) \in G(t, z(h_3(t))) \text{ for a.e. } t \in [0,T]\}$$

for fixed $z \in \mathcal{C}$, is nonempty;
(ii) There exists a continuous function $m_G : [0,T] \to [0,\infty)$ and an increasing function $W_G : \mathbb{R} \to (0,\infty)$ such that

$$\|G(t,z)\|^2_{\mathbb{U}} = \sup\{\|g\|^2_{\mathbb{U}} : g \in G(t,z)\} \leq m_G(t)W_G(\mathbb{E}\|z\|^2), \quad \text{a.e. } t \in [0,T],\ z \in \mathbb{U}.$$

(iii)There exists a continuous function $m_K : [0,T] \to [0,\infty)$ and an increasing function $W_K : \mathbb{R} \to (0,\infty)$ such that

$$\|K(t,z)\|^2 \leq m_K(t)W_K(\mathbb{E}\|z\|^2), \ \ \forall\ z \in \mathbb{U},\ t \in J, \tag{3.1}$$

and $\int_0^\infty \frac{\mathrm{d}s}{W_K(s)+W_G(s)} < \infty$.

(A4) $h, g : \mathbb{U} \to \mathbb{U}$ are completely continuous functions and there exist positive constants $\mathcal{C}_1$,$\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ such that

$$\begin{aligned} \|h(z)\|^2_{\mathbb{U}} &\leq \mathcal{C}_1\mathbb{E}\|z\|^2_{\mathbb{U}} + \mathcal{C}_2, \\ \|g(z)\|^2 &\leq \mathcal{C}_3\mathbb{E}\|z\|^2_{\mathbb{U}} + \mathcal{C}_4, \quad z \in \mathbb{U}. \end{aligned} \tag{3.2}$$

(A5) $I_i, J_i : \mathbb{U} \to \mathbb{U}$ are continuous functions and there exist constants $L_{I_i}, L_{J_i} > 0$ such that

$$\begin{aligned} \|I_i(z)\|^2 &\leq L_{I_i}(\mathbb{E}\|z\|^2), \\ \|J_i(z)\|^2 &\leq L_{J_i}(\mathbb{E}\|z\|^2). \end{aligned}$$

THEOREM 3.1. *Let us assume that* (A1)–(A5) *are satisfied, then the system (1.2)-(1.3) has a mild solution on* $[0,T]$.

Proof. To prove the theorem, we first define the operator $\Psi : \mathcal{C} \to \mathcal{C}$ by Ψy the set of $\rho \in \mathcal{C}$ such that

$$\begin{aligned} \rho(t) = {} & \mathcal{T}_\alpha(t)[y_0 - h(y)] + \mathcal{K}_\alpha(t)[y_1 - g(y) - F(0, y(h_1(0)), 0)] \\ & + \int_0^t \mathcal{T}_\alpha(t-s)F(s, y(h_1(s)), \int_0^s a_1(s,\tau,y(h_2(\tau)))\mathrm{d}\tau)\mathrm{d}s \end{aligned} \tag{3.3}$$

$$
\begin{aligned}
&+\int_0^t \mathcal{R}_\alpha(t-s)Bu^\kappa(s)\mathrm{d}s+\int_0^t \mathcal{R}_\alpha(t-s)K(s,y(h_3(s)))\mathrm{d}s\\
&+\int_0^t \mathcal{R}_\alpha(t-s)g(s)\mathrm{d}w(s)+\sum_{0<t_i<t}\mathcal{T}_\alpha(t-t_i)I_i(y(t_i))\\
&+\sum_{0<t_i<t}\mathcal{K}_\alpha(t-t_i)J_i(y(t_i)), \qquad t\in[0,T],
\end{aligned}
$$

where $g \in \mathcal{N}_{G,y(h_3(t))} = \{g \in L^2(L(\mathbb{V},\mathbb{U})) : g(t) \in G(t,y(h_3(t)))$ a.e. $t \in [0,T]\}$. It is clear that the map Ψ is a well defined map from $\mathcal{C}$ into $\mathcal{P}(\mathcal{C})$ by using the facts that F, g, K, h are continuous functions. In order to show that there exists a mild solution for the problem (1.2)–(1.3), it is sufficient to prove that Ψ has a fixed point. We establish the proof through the following steps.

Step 1. We show that there exists an open set $V \subset \mathcal{C}$ with $y \in \lambda\Psi y$ for $\lambda \in (0,1)$ and $y \in \partial V$. Let $y \in \mathcal{C}$. Then, there exists $g \in \mathcal{N}_{G,y(h_3(t))}$ such that

$$
\begin{aligned}
y(t) &= \lambda\mathcal{T}_\alpha(t)[y_0-h(y)]+\lambda\mathcal{K}_\alpha(t)[y_1-g(y)-F(0,y(h_1(0)),0)]\\
&+\lambda\int_0^t \mathcal{T}_\alpha(t-s)F(s,y(h_1(s)),\int_0^s a_1(s,\tau,y(h_2(\tau)))\mathrm{d}\tau)\mathrm{d}s+\lambda\int_0^t \mathcal{R}_\alpha(t-s)B\\
&\times\Bigg\{B^*\mathcal{R}_\alpha^*(T-t)(\kappa I+\Gamma_0^T)^{-1}\Bigg[E\hat{y}_T+\int_0^T \hat{\phi}(s)dw(s)-\mathcal{T}_\alpha(T)(y_0-h(y))\\
&-\mathcal{K}_\alpha(T)\Big(y_1-g(y)-F(0,y(h_1(0)),0)\Big)\Bigg]\\
&-B^*\mathcal{R}_\alpha^*(T-t)\int_0^T(\kappa I+\Gamma_s^T)^{-1}\mathcal{T}_\alpha(T-s)\\
&\times F(s,y(h_1(s)),\int_0^s a_1(s,\tau,y(h_2(\tau)))\mathrm{d}\tau)\mathrm{d}s-B^*\mathcal{R}_\alpha^*(T-t)\\
&\times\int_0^T(\kappa I+\Gamma_s^T)^{-1}\mathcal{R}_\alpha(T-s)K(s,y(h_3(s)))\mathrm{d}s-B^*\mathcal{R}_\alpha^*(T-t)\\
&\times\int_0^T(\kappa I+\Gamma_s^T)^{-1}\mathcal{R}_\alpha(T-s)g(s)\mathrm{d}w(s)\\
&-B^*\mathcal{R}_\alpha^*(T-t)\sum_{0<t_i<t}(\kappa I+\Gamma_s^T)^{-1}\mathcal{T}_\alpha(T-t_i)I_i(y(t_i))\\
&-B^*\mathcal{R}_\alpha^*(T-t)\sum_{0<t_i<t}(\kappa I+\Gamma_s^T)^{-1}\mathcal{K}_\alpha(T-t_i)J_i(y(t_i))\Bigg\}\mathrm{d}s\\
&+\lambda\int_0^t \mathcal{R}_\alpha(t-s)K(s,y(h_3(s)))\mathrm{d}s+\lambda\int_0^t \mathcal{R}_\alpha(t-s)g(s)\mathrm{d}w(s)
\end{aligned}
\tag{3.4}
$$

$$+\lambda\sum_{0<t_i<t}\mathcal{T}_\alpha(t-t_i)I_i(y^r(t_i))+\lambda\sum_{0<t_i<t}\mathcal{K}_\alpha(t-t_i)J_i(y^r(t_i)),\ t\in[0,T].$$

Therefore, we get

$$\begin{aligned}
&\mathbb{E}\|y(t)\|^2\\
\leq&8\Bigg\{\mathbb{E}\|\mathcal{T}_\alpha(t)[y_0-h(y)]\|^2+\mathbb{E}\|\mathcal{K}_\alpha(t)[y_1-g(y)-F(0,y(h_1(0)),0)]\|^2\\
&+\mathbb{E}\|\int_0^t\mathcal{T}_\alpha(t-s)F(s,y(h_1(s)),\int_0^s a_1(s,\tau,y(h_2(\tau)))\mathrm{d}\tau)\mathrm{d}s\|^2\\
&+\mathbb{E}\Big\|\int_0^t\mathcal{R}_\alpha(t-s)B\Bigg\{B^*\mathcal{R}_\alpha^*(T-t)(\kappa I+\Gamma_0^T)^{-1}\Bigg[E\hat{y}_T+\int_0^T\hat{\phi}(s)\mathrm{d}w(s)\\
&-\mathcal{T}_\alpha(T)(y_0-h(y))-\mathcal{K}_\alpha(T)\Big(y_1-g(y)-F(0,y(h_1(0)),0)\Big)\Bigg]\\
&-B^*\mathcal{R}_\alpha^*(T-t)\int_0^T(\kappa I+\Gamma_s^T)^{-1}\mathcal{T}_\alpha(T-s)\\
&\times F(s,y(h_1(s)),\int_0^s a_1(s,\tau,y(h_2(\tau)))\mathrm{d}\tau)\mathrm{d}s-B^*\mathcal{R}_\alpha^*(T-t)\\
&\times\int_0^T(\kappa I+\Gamma_s^T)^{-1}\mathcal{R}_\alpha(T-s)K(s,y(h_3(s)))\mathrm{d}s-B^*\mathcal{R}_\alpha^*(T-t)\\
&\times\int_0^T(\kappa I+\Gamma_s^T)^{-1}\mathcal{R}_\alpha(T-s)g(s)\mathrm{d}w(s)\\
&-B^*\mathcal{R}_\alpha^*(T-t)\sum_{0<t_i<t}(\kappa I+\Gamma_s^T)^{-1}\mathcal{T}_\alpha(T-t_i)I_i(y(t_i))\\
&-B^*\mathcal{R}_\alpha^*(T-t)\sum_{0<t_i<t}(\kappa I+\Gamma_s^T)^{-1}\mathcal{K}_\alpha(T-t_i)J_i(y(t_i))\Bigg\}\mathrm{d}s\Big\|\\
&+\mathbb{E}\|\int_0^t\mathcal{R}_\alpha(t-s)K(s,y(h_3(s)))\mathrm{d}s\|^2+\mathbb{E}\|\int_0^t\mathcal{R}_\alpha(t-s)g(s)\mathrm{d}w(s)\|^2\\
&+\sum_{0<t_i<t}\mathbb{E}\|\mathcal{T}_\alpha(t-t_i)I_i(y^r(t_i))\|^2+\sum_{0<t_i<t}\mathbb{E}\|\mathcal{K}_\alpha(t-t_i)J_i(y^r(t_i))\|^2\Bigg\},
\end{aligned}$$

(3.5)

$$\begin{aligned}
\leq&8\Bigg[2\widetilde{M}^2\Big(\mathbb{E}\|y_0\|^2+\mathcal{C}_1\mathbb{E}\|y\|^2+\mathcal{C}_2\Big)+3\widetilde{M}^2\Big(\mathbb{E}\|y_1\|^2+\mathcal{C}_3\mathbb{E}\|y\|^2+\mathcal{C}_4\\
&+2(L_F\mathbb{E}\|y\|^2+\mathbb{D}_1)\Big)+\widetilde{M}^2T^2\Big[2L_F(1+2L_{a_1})\mathbb{E}\|y\|^2+4L_FL_{a_1}^1+2\mathbb{D}_1\Big]
\end{aligned}$$

$$
\begin{aligned}
&+ M_B^2 \int_0^t \|\mathcal{R}_\alpha(t-s)\| ds \int_0^t \|\mathcal{R}_\alpha(t-s)\| 9 \frac{1}{\kappa^2} M_B^2 \widetilde{M}^2 \\
&\times \Bigg\{ \|\mathbb{E}\hat{y}_T + \int_0^T \hat{\phi}(s) \mathrm{d}w(s)\|^2 + 2\widetilde{M}^2 (\mathbb{E}\|y_0\|_{\mathbb{U}}^2 + \mathcal{C}_1 \mathbb{E}\|y\|_{\mathbb{U}}^2 + \mathcal{C}_3) + 3\widetilde{M}^2 \\
&\times \Big(\mathbb{E}\|y_1\|_{\mathbb{U}}^2 + \mathcal{C}_3 \mathbb{E}\|y\|_{\mathbb{U}}^2 + \mathcal{C}_4 + 2(L_F \mathbb{E}\|y\|_{\mathbb{U}}^2 + \mathbb{D}_1) + \Big) \\
&+ \widetilde{M}^2 T^2 \Big[2L_F(1+2L_{a_1}) \mathbb{E}\|y\|^2 + 4L_F L_{a_1}^1 + 2\mathbb{D}_1 \Big] \\
&+ \widetilde{M}^2 T^2 \int_0^T m_K(s) W_K(\mathbb{E}\|y(s)\|^2) \mathrm{d}s + \widetilde{M}^2 Tr(\mathcal{Q}) \int_0^T m_G(s) W_G(\mathbb{E}\|y(s)\|^2) \mathrm{d}s \\
&+ \widetilde{M}^2 \sum_{i=1}^m L_{I_i} \|y\|_{\mathbb{U}}^2 + \widetilde{M}^2 \sum_{i=1}^m L_{J_i} \|y\|_{\mathbb{U}}^2 \Bigg\} \mathrm{d}s + \int_0^t \|\mathcal{R}_\alpha(t-s)\| \mathrm{d}s \\
&\times \int_0^t \|\mathcal{R}_\alpha(t-s)\| \mathbb{E}\|K(s, y^r(h_3(s)))\|^2 \mathrm{d}s + \int_0^t \|\mathcal{R}_\alpha(t-s)\|^2 \mathbb{E}\|g^r(s)\|^2 \mathrm{d}s \\
&+ \sum_{0<t_i<t} \|\mathcal{T}_\alpha(t-t_i)\|^2 \mathbb{E}\|I_i(y(t_i))\|^2 + \sum_{0<t_i<t} \|\mathcal{K}_\alpha(t-t_i)\|^2 \mathbb{E}\|J_i(y(t_i))\| \Bigg],
\end{aligned}
$$

(3.6)

$$
\begin{aligned}
\leq & 8\Bigg[2\widetilde{M}^2 \Big(\mathbb{E}\|y_0\|^2 + \mathcal{C}_1 \mathbb{E}\|y\|^2 + \mathcal{C}_2 \Big) + 3\widetilde{M}^2 \Big(\mathbb{E}\|y_1\|^2 + \mathcal{C}_3 \mathbb{E}\|y\|^2 + \mathcal{C}_4 \\
&+ 2(L_F \mathbb{E}\|y\|^2 + \mathbb{D}_1) \Big) + \widetilde{M}^2 T^2 \Big[2L_F(1+2L_{a_1}) \|y\|_{\mathbb{U}}^2 + 4L_F L_{a_1}^1 + 2\mathbb{D}_1 \Big] \\
&+ 9M_B^2 \widetilde{M}^2 T^2 \frac{1}{\kappa^2} M_B^2 \widetilde{M}^2 \Bigg\{ \|\mathbb{E}\hat{y}_T + \int_0^T \hat{\phi}(s) \mathrm{d}w(s)\|^2 + 2\widetilde{M}^2 \mathbb{E}(\|y_0\|_{\mathbb{U}}^2 \\
&+ \mathcal{C}_1 \|y\|_{\mathbb{U}}^2 + \mathcal{C}_3) + 3\widetilde{M}^2 \Big(\mathbb{E}\|y_1\|_{\mathbb{U}}^2 + \mathcal{C}_3 \|y\|_{\mathbb{U}}^2 + \mathcal{C}_4 + 2(L_F \|y\|_{\mathbb{U}}^2 + \mathbb{D}_1) + \Big) \\
&+ \widetilde{M}^2 T^2 \Big[2L_F(1+2L_{a_1}) \|y\|_{\mathbb{U}}^2 + 4L_F L_{a_1}^1 + 2\mathbb{D}_1 \Big] \\
&+ \widetilde{M}^2 T^2 \int_0^T m_K(s) W_K(\mathbb{E}\|y(s)\|^2) \mathrm{d}s + \widetilde{M}^2 Tr(\mathcal{Q}) \int_0^T m_G(s) W_G(\|y(s)\|^2) \mathrm{d}s \\
&+ \widetilde{M}^2 \sum_{i=1}^m L_{I_i} \|y\|_{\mathbb{U}}^2 + \widetilde{M}^2 \sum_{i=1}^m L_{J_i} \|y\|_{\mathbb{U}}^2 \Bigg\} + \widetilde{M}^2 T^2 \int_0^t m_K(s) W_K(\mathbb{E}\|y(s)\|) \mathrm{d}s \\
&+ \widetilde{M}^2 Tr(\mathcal{Q}) T \int_0^t m_G(s) W_G(\mathbb{E}\|y(s)\|^2) \mathrm{d}s + \widetilde{M}^2 \sum_{i=1}^m L_{I_i} \|y\|_{\mathbb{U}}^2 + \widetilde{M}^2 \sum_{i=1}^m L_{J_i} \|y\|_{\mathbb{U}}^2 \Bigg], \\
\leq & \frac{\widehat{\mathbb{M}}}{1-\widehat{\mathbb{L}}} + \frac{\widetilde{M}^2 T^2 \widehat{\mathbb{M}}}{1-\widehat{\mathbb{L}}} \int_0^t m_K(s) W_K(\mathbb{E}\|y(s)\|) \mathrm{d}s
\end{aligned}
$$

$$+\frac{\widetilde{M}^2Tr(\mathcal{Q})T\widehat{\mathbb{M}}}{1-\widehat{\mathbb{L}}}\int_0^t m_G(s)W_G(\mathbb{E}\|y(s)\|^2)\mathrm{d}s,$$

where

$$\widehat{\mathbb{M}} = 16\widetilde{M}^2(\mathbb{E}\|y_0\|^2\mathbb{U}+\mathcal{C}_2)+24\widetilde{M}^2(\mathbb{E}\|y_1\|^2_{\mathbb{U}}+\mathcal{C}_4+2\mathbb{D}_1)+16\widetilde{M}^2T^2(2L_FL^1_{a_1}+\mathbb{D}_1)$$
$$+\frac{72M_B^4\widetilde{M}^4T^2}{\kappa^2}\Bigg(\|\mathbb{E}\hat{y}_T+\int_0^T\hat{\phi}(s)dw(s)\|^2+2\widetilde{M}^2(\mathbb{E}\|y_0\|^2+\mathcal{C}_2)$$
$$+3\widetilde{M}^2(\mathbb{E}\|y_1\|^2_{\mathbb{U}}+\mathcal{C}_4+2\mathbb{D}_1)+\widetilde{M}^2T^2(L_FL^1_{a_1}+2\mathbb{D}_1)+\widetilde{M}^2T^2$$
$$\times\int_0^T m_K(s)W_K(\mathbb{E}\|y(s)\|^2)\mathrm{d}s+\widetilde{M}^2Tr(\mathcal{Q})\int_0^T m_G(s)W_G(\mathbb{E}\|y(s)\|^2)\mathrm{d}s\Bigg),$$

$$\widehat{\mathbb{L}} = \max_{i=1,\cdots,m}\Bigg\{16\widetilde{M}^2\mathcal{C}_1+24\widetilde{M}^2\mathcal{C}_3+48\widetilde{M}^2L_F+16\widetilde{M}^2T^2L_F(1+2L_{a_1})$$
$$+\frac{72M_B^4\widetilde{M}^4T^2}{\kappa^2}\Bigg(2\widetilde{M}^2\mathcal{C}_1+3\widetilde{M}^2(\mathcal{C}_3+2L_F)+2\widetilde{M}^2T^2L_F(1+2L_{a_1})$$
$$+\widetilde{M}^2\sum_{i=1}^m(L_{I_i}+L_{J_i})\Bigg)+\widetilde{M}^2\sum_{i=1}^m(L_{I_i}+L_{J_i})\Bigg\}.$$

Denoting by $\zeta(t)$ the right-hand side of the inequality (3.6), we have

$$\eta(t)\le\zeta(t), \tag{3.7}$$

where $\eta(t)=\sup_{s\in[0,T]}\mathbb{E}\|y(s)\|^2_{\mathbb{U}}$ for $t\in[0,T]$ and $\zeta(0)=\frac{\widehat{\mathbb{M}}}{1-\widehat{\mathbb{L}}}$. Moreover,

$$\begin{aligned}
\zeta'(t) &= \frac{\widetilde{M}^2T^2\widehat{\mathbb{M}}}{1-\widehat{\mathbb{L}}}m_K(t)W_K(\eta(t))+\frac{\widetilde{M}^2Tr(\mathcal{Q})T\widehat{\mathbb{M}}}{1-\widehat{\mathbb{L}}}m_G(t)W_G(\eta(t)),\\
&\le \frac{\widetilde{M}^2T^2\widehat{\mathbb{M}}}{1-\widehat{\mathbb{L}}}m_K(t)W_K(\zeta(t))+\frac{\widetilde{M}^2Tr(\mathcal{Q})T\widehat{\mathbb{M}}}{1-\widehat{\mathbb{L}}}m_G(t)W_G(\zeta(t)),\\
&\le m^*(t)[W_K(\zeta(t))+W_G(\zeta(t))],\ t\in[0,T]. \qquad (3.8)
\end{aligned}$$

where $m^*(t)=\max\{\frac{\widetilde{M}^2T^2\widehat{\mathbb{M}}}{1-\widehat{\mathbb{L}}}m_K(t),\frac{\widetilde{M}^2Tr(\mathcal{Q})T\widehat{\mathbb{M}}}{1-\widehat{\mathbb{L}}}m_G(t)\}$. This gives

(3.9)
$$\int_{\zeta(0)}^{\zeta(t)}\frac{\mathrm{d}s}{W_K(s)+W_G(s)}\le\int_0^T m(s)\mathrm{d}s<\int_{\zeta(0)}^{\infty}\frac{\mathrm{d}s}{W_K(s)+W_G(s)}<\infty,\quad t\in[0,T].$$

The above inequality demonstrates that $\zeta(t)<\infty$, *i.e.*, there exists a constant $\mathcal{L}$ such that $\zeta(t)<\mathcal{L}$ for $t\in[0,T]$, where $\mathcal{L}$ depends upon constants $\widetilde{M}$, T and functions W_K, W_G, m_K, m_G. Hence $\eta(t)\le\mathcal{L}$. In addition, we conclude that $\|y(t)\|^2\le\eta(t)\le\mathcal{L}$, $t\in[0,T]$. Therefore, there exists r such that $\|y\|^2_{\mathcal{C}}\ne r$.

Set $V = \{y \in \mathcal{C} : \|y\|_{\mathcal{C}}^2 \le r\}$. From the choice of V, there is no $y \in \partial V$ such that $y \in \lambda \Psi y$ for each $0 < \lambda < 1$.

Step 2. Ψ has a closed graph. Let $y_n \to y_*$, $\rho_n \in \Psi y_n$, $y_n \in \overline{V}$ and $\rho_n \to \rho_*$. We show that $\rho_* \in \Psi y_*$. Now, $\rho_n \in \Psi y_n$ implies that there exists $g_n \in \mathcal{N}_{G, y_n(h_3(t))}$ such that

(3.10)

$$\begin{aligned}
\rho_n(t) &= \mathcal{T}_\alpha(t)[y_0 - h(y_n)] + \mathcal{K}_\alpha(t)[y_1 - g(y_n) - F(0, y_n(h_1(0)), 0)] \\
&\quad + \int_0^t \mathcal{T}_\alpha(t-s) F(s, y_n(h_1(s)), \int_0^s a_1(s, \tau, y_n(h_2(\tau)))\mathrm{d}\tau)\mathrm{d}s + \int_0^t \mathcal{R}_\alpha(t-s) B \\
&\quad \times \Big\{ B^* \mathcal{R}_\alpha^*(T-t)(\kappa I + \Gamma_0^T)^{-1} \Big[E\hat{y}_T + \int_0^T \hat{\phi}(s)\mathrm{d}w(s) - \mathcal{T}_\alpha(T)(y_0 - h(y_n)) \\
&\quad - \mathcal{K}_\alpha(T)\Big(y_1 - g(y_n) - F(0, y_n(h_1(0)), 0)\Big)\Big] \\
&\quad - B^* \mathcal{R}_\alpha^*(T-t) \int_0^T (\kappa I + \Gamma_s^T)^{-1} \mathcal{T}_\alpha(T-s) \\
&\quad \times F(s, y_n(h_1(s)), \int_0^s a_1(s, \tau, y_n(h_2(\tau)))\mathrm{d}\tau)\mathrm{d}s - B^* \mathcal{R}_\alpha^*(T-t) \\
&\quad \times \int_0^T (\kappa I + \Gamma_s^T)^{-1} \mathcal{R}_\alpha(T-s) K(s, y_n(h_3(s)))\mathrm{d}s - B^* \mathcal{R}_\alpha^*(T-t) \\
&\quad \times \int_0^T (\kappa I + \Gamma_s^T)^{-1} \mathcal{R}_\alpha(T-s) g_n(s)\mathrm{d}w(s) \\
&\quad - B^* \mathcal{R}_\alpha^*(T-t) \sum_{0<t_i<t} (\kappa I + \Gamma_s^T)^{-1} \mathcal{T}_\alpha(T-t_i) I_i(y_n(t_i)) \\
&\quad - B^* \mathcal{R}_\alpha^*(T-t) \sum_{0<t_i<t} (\kappa I + \Gamma_s^T)^{-1} \mathcal{K}_\alpha(T-t_i) J_i(y_n(t_i)) \Big\} \mathrm{d}s \\
&\quad + \int_0^t \mathcal{R}_\alpha(t-s) K(s, y_n(h_3(s)))\mathrm{d}s + \int_0^t \mathcal{R}_\alpha(t-s) g_n(s)\mathrm{d}w(s) \\
&\quad + \sum_{0<t_i<t} \mathcal{T}_\alpha(t-t_i) I_i(y_n(t_i)) + \sum_{0<t_i<t} \mathcal{K}_\alpha(t-t_i) J_i(y_n(t_i)), \ t \in [0, T].
\end{aligned}$$

We must prove that there exists $g_* \in \mathcal{N}_{G, y_*(h_3(t))}$ such that

(3.11)

$$\rho_*(t) = \mathcal{T}_\alpha(t)[y_0 - h(y_*)] + \mathcal{K}_\alpha(t)[y_1 - g(y_*) - F(0, y_*(h_1(0)), 0)]$$

$$+\int_0^t \mathcal{T}_\alpha(t-s)F(s,y_*(h_1(s)),\int_0^s a_1(s,\tau,y_*(h_2(\tau)))\mathrm{d}\tau)\mathrm{d}s+\int_0^t \mathcal{R}_\alpha(t-s)B$$
$$\times\Bigg\{B^*\mathcal{R}_\alpha^*(T-t)(\kappa I+\Gamma_0^T)^{-1}\Bigg[E\hat{y}_T+\int_0^T\hat{\phi}(s)\mathrm{d}w(s)-\mathcal{T}_\alpha(T)(y_0-h(y_*))-$$
$$\mathcal{K}_\alpha(T)\Big(y_1-g(y_*)-F(0,y_*(h_1(0)),0)\Big)\Bigg]$$
$$-B^*\mathcal{R}_\alpha^*(T-t)\int_0^T(\kappa I+\Gamma_s^T)^{-1}\mathcal{T}_\alpha(T-s)$$
$$\times F(s,y_*(h_1(s)),\int_0^s a_1(s,\tau,y_*(h_2(\tau)))\mathrm{d}\tau)\mathrm{d}s-B^*\mathcal{R}_\alpha^*(T-t)$$
$$\times\int_0^T(\kappa I+\Gamma_s^T)^{-1}\mathcal{R}_\alpha(T-s)K(s,y_*(h_3(s)))\mathrm{d}s-B^*\mathcal{R}_\alpha^*(T-t)$$
$$\times\int_0^T(\kappa I+\Gamma_s^T)^{-1}\mathcal{R}_\alpha(T-s)g_*(s)\mathrm{d}w(s)$$
$$-B^*\mathcal{R}_\alpha^*(T-t)\sum_{0<t_i<t}(\kappa I+\Gamma_s^T)^{-1}\mathcal{T}_\alpha(T-t_i)I_i(y_*(t_i))$$
$$-B^*\mathcal{R}_\alpha^*(T-t)\sum_{0<t_i<t}(\kappa I+\Gamma_s^T)^{-1}\mathcal{K}_\alpha(T-t_i)J_i(y_*(t_i))\Bigg\}\mathrm{d}s$$
$$+\int_0^t\mathcal{R}_\alpha(t-s)K(s,y_*(h_3(s)))ds+\int_0^t\mathcal{R}_\alpha(t-s)g_*(s)\mathrm{d}w(s)$$
$$+\sum_{0<t_i<t}\mathcal{T}_\alpha(t-t_i)I_i(y_*(t_i))+\sum_{0<t_i<t}\mathcal{K}_\alpha(t-t_i)J_i(y_*(t_i)),\ t\in[0,T]$$

Now, for every $t\in[0,T]$, we have

(3.12)
$$\Bigg\|\rho_n(t)-\mathcal{T}_\alpha(t)[y_0-h(y_n)]-\mathcal{K}_\alpha(t)[y_1-g(y_n)-F(0,y_n(h_1(0)),0)]$$
$$-\int_0^t\mathcal{T}_\alpha(t-s)F(s,y_n(h_1(s)),\int_0^s a_1(s,\tau,y_n(h_2(\tau)))\mathrm{d}\tau)\mathrm{d}s-\int_0^t\mathcal{R}_\alpha(t-s)B$$
$$\times\Bigg\{B^*\mathcal{R}_\alpha^*(T-t)(\kappa I+\Gamma_0^T)^{-1}\Bigg[E\hat{y}_T+\int_0^T\hat{\phi}(s)\mathrm{d}w(s)-\mathcal{T}_\alpha(T)(y_0-h(y_n))$$
$$-\mathcal{K}_\alpha(T)\Big(y_1-g(y_n)-F(0,y_n(h_1(0)),0)\Big)\Bigg]-B^*\mathcal{R}_\alpha^*(T-t)\int_0^T(\kappa I+\Gamma_s^T)^{-1}\mathcal{T}_\alpha(T-s)$$

$$\times F(s, y_n(h_1(s)), \int_0^s a_1(s, \tau, y_n(h_2(\tau)))\mathrm{d}\tau)\mathrm{d}s - B^*\mathcal{R}_\alpha^*(T-t)$$
$$\times \int_0^T (\kappa I + \Gamma_s^T)^{-1}\mathcal{R}_\alpha(T-s)K(s, y_n(h_3(s)))\mathrm{d}s$$
$$- B^*\mathcal{R}_\alpha^*(T-t) \sum_{0<t_i<t} (\kappa I + \Gamma_s^T)^{-1}\mathcal{T}_\alpha(T-t_i)I_i(y_n(t_i))$$
$$- B^*\mathcal{R}_\alpha^*(T-t) \sum_{0<t_i<t} (\kappa I + \Gamma_s^T)^{-1}\mathcal{K}_\alpha(T-t_i)J_i(y_n(t_i)) \Bigg\}\mathrm{d}s$$
$$- \int_0^t \mathcal{R}_\alpha(t-s)K(s, y_n(h_3(s)))\mathrm{d}s - \sum_{0<t_i<t} \mathcal{T}_\alpha(t-t_i)I_i(y_n(t_i))$$
$$- \sum_{0<t_i<t} \mathcal{K}_\alpha(t-t_i)J_i(y_n(t_i)) - \Bigg(\rho_*(t) - \mathcal{T}_\alpha(t)[y_0 - h(y_*)] - \mathcal{K}_\alpha(t)[y_1 - g(y_*)$$
$$- F(0, y_*(h_1(0)), 0)] - \int_0^t \mathcal{T}_\alpha(t-s)F(s, y_*(h_1(s)), \int_0^s a_1(s, \tau, y_*(h_2(\tau)))\mathrm{d}\tau)\mathrm{d}s$$
$$- \int_0^t \mathcal{R}_\alpha(t-s)B\Bigg\{B^*\mathcal{R}_\alpha^*(T-t)(\kappa I + \Gamma_0^T)^{-1}\Bigg[E\hat{y}_T + \int_0^T \hat{\phi}(s)\mathrm{d}w(s)$$
$$- \mathcal{T}_\alpha(T)(y_0 - h(y_*)) - \mathcal{K}_\alpha(T)\Big(y_1 - g(y_*) - F(0, y_*(h_1(0)), 0)\Big)\Bigg]$$
$$- B^*\mathcal{R}_\alpha^*(T-t)\int_0^T(\kappa I+\Gamma_s^T)^{-1}\mathcal{T}_\alpha(T-s)F(s, y_*(h_1(s)),\int_0^s a_1(s, \tau, y_*(h_2(\tau)))\mathrm{d}\tau)\mathrm{d}s$$
$$- B^*\mathcal{R}_\alpha^*(T-t) \int_0^T (\kappa I + \Gamma_s^T)^{-1}\mathcal{R}_\alpha(T-s)K(s, y_*(h_3(s)))\mathrm{d}s$$
$$- B^*\mathcal{R}_\alpha^*(T-t) \sum_{0<t_i<t} (\kappa I + \Gamma_s^T)^{-1}\mathcal{T}_\alpha(T-t_i)I_i(y_*(t_i))$$
$$- B^*\mathcal{R}_\alpha^*(T-t) \sum_{0<t_i<t} (\kappa I + \Gamma_s^T)^{-1}\mathcal{K}_\alpha(T-t_i)J_i(y_*(t_i)) \Bigg\}\mathrm{d}s$$
$$- \int_0^t \mathcal{R}_\alpha(t-s)K(s, y_*(h_3(s)))\mathrm{d}s - \sum_{0<t_i<t} \mathcal{T}_\alpha(t-t_i)I_i(y_*(t_i))$$
$$- \sum_{0<t_i<t} \mathcal{K}_\alpha(t-t_i)J_i(y_*(t_i))\Bigg)\Bigg\| \to 0, \text{ as } n \to \infty.$$

Define the linear continuous operator $\Xi : L^2([0,T],\mathbb{U}) \to C([0,T],\mathbb{U})$ as

(3.13) $$\Xi g(t) = \int_0^t \mathcal{R}_\alpha(t-s)BB^*\mathcal{R}_\alpha^*(T-t) \times \left(\int_0^T (\kappa I+\Gamma_s^T)^{-1}\mathcal{R}_\alpha(T-s)g(s)\mathrm{d}w(s)\right)\mathrm{d}s+\int_0^t \mathcal{R}_\alpha(t-s)g(s)\mathrm{d}s.$$

From Lemma 2.2, it follows that $\Xi \circ \mathcal{N}_G$ is a closed graph mapping. By the definition of Ξ, we also have that, for $t \in [0,T]$

(3.14)

$$\begin{aligned}
&\rho_n(t) - \mathcal{T}_\alpha(t)[y_0 - h(y_n)] - \mathcal{K}_\alpha(t)[y_1 - g(y_n) - F(0, y_n(h_1(0)), 0)] \\
&- \int_0^t \mathcal{T}_\alpha(t-s)F(s, y_n(h_1(s)), \int_0^s a_1(s,\tau,y_n(h_2(\tau)))\mathrm{d}\tau)\mathrm{d}s - \int_0^t \mathcal{R}_\alpha(t-s)B \\
&\times \Bigg\{ B^*\mathcal{R}_\alpha^*(T-t)(\kappa I + \Gamma_0^T)^{-1}\Bigg[E\hat{y}_T + \int_0^T \hat{\phi}(s)\mathrm{d}w(s) - \mathcal{T}_\alpha(T)(y_0 - h(y_n)) \\
&- \mathcal{K}_\alpha(T)\Big(y_1 - g(y_n) - F(0, y_n(h_1(0)), 0)\Big)\Bigg] \\
&- B^*\mathcal{R}_\alpha^*(T-t)\int_0^T (\kappa I+\Gamma_s^T)^{-1}\mathcal{T}_\alpha(T-s)F(s, y_n(h_1(s))),\int_0^s a_1(s,\tau,y_n(h_2(\tau)))\mathrm{d}\tau)\mathrm{d}s \\
&- B^*\mathcal{R}_\alpha^*(T-t)\int_0^T (\kappa I+\Gamma_s^T)^{-1}\mathcal{R}_\alpha(T-s)K(s, y_n(h_3(s)))\mathrm{d}s \\
&- B^*\mathcal{R}_\alpha^*(T-t)\sum_{0<t_i<t}(\kappa I+\Gamma_s^T)^{-1}\mathcal{T}_\alpha(T-t_i)I_i(y_n(t_i)) \\
&- B^*\mathcal{R}_\alpha^*(T-t)\sum_{0<t_i<t}(\kappa I+\Gamma_s^T)^{-1}\mathcal{K}_\alpha(T-t_i)J_i(y_n(t_i))\Bigg\}\mathrm{d}s \\
&- \int_0^t \mathcal{R}_\alpha(t-s)K(s, y_n(h_3(s)))\mathrm{d}s - \sum_{0<t_i<t}\mathcal{T}_\alpha(t-t_i)I_i(y_n(t_i)) \\
&- \sum_{0<t_i<t}\mathcal{K}_\alpha(t-t_i)J_i(y_n(t_i)) \in \Xi(\mathcal{N}_{y_n,G}).
\end{aligned}$$

Since $y_n \to y_*$, for some $g_* \in \mathcal{N}_{G,y_*(h_3(t))}$, thus we have

(3.15)

$$\begin{aligned}
&\rho_*(t) - \mathcal{T}_\alpha(t)[y_0 - h(y_*)] - \mathcal{K}_\alpha(t)[y_1 - g(y_*) - F(0, y_*(h_1(0)), 0)] \\
&- \int_0^t \mathcal{T}_\alpha(t-s)F(s, y_*(h_1(s)), \int_0^s a_1(s,\tau,y_*(h_2(\tau)))\mathrm{d}\tau)\mathrm{d}s - \int_0^t \mathcal{R}_\alpha(t-s)B
\end{aligned}$$

$$\times \Bigg\{ B^*\mathcal{R}_\alpha^*(T-t)(\kappa I+\Gamma_0^T)^{-1}\Bigg[E\hat{y}_T + \int_0^T \hat{\phi}(s)\mathrm{d}w(s) - \mathcal{T}_\alpha(T)(y_0 - h(y_*))$$

$$- \mathcal{K}_\alpha(T)\Big(y_1 - g(y_*) - F(0, y_*(h_1(0)), 0)\Big)\Bigg] - B^*\mathcal{R}_\alpha^*(T-t)\int_0^T (\kappa I + \Gamma_s^T)^{-1}\mathcal{T}_\alpha(T-s)$$

$$\times F(s, y_*(h_1(s)), \int_0^s a_1(s,\tau,y_*(h_2(\tau)))\mathrm{d}\tau)\mathrm{d}s - B^*\mathcal{R}_\alpha^*(T-t)$$

$$\times \int_0^T (\kappa I + \Gamma_s^T)^{-1}\mathcal{R}_\alpha(T-s)K(s, y_*(h_3(s)))\mathrm{d}s$$

$$- B^*\mathcal{R}_\alpha^*(T-t) \sum_{0<t_i<t} (\kappa I + \Gamma_s^T)^{-1}\mathcal{T}_\alpha(T-t_i)I_i(y_*(t_i))$$

$$- B^*\mathcal{R}_\alpha^*(T-t) \sum_{0<t_i<t} (\kappa I + \Gamma_s^T)^{-1}\mathcal{K}_\alpha(T-t_i)J_i(y_*(t_i))\Bigg\}\mathrm{d}s$$

$$- \int_0^t \mathcal{R}_\alpha(t-s)K(s, y_*(h_3(s)))\mathrm{d}s - \sum_{0<t_i<t} \mathcal{T}_\alpha(t-t_i)I_i(y_*(t_i))$$

$$- \sum_{0<t_i<t} \mathcal{K}_\alpha(t-t_i)J_i(y_*(t_i)) = \int_0^t \mathcal{R}_\alpha(t-s)g_*(s)\mathrm{d}w(s)$$

$$+ \int_0^t \mathcal{R}_\alpha(t-s)BB^*\mathcal{R}_\alpha^*(T-t)\Bigg(\int_0^T (\kappa I + \Gamma_s^T)^{-1}\mathcal{R}_\alpha(T-s)g_*(s)\mathrm{d}w(s)\Bigg)\mathrm{d}s.$$

Therefore, Ψ has a closed graph.

Next, we are going to show that the operator Ψ is condensing. To this end, operator Ψ is decomposed as $\Psi_1 + \Psi_2$, where the map $\Psi_1 : \overline{V} \to \mathcal{C}$ be defined as $\Psi_1 y$, the set $\rho_1 \in \mathcal{C}$ such that for $t \in [0,T]$,

$$\begin{aligned} \rho_1(t) = &- \mathcal{K}_\alpha(t)F(0, y(h_1(0)), 0) \\ &+ \int_0^t \mathcal{T}_\alpha(t-s)F(s, y(h_1(s)), \int_0^s a_1(s,\tau,y(h_2(\tau)))\mathrm{d}\tau)\mathrm{d}s, \end{aligned} \tag{3.16}$$

and the map $\Psi_2 : \overline{V} \to \mathcal{C}$ be given by $\Psi_2 y$, the set $\rho_2 \in \mathcal{C}$ such that

$$\begin{aligned} \rho_2(t) = {}& \mathcal{T}_\alpha(t)[y_0 - h(y)] + \mathcal{K}_\alpha(t)[y_1 - g(y)] + \int_0^t \mathcal{R}_\alpha(t-s)Bu^\kappa(s)\mathrm{d}s \\ &+ \int_0^t \mathcal{R}_\alpha(t-s)K(s, y(h_3(s)))\mathrm{d}s + \int_0^t \mathcal{R}_\alpha(t-s)g_j(s)\mathrm{d}w(s) \\ &+ \sum_{0<t_i<t} \mathcal{T}_\alpha(t-t_i)I_i(y(t_i)) + \sum_{0<t_i<t} \mathcal{K}_\alpha(t-t_i)J_i(y(t_i)),\ t \in [0,T]. \end{aligned} \tag{3.17}$$

Now, we prove that Ψ_1 is a contraction while Ψ_2 is a completely continuous operator.

Step 3. Ψ_1 is a contraction on $\mathcal{C}$. Let $t \in [0,T]$ and $y^*, y^{**} \in \mathcal{C}$. Then, we have

$$\begin{aligned}
\mathbb{E}\|\Psi_1 y^*(t) - \Psi_1 y^{**}(t)\|_{\mathbb{U}}^2 &\leq 2\|\mathcal{K}_\alpha(t)[F(0, y^*(h_1(0)), 0) - F(0, y^{**}(h_1(0)), 0)]\|^2 \\
&\quad + 2\Big\| \int_0^t \mathcal{T}_\alpha(t-s) \times [F(s, y^*(h_1(s)), \int_0^s a_1(s,\tau, y^*(h_2(\tau)))\mathrm{d}\tau) \\
&\quad - F(s, y^{**}(h_1(s)), \int_0^s a_1(s,\tau, y^{**}(h_2(\tau)))\mathrm{d}\tau)]\mathrm{d}s \Big\|^2, \\
&\leq 2\widetilde{M}^2 L_F \|u_1 - u_2\|_{\mathcal{C}}^2 + \widetilde{M}^2 T^2 L_F [1 + 2L_{a_1}]\|u_1 - u_2\|_{\mathcal{C}}^2, \\
&= \widetilde{M}^2 L_F [1 + (1 + 2L_{a_1})] \times \|u_1 - u_2\|_{\mathcal{C}}^2. \qquad (3.18)
\end{aligned}$$

Taking supremum over t and getting

$$\|\Psi_1 y^* - \Psi_1 y^{**}\|_{\mathcal{C}}^2 \leq \widetilde{M}^2 L_F [1 + (1 + 2L_{a_1})] \times \|u_1 - u_2\|_{\mathcal{C}}^2, \qquad (3.19)$$

where $\widetilde{M}^2 L_F[1 + (1 + 2L_{a_1})] < 1$. Hence, we conclude that Ψ_1 is a contraction on $\mathcal{C}$.

Step 4. Φ_2 is convex for each $y \in \overline{V}$. Let $\widehat{\rho}_1, \widehat{\rho}_2 \in \Psi_2 y$. Then there exist $g_1, g_2 \in \mathcal{N}_{G, y(h_3(t))}$ such that

$$\begin{aligned}
\widehat{\rho}_j(t) &= \mathcal{T}_\alpha(t)[y_0 - h(y)] + \mathcal{K}_\alpha(t)[y_1 - g(y)] + \int_0^t \mathcal{R}_\alpha(t-s) B u^\kappa(s)\mathrm{d}s \\
&\quad + \int_0^t \mathcal{R}_\alpha(t-s) K(s, y(h_3(s)))\mathrm{d}s + \int_0^t \mathcal{R}_\alpha(t-s) g_j(s)\mathrm{d}w(s) \\
&\quad + \sum_{0<t_i<t} \mathcal{T}_\alpha(t-t_i) I_i(y(t_i)) + \sum_{0<t_i<t} \mathcal{K}_\alpha(t-t_i) J_i(y(t_i)), \quad j = 1, 2, \qquad (3.20)
\end{aligned}$$

for each $t \in [0,T]$.
Let $\mu \in [0,1]$. Thus, for each $t \in [0,T]$, we get

(3.21)

$$\begin{aligned}
&\mu\widehat{\rho}_1(t) + (1-\mu)\widehat{\rho}_2(t) \leq \mathcal{T}_\alpha(t)[y_0 - h(y)] + \mathcal{K}_\alpha(t)[y_1 - g(y)] \\
&+ \int_0^t \mathcal{R}_\alpha(t-s) \times \Bigg\{ BB^*\mathcal{R}_\alpha^*(T-t)(\kappa I + \Gamma_0^T)^{-1}\Bigg[E\hat{y}_T + \int_0^T \hat{\phi}(s)\mathrm{d}w(s) \\
&- \mathcal{T}_\alpha(T)(y_0 - h(y)) - \mathcal{K}_\alpha(T)\Big(y_1 - g(y) - F(0, y(h_1(0)), 0)\Big)\Bigg]
\end{aligned}$$

$$- BB^*\mathcal{R}^*_\alpha(T-t)\int_0^T(\kappa I + \Gamma_s^T)^{-1}\mathcal{T}_\alpha(T-s)\times F(s,y(h_1(s)),\int_0^s a_1(s,\tau,y(h_2(\tau)))\mathrm{d}\tau)\mathrm{d}s$$
$$- BB^*\mathcal{R}^*_\alpha(T-t)\times\int_0^T(\kappa I + \Gamma_s^T)^{-1}\mathcal{R}_\alpha(T-s)\times K(s,y(h_3(s)))\mathrm{d}s$$
$$- BB^*\mathcal{R}^*_\alpha(T-t)\times\int_0^T(\kappa I + \Gamma_s^T)^{-1}\mathcal{R}_\alpha(T-s)[\mu g_1(s)+(1-\mu)g_2(s)]\mathrm{d}w(s)$$
$$- BB^*\mathcal{R}^*_\alpha(T-t)\times\sum_{0<t_i<t}(\kappa I + \Gamma_s^T)^{-1}\mathcal{T}_\alpha(T-t_i)I_i(y(t_i)) - BB^*\mathcal{R}^*_\alpha(T-t)$$
$$\times\sum_{0<t_i<t}(\kappa I + \Gamma_s^T)^{-1}\mathcal{K}_\alpha(T-t_i)J_i(y(t_i))\Bigg\}\mathrm{d}s + \int_0^t\mathcal{R}_\alpha(t-s)K(s,y(h_3(s)))\mathrm{d}s$$
$$+\int_0^t\mathcal{R}_\alpha(t-s)(\mu g_1(s)+(1-\mu)g_2(s))\mathrm{d}w(s)+\sum_{0<t_i<t}\mathcal{T}_\alpha(t-t_i)I_i(y(t_i))$$
$$+\sum_{0<t_i<t}\mathcal{K}_\alpha(t-t_i)J_i(y(t_i)).$$

Since $\mathcal{N}_{G,y(h_3(t))}$ is convex, therefore $\mu\widehat{\rho}_1+(1-\mu)\widehat{\rho}_2\in\mathcal{N}_{G,y(h_3(t))}$. Hence, $\lambda\widehat{\rho}_1+(1-\lambda)\widehat{\rho}_2\in\Psi y$.

Step 5. Ψ_2 maps $\overline{V}$ into an equicontinuous family. Let $t_1,t_2\in[0,T]$ with $0<t_1<t_2\le T$ and $\epsilon>0$. Then we have for each $y\in\overline{V}$ and $\rho_2\in\Psi_2 y$, there is $g\in\mathcal{N}_{G,y(h_3(t))}$ such that

$$\begin{aligned}\rho_2 &= \mathcal{T}_\alpha(t)[y_0-h(y)]+\mathcal{K}_\alpha(t)[y_1-g(y)]+\int_0^t\mathcal{R}_\alpha(t-s)Bu^\kappa(s)\mathrm{d}s\\ &\quad+\int_0^t\mathcal{R}_\alpha(t-s)K(s,y(h_3(s)))\mathrm{d}s+\int_0^t\mathcal{R}_\alpha(t-s)g(s)\mathrm{d}w(s)\\ &\quad+\sum_{0<t_i<t}\mathcal{T}_\alpha(t-t_i)I_i(y(t_i))+\sum_{0<t_i<t}\mathcal{K}_\alpha(t-t_i)J_i(y(t_i)),\ t\in[0,T].\end{aligned}\tag{3.22}$$

Then, we have

(3.23)

$$\mathbb{E}\|\rho_2(t_2)-\rho_2(t_1)\|^2_{\mathbb{U}}\le 13\Bigg\{\mathbb{E}\|[\mathcal{T}_\alpha(t_2)-\mathcal{T}_\alpha(t_1)][y_0-h(y)]\|^2_{\mathbb{U}}$$
$$+\mathbb{E}\|[\mathcal{K}_\alpha(t_2)-\mathcal{K}_\alpha(t_1)][y_1-g(y)]\|^2_{\mathbb{U}}+\mathbb{E}\|\int_{t_1}^{t_2}\mathcal{R}_\alpha(t_2-s)Bu^\kappa(s)\mathrm{d}s\|^2_{\mathbb{U}}$$
$$+\mathbb{E}\|\int_0^{t_1-\epsilon}[\mathcal{R}_\alpha(t_2-s)-\mathcal{R}_\alpha(t_1-s)]Bu^\kappa(s)\mathrm{d}s\|^2+\mathbb{E}\|\int_{t_1-\epsilon}^{t_1}[\mathcal{R}_\alpha(t_2-s)$$

$$
\begin{aligned}
&- \mathcal{R}_\alpha(t_1 - s)]Bu^\kappa(s)\mathrm{d}s\|^2 + \mathbb{E}\|\int_{t_1}^{t_2} \mathcal{R}_\alpha(t_2 - s)K(s, y(h_3(s)))\mathrm{d}s\|_{\mathbb{U}}^2 \\
&+ \mathbb{E}\|\int_0^{t_1-\epsilon} [\mathcal{R}_\alpha(t_2 - s) - \mathcal{R}_\alpha(t_1 - s)]K(s, y(h_3(s)))\mathrm{d}s\|_{\mathbb{U}}^2 \\
&+ \mathbb{E}\|\int_{t_1-\epsilon}^{t_1} [\mathcal{R}_\alpha(t_2 - s) - \mathcal{R}_\alpha(t_1 - s)] \times K(s, y(h_3(s)))\mathrm{d}s\|_{\mathbb{U}}^2 \\
&+ \mathbb{E}\|\int_{t_1}^{t_2} \mathcal{R}_\alpha(t_2 - s)g(s)\mathrm{d}w(s)\|_{L_2}^2 + \mathbb{E}\|\int_0^{t_1-\epsilon} [\mathcal{R}_\alpha(t_2 - s) \\
&- \mathcal{R}_\alpha(t_1-s)]g(s)\mathrm{d}w(s)\|_{L_2}^2 + \mathbb{E}\|\int_{t_1-\epsilon}^{t_1} [\mathcal{R}_\alpha(t_2-s) - \mathcal{R}_\alpha(t_1-s)]g(s)\mathrm{d}w(s)\|_{L_2}^2 \\
&+ \mathbb{E}\|\sum_{0<t_i<t} [\mathcal{T}_\alpha(t_2 - t_i) - \mathcal{T}_\alpha(t_1 - t_i)]I_i(y(t_i))\|_{\mathbb{U}}^2 \\
&+ \mathbb{E}\|\sum_{0<t_i<t} [\mathcal{K}_\alpha(t_2 - t_i) - \mathcal{K}_\alpha(t_1 - t_i)]J_i(y(t_i))\|_{\mathbb{U}}^2 \Bigg\}, \\
\leq 13 \Bigg\{ & 2\|\mathcal{T}_\alpha(t_2) - \mathcal{T}_\alpha(t_1)\|^2(\mathbb{E}\|y_0\|_{\mathbb{U}}^2 + \mathcal{C}_1 r + \mathcal{C}_2) + 2\|\mathcal{K}_\alpha(t_2) - \mathcal{K}_\alpha(t_1)\|^2(\|y_1\|_{\mathbb{U}}^2 \\
&+ \mathcal{C}_3 r + \mathcal{C}_4) + M_B^2 \widehat{M}^2(t_2 - t_1)\int_{t_1}^{t_2} \mathbb{E}\|u^\kappa(s)\|^2\mathrm{d}s \\
&+ M_B^2(t_1 - \epsilon)\int_0^{t-\epsilon} \|\mathcal{R}_\alpha(t_2 - s) - \mathcal{R}_\alpha(t_1 - s)\|^2\mathbb{E}\|u^\kappa(s)\|^2\mathrm{d}s \\
&+ M_B^2\epsilon\int_{t_1-\epsilon}^{t_1} \|\mathcal{R}_\alpha(t_2 - s) - \mathcal{R}_\alpha(t_1 - s)\|^2\mathbb{E}\|u^\kappa(s)\|^2\mathrm{d}s \\
&+ (t_2 - t_1)\widetilde{M}^2\int_{t_1}^{t_2} m_K(s)W_K(\mathbb{E}\|y(s)\|^2)\mathrm{d}s \\
&+ (t_1 - \epsilon)\int_0^{t_1-\epsilon} \|\mathcal{R}_\alpha(t_2 - s) - \mathcal{R}_\alpha(t_1 - s)\|^2 m_K(s)W_K(\mathbb{E}\|y(s)\|^2)\mathrm{d}s \\
&+ \epsilon\int_{t_1-\epsilon}^{t_1} \|\mathcal{R}_\alpha(t_2 - s) - \mathcal{R}_\alpha(t_1 - s)\|^2 \times m_K(s)W_K(\mathbb{E}\|y(s)\|^2)\mathrm{d}s \\
&+ \widetilde{M}^2\int_{t_1}^{t_2} m_G(s)W_G(\mathbb{E}\|y(s)\|^2)\mathrm{d}s \\
&+ \int_0^{t_1-\epsilon} \|\mathcal{R}_\alpha(t_2 - s) - \mathcal{R}_\alpha(t_1 - s)\|^2 \times m_G(s)W_G(\mathbb{E}\|y(s)\|^2)\mathrm{d}s \\
&+ \int_{t_1-\epsilon}^{t_1} \|\mathcal{R}_\alpha(t_2 - s) - \mathcal{R}_\alpha(t_1 - s)\|^2 m_G(s)W_G(\mathbb{E}\|y(s)\|^2)\mathrm{d}s
\end{aligned}
$$

$$+ \sum_{0<t_i<t} \|\mathcal{T}_\alpha(t_2 - t_i) - \mathcal{T}_\alpha(t_1 - t_i)\| \times \mathbb{E}\|I_i(y(t_i))\|^2_{\mathbb{U}}$$

$$+ \sum_{0<t_i<t} \|\mathcal{K}_\alpha(t_2 - t_i) - \mathcal{K}_\alpha(t_1 - t_i)\|^2 \times \mathbb{E}\|J_i(y(t_i))\|^2_{\mathbb{U}} \Bigg\}.$$

The right-hand side of the above inequality is independent of $y \in \overline{V}$ and tends to zero as $t_2 \to t_1$ and ϵ sufficiently small, since the compactness of $\mathcal{T}_\alpha(t), \mathcal{K}_\alpha(t), \mathcal{R}_\alpha(t)$ for $t > 0$ provides the continuity in the uniform operator topology. Therefore, the set $\{\Psi_2 y : y \in \overline{V}\}$ is equicontinuous. Hence, Ψ_2 maps $\overline{V}$ into an equicontinuous family of functions.

Step 6. Ψ_2 maps $\overline{V}$ into a precompact set. For this, we decompose Ψ_2 into Ψ_2^1 and Ψ_2^2, where the map $\Psi_2^1 : \overline{V} \to \mathcal{C}$ be defined as $\Psi_2^1 y$, the set $\rho_2^1 \in \mathcal{C}$ such that

$$\begin{aligned}\Psi_2^1 y(t) &= \mathcal{T}_\alpha(t)[y_0 - h(y)] + \mathcal{K}_\alpha(t)[y_1 - g(y)] + \int_0^t \mathcal{R}_\alpha(t-s)Bu^\kappa(s)\mathrm{d}s \\ &+ \int_0^t \mathcal{R}_\alpha(t-s)K(s, y(h_3(s)))\mathrm{d}s + \int_0^t \mathcal{R}_\alpha(t-s)g(s)\mathrm{d}w(s), \quad t \in [0,T],\end{aligned} \tag{3.24}$$

where $g \in \mathcal{N}_{G,y(h_3(t))}$ and the map $\Psi_2^2 : \overline{V} \to \mathcal{C}$ be given by Ψ_2^2, the set $\rho_2^2 \in \mathcal{C}$ such that

$$\rho_2^2(t) = \sum_{0<t_i<t} \mathcal{T}_\alpha(t - t_i)I_i(y(t_i)) + \sum_{0<t_i<t} \mathcal{K}_\alpha(t - t_i)J_i(y(t_i)), \quad t \in [0,T]. \tag{3.25}$$

We now prove that $\{\rho_2^1(t) \in: y \in \overline{V}\}$ is relatively compact on $\mathbb{U}$ for all $t \in [0,T]$. Let $0 < t \leq s \leq T$ be fixed and ϵ be a real number that satisfies $0 < \epsilon < t$ and $\delta > 0$. For $y \in \overline{V}$, we define

$$\begin{aligned}\Psi_2^{1\epsilon} y(t) &= \mathcal{T}_\alpha(t)[y_0 - h(y)] + \mathcal{K}_\alpha(t)[y_1 - g(y)] + \int_0^{t-\epsilon} \mathcal{R}_\alpha(t - s)Bu^\kappa(s)\mathrm{d}s \\ &+ \int_0^{t-\epsilon} \mathcal{R}_\alpha(t-s)K(s, y(h_3(s)))\mathrm{d}s + \int_0^{t-\epsilon} \mathcal{R}_\alpha(t-s)g(s)\mathrm{d}w(s), \ t \in [0,T],\end{aligned}$$

where $g \in \mathcal{N}_{G,y(h_3(t))}$. Utilizing the compactness of $\mathcal{T}_\alpha(t), \mathcal{K}_\alpha(t), \mathcal{R}_\alpha(t)$ for $t > 0$, it is deduced that the set $U_\epsilon(t) = \{\rho_2^{1\epsilon}(t) : y \in \overline{V}\}$ is precompact in $\mathbb{U}$ for every ϵ, $\epsilon \in (0,t)$. Moreover, for each $y \in \overline{V}$, we get

$$\begin{aligned}&\mathbb{E}\|\rho_2^1(t) - \rho_2^{1\epsilon}\|^2_{\mathbb{U}} \\ &\leq 3\Big\|\int_{t-\epsilon}^t \mathcal{R}_\alpha(t - s)Bu^\kappa(s)\mathrm{d}s\Big\|^2_{\mathbb{U}} + 3\Big\|\int_{t-\epsilon}^0 \mathcal{R}_\alpha(t - s)K(s, y(h_3(s)))\mathrm{d}s\Big\|^2_{\mathbb{U}} \\ &+3\Big\|\int_{t-\epsilon}^t \mathcal{R}_\alpha(t - s)g(s)\mathrm{d}w(s)\Big\|^2_{L_2},\end{aligned} \tag{3.26}$$

Therefore, taking $\epsilon \to 0$, we can see that there are relatively compact sets arbitrarily close to the set $U(t) = \{\rho_2^1(t) : y \in \overline{V}\}$. Hence, $U(t)$ is precompact in $\mathbb{U}$. We next prove that the set $\{\rho_2^2(t) : y \in \overline{V}\}$ is relatively compact in $\mathbb{U}$ for all $t \in [0, T]$. For $t \in [0, T]$, we have, $\rho_2^2(t) = \sum_{0<t_i<t} \mathcal{T}_\alpha(t - t_i)I_i(y(t_i)) + \sum_{0<t_i<t} \mathcal{K}_\alpha(t - t_i)J_i(y(t_i))$, which is equicontinuous and bounded. By compactness of operators $\mathcal{T}_\alpha(t), \mathcal{K}_\alpha(t)$, it follows that $\{\rho_2^2(t) : y \in \overline{V}\}$ is relatively compact subset of $\mathbb{U}$ for all $t \in [0, T]$. Hence, by Arzelá-Ascoli theorem, we conclude that $\Psi_2 = \Psi_2^1 + \Psi_2^2 : \overline{V} \to P(\mathcal{C})$ is completely continuous.

As a result of the above steps $1 - 6$, it can be concluded that Ψ is a condensing operator. All of the conditions of Lemma 2.5 are fulfilled and we deduce that Ψ has a fixed point in $\mathcal{C}$ which is a mild solution of the system (1.2)–(1.3). This completes the proof of the theorem. $\square$

In order to prove the approximate controllability result, the following additional conditions are needed:

(B1) The fractional linear differential inclusion (1.2) is approximately controllable.

(B2) The functions $F : [0, T] \times \mathbb{U} \times \mathbb{U} \to \mathbb{U}$, $K : [0, T] \times \mathbb{U} \times \mathbb{U} \to \mathbb{U}$ and $F : [0, T] \times \mathbb{U} \times \mathbb{U} \to \mathcal{P}(L(\mathbb{V}, \mathbb{U}))$ are uniformly bounded.

Theorem 3.2. *Assume that the assumptions of Theorem 3.1 are fulfilled and* $(B1)$-$(B2)$ *are also satisfied. Then, the fractional stochastic control system (1.2) is approximately controllable on interval* $[0, T]$.

Proof. Let y^κ be a fixed point of the operator Ψ in $\mathcal{C}$. From Theorem 3.1, the fixed point of operator Ψ is a mild solution of system (1.2)-(1.3) under the control function $u^\kappa(s)$. Utilizing the stochastic Fubini Theorem, it is easy to see that for $g^\kappa \in \mathcal{N}_{G,y^\kappa}$

$$
\begin{aligned}
(3.27)\quad y^\kappa(t) = \hat{y}_T - \kappa(\kappa I + \Gamma_0^T)^{-1} \Bigg[& E\hat{y}_T + \int_0^T \hat{\phi}(s)\mathrm{d}w(s) - \mathcal{T}_\alpha(T)(y_0 - h(y^\kappa)) \\
& - \mathcal{K}_\alpha(T) \times \Big(y_1 - g(y^\kappa) - F(0, y(h_1(0)), 0) \Big) \Bigg] \\
& + \kappa \int_0^T (\kappa I + \Gamma_s^T)^{-1} \mathcal{T}_\alpha(T - s) \times F(s, y^\kappa(h_1(s)), \int_0^s a_1(s, \tau, y^\kappa(h_2(\tau)))\mathrm{d}\tau)\mathrm{d}s \\
& + \kappa \int_0^T (\kappa I + \Gamma_s^T)^{-1} \mathcal{R}_\alpha(T - s) K(s, y^\kappa(h_3(s)))\mathrm{d}s \\
& + \kappa \int_0^T (\kappa I + \Gamma_s^T)^{-1} \mathcal{R}_\alpha(T - s) g^\kappa(s)\mathrm{d}w(s)
\end{aligned}
$$

$$+\kappa\sum_{0<t_i<t}(\kappa I+\Gamma_s^T)^{-1}\mathcal{T}_\alpha(T-t_i)I_i(y^\kappa(t_i))$$
$$+\kappa\sum_{0<t_i<t}(\kappa I+\Gamma_s^T)^{-1}\mathcal{K}_\alpha(T-t_i)J_i(y^\kappa(t_i)),\quad t\in[0,T].$$

By uniformly boundedness of F, K and G, there are subsequences still denoted by $F(t,y^\kappa(h_1(t)),\int_0^t a_1(t,s,y^\kappa(h_2(s)))ds)$, $K(t,y^\kappa(h_3(t)))$ and $g^\kappa(t)$, which converge weakly to $f(s)\in\mathbb{U}$, $k(s)\in\mathbb{U}$ and $g(s)\in L(\mathbb{V},\mathbb{U})$. Now, we have

(3.28)
$$\mathbb{E}\|y^\kappa(T)-\hat{y}_T\|^2\le 7\left\|\kappa(\kappa I+\Gamma_0^T)^{-1}\left[E\hat{y}_T-\mathcal{T}_\alpha(T)(y_0-h(y^\kappa))\right.\right.$$
$$\left.\left.-\mathcal{K}_\alpha(T)\Big(y_1-g(y^\kappa)-F(0,y(h_1(0)),0)\Big)\right]\right\|^2+7\|\kappa(\kappa I+\Gamma_0^T)^{-1}\int_0^T\hat{\phi}(s)\mathrm{d}w(s)\|^2$$
$$+7\|\kappa\int_0^T(\kappa I+\Gamma_s^T)^{-1}\mathcal{T}_\alpha(T-s)\times F(s,y^\kappa(h_1(s)),\int_0^s a_1(s,\tau,y^\kappa(h_2(\tau)))\mathrm{d}\tau)\mathrm{d}s\|^2$$
$$+7\|\kappa\int_0^T(\kappa I+\Gamma_s^T)^{-1}\mathcal{R}_\alpha(T-s)K(s,y^\kappa(h_3(s)))\mathrm{d}s\|^2$$
$$+7\|\kappa\int_0^T(\kappa I+\Gamma_s^T)^{-1}\mathcal{R}_\alpha(T-s)g^\kappa(s)\mathrm{d}w(s)\|^2$$
$$+7\|\kappa\sum_{0<t_i<t}(\kappa I+\Gamma_s^T)^{-1}\mathcal{T}_\alpha(T-t_i)I_i(y^\kappa(t_i))\|^2$$
$$+7\|\kappa\sum_{0<t_i<t}(\kappa I+\Gamma_s^T)^{-1}\mathcal{K}_\alpha(T-t_i)J_i(y^\kappa(t_i))\|^2.$$

By assumption (B1), for all $s\in[0,T]$, the operator $\kappa(\kappa I+\Gamma_s^T)^{-1}\to 0$ strongly as $\kappa\to 0^+$ and moreover $\|\kappa(\kappa I+\Gamma_s^T)^{-1}\|\le 1$. Thus, by compactness of operator $\mathcal{R}_\alpha$ and Lebesgue dominated convergence theorem, we get that $\mathbb{E}\|y^\kappa(T)-\hat{y}_T\|^2\to 0$ as $\kappa\to 0^+$. This implies the approximate controllability of (1.2). □

4. EXAMPLE

Let us consider following partial neutral fractional order stochastic differential equation with nonlocal conditions

$$\begin{aligned}(4.1)\quad & D_t^{\alpha-1}[y'(t,x)-\int_0^t\int_0^\pi a(s,x,\zeta)y(\sin t,\zeta)\mathrm{d}\zeta\mathrm{d}s]\in\frac{\partial^2}{\partial x^2}y(t,x)\\ &+\mathcal{K}(t,\frac{\partial}{\partial x}y(\sin t,x))+\mu(t,x)+\chi(t,\frac{\partial}{\partial x}y(\sin t,x))\frac{\mathrm{d}w(t)}{\mathrm{d}t},\ t\in[0,1],\end{aligned}$$

$$\Delta y(t_i, x) = I_i(y(t_i, x)), \quad i = 1, \cdots, m, \tag{4.2}$$
$$\Delta y'(t_i, x) = J_i(y(t_i, x)), \quad i = 1, \cdots, m,$$

$$y(t,0) = y(t,\pi) = 0 = y'(t,0) = y'(t,\pi), \ \ t \in [0,1], \tag{4.3}$$

$$y(0,x) + \int_0^{\pi} b(x,\theta) y(0,\theta) \mathrm{d}\theta = y_0(x), \tag{4.4}$$
$$y'(0,x) + \int_0^{\pi} d(x,\theta) y(0,\theta) \mathrm{d}\theta = y_1(x), \ x \in [0,\pi],$$

where $w(t)$ represents a one-dimensional standard Wiener process defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, D_t^{α} denotes the generalized Caputo fractional derivative of order $\alpha \in [1,2]$ and $t_i, \ i = 1, \cdots, m, \ m \in \mathbb{N}$ are fixed numbers such that $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$. The nonlinear function $b, d : [0,\pi] \times [0,\pi] \to \mathbb{R}$, $a : [0,1] \times [0,\pi] \times [0,\pi] \to \mathbb{R}$, $K : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $\chi : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ are continuous mappings and $y_0, y_1 \in L^2([0,\pi])$.
Let $\mathbb{U} = \mathcal{H} = L^2([0,\pi])$. We consider the operator $A : D(A) \subset \mathbb{U} \to \mathbb{U}$ defined by $Ay = y''$ with the domain

$$D(A) = \{y \in \mathbb{U} \colon y, \ y' \text{ are absolutely continuous}, \ y'' \in \mathbb{U}, \text{ and } y(0) = y(\pi) = 0\}.$$

Therefore, it is clear that the densely defined operator A is the infinitesimal generator of a resolvent family $\mathcal{R}_{\alpha}, \ t \geq 0$. Moreover, A has a discrete spectrum with eigenvalues of the form $-n^2$, $n = 1, 2, \cdots$ and corresponding normalized eigenfunctions are given by $y_n(x) = \sqrt{2/\pi} \sin(nx)$. Additionally, the set $\{y_n : n \in \mathbb{N}\}$ is an orthonormal basis for $\mathbb{U}$

$$T(t)y = \sum_{n=1}^{\infty} e^{-n^2 t} (y, y_n) y_n, \ \text{ for all } \ y \in \mathbb{U}, \ t > 0.$$

Take $y(t)x = y(t,x)$ and consider the bounded linear operator $B : \mathcal{H} \to \mathbb{U}$ by $Bu(t) = \mu(t,x), \ x \in [0,\pi], \ u \in \mathcal{H}$.
Therefore, let us consider

$$F(t,u)(\cdot) = \int_0^{\pi} a(t, \cdot, \vartheta) u(\vartheta) \mathrm{d}\vartheta, \tag{4.5}$$

$$K(t,u)(\cdot) = \mathcal{K}(t, u'(\cdot)), \tag{4.6}$$

$$G(t,u)(\cdot) = \chi(t, u'(\cdot)), \tag{4.7}$$

$$h(u)(\cdot) = \int_0^{\pi} b(\cdot, \vartheta) u(\vartheta) \mathrm{d}\vartheta, \tag{4.8}$$

$$g(u)(\cdot) = \int_0^{\pi} d(\cdot, \vartheta) u(\vartheta) \mathrm{d}\vartheta. \tag{4.9}$$

Take $h_1(t) = h_3(t) = h_4(t) = \sin(t)$. Thus, the system (1.2)-(1.3) can be written as

$$D_t^{\alpha-1}[y'(t) - F(t, y(h_1(t)))] \in Ay(t) + Bu(t) + K(t, y(h_3(t)))$$

$$+G(t,y(h_4(t)))\frac{\mathrm{d}w(t)}{\mathrm{d}t},\ \ t\in[0,T],\ \ t\neq t_i,\ i=1,\cdots,m,\ m\in\mathbb{N}, \tag{4.10}$$

$$\Delta y(t_i)=y(t_i^+)-y(t_i^-)=I_i(y(t_i^-)),\ \Delta y(t_i)=J_i(y(t_i)), \tag{4.11}$$

$$y(0)+h(y)=u_0,\ \ y'(0)+g(y)=y_1. \tag{4.12}$$

Furthermore, $F, K : J\times\mathbb{U}\to\mathbb{U}$ $G:[0,T]\times\mathbb{U}\to L(\mathbb{V},\mathbb{U})$. On the other hand, the linear fractional stochastic system corresponding to (4.1) is approximately controllable. Hence, there exists a mild solution for (4.1)–(4.4) under appropriate functions G, F, K, h, g and I_i satisfying suitable conditions to verify the assumptions on Theorem 3.2.

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REFERENCES

[1] P. Balasubramaniam and P. Tamilalagan, *Approximate controllability of a class of fractional neutral stochastic integro-differential inclusions with infinite delay by using Mainardi's function.* Appl. Math. Comput. **256** (2015), 232–246.

[2] P. Balasubramaniam, V. Vembarasan, and T. Senthilkumar, *Approximate controllability of impulsive fractional integro-differential systems with nonlocal conditions in Hilbert space.* Numer. Funct. Anal. Optim. **35** (2014), 177–197.

[3] M. Benchohra, J. Henderson, and S. K. Ntouyas, *Impulsive Differential Equations and Inclusions.* Contemporary Mathematics and Its Applications,Vol.2, Hindawi Publishing Corporation, New York, 2006.

[4] M. Benchohra and S. Ntouyas, *Existence and controllability results for multivalued semilinear differential equations with nonlocal conditions.* Soochow J. Math. **29**(2003), 157–170.

[5] M. Benchohra and M. Ziane, *Impulsive evolution inclusions with state-dependent delay and multivalued jumps.* Electron. J. Qual. Theory Differ. Equ. **2013** (2013), 1–21.

[6] L. Byszewski, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem.* J. Math. Anal. Appl. **162** (1991), 497–505.

[7] L. Byszewski and V. Lakshmikantham, *Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space.* Appl. Anal. **40** (1990), 11–19.

[8] A. Chadha and D.N. Pandey, *Mild solution for impulsive neutral fractional partial differential inclusions with nonlocal conditions.* Collect. Math. **67** (2016), 85–111.

[9] A. Chadha and D.N. Pandey, *Existence of the mild solution for impulsive neutral stochastic fractional integro-differential inclusions with nonlocal conditions.* Mediterr. J. Math. **13** (2016), 1005–1031.

[10] A. Chadha and D.N. Pandey, *Approximate Controllability of a Neutral Stochastic Fractional Integro-Differential Inclusion with Nonlocal Conditions.* J. Theoret. Probab. **31** (2018), 705–740.

[11] Y.-K. Chang and J.J. Nieto, *Existence of solutions for impulsive neutral integro-differential inclusions with nonlocal initial conditions via fractional operators.* Numer. Funct. Anal. Optim. **30** (2009), 227–244.

[12] Y. K. Chang, Z. H. Zhao, and G.M. N'Guérékata, *Square-mean almost automorphic mild solutions to nonautonomous stochastic differential equations in Hilbert spaces.* Comput. Math. Appl. **61** (2011), 384–391.

[13] G. Da. Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions.* Cambridge University Press, Cambridge, 1992.

[14] A. Debbouche and D. F.M. Torres, *Approximate controllability of fractional delay dynamic inclusions with nonlocal control conditions.* Appl. Math. Comput. **243** (2014) 161–175.

[15] K. Deimling, *Multivalued Differential Equations.* De Gruyter, Berlin, 1992.

[16] K. Ezzinbi, X. Fu, and K. Hilal, *Existence and regularity in the α-norm for some neutral partial differential equations with nonlocal conditions.* Nonlinear Anal.: TMA **67** (2007), 1613–1622.

[17] Z. Fan, *Approximate controllability of fractional differential equations via resolvent operators.* Adv. Diff. Equ. 2014 (2014), 1–11.

[18] M. Fečkan, J.-R.Wang, and Y. Zhou, *Controllability of fractional functional evolution equations of Sobolev type via characteristic solution operators.* J. Optim. Theory Appl. **165** (2013), 79–95.

[19] M. Fečkan, Y. Zhou, and J.-R. Wang, *On the concept and existence of solution for impulsive fractional differential equations.* Commun. Nonlinear Sci. Numer. Simul. **17** (2012) 3050–3060.

[20] T. Guendouzi and S. Farahi, *Approximate controllability of semilinear fractional stochastic dynamic systems with nonlocal conditions in Hilbert spaces.* Mediterr. J. Math. **13** (2016), 637–656.

[21] M. Kerboua, A. Debbouche, and D. Baleanu, *Approximate controllability of Sobolev type nonlocal fractional stochastic dynamic systems in Hilbert spaces.* Abstr. Appl. Anal. **2013** (2013), 1–10.

[22] S. Kumar and N. Sukavanam, *Approximate controllability of fractional order semilinear systems with bounded delay.* J. Differential Equations **252** (2012), 6163–6174.

[23] V. Lakshmikantham, D. Bainov, and P.S. Simeonov, *Theory of Impulsive Differential Equations, Series in Modern Applied Mathematics.* World Scientific Publishing Co. Inc., Teaneck, NJ, 1989.

[24] K. Li, J. Peng, and J. Gao, *Controllability of nonlocal fractional differential systems of order $\alpha \in (1, 2]$ in Banach spaces.* Rep. Math. Phys. **71** (2013), 1–11.

[25] A. Lin and L. Hu, *Existence results for impulsive neutral stochastic functional integro-differential inclusions with nonlocal initial conditions.* Comput. Math. Appl. **59** (2011), 64–73.

[26] Z. Liu and X. Li, *On the controllability of impulsive fractional evolution inclusions in Banach spaces.* J. Optim. Theory Appl. **156** (2013), 167–182.

[27] N. I. Mahmudov and A. Denker, *On controllability of linear stochastic systems.* Internat. J. Control **73** (2000), 144–151.

[28] N.I. Mahmudov and S. Zorlu, *Approximate controllability of fractional integro-differential equations involving nonlocal initial conditions*. Bound. Value Probl. **2013** (2013), 118.

[29] X. R. Mao, *Stochastic Differential Equations and Applications*. Horwood, Chichester, UK, 1997.

[30] B. Oksendal, *Stochastic Differential Equations*. Springer, Berlin, Germany, 5th edition, 2002.

[31] A. Pazy, *Semi-groups of Linear Operator and Applications of Partial Differential Equations*. Springer Verlag, New York, 1983.

[32] I. Podlubny, *Fractional Differential Equations*. Acadmic press, New York, 1993.

[33] D. O'Regan, *Nonlinear alternatives for multivaluedmapswith applications to operator inclusions in abstract spaces.* Proc. Amer. Math. Soc.**127** (1999), 3557–3564.

[34] R. Sakthivel, R. Ganesh, and S. Suganya, *Approximate controllability of fractional neutral stochastic system with infinite delay.* Reports Math. Phys. **70** (2012), 291–311.

[35] R. Sakthivel and J. Luo, *Asymptotic stability of nonlinear impulsive stochastic differential eqnarrays*. Stat. Probab. Lett. **79** (2009), 1219–1223.

[36] R. Sakthivel, Y. Ren, A. Debbouchec, and N.I. Mahmudov, *Approximate controllability of fractional stochastic differential inclusions with nonlocal conditions*. Appl. Anal. **95** (2016), 2361–2382.

[37] R. Sakthivel, S. Suganya, and S.M. Anthoni, *Approximate controllability of fractional stochastic evolution equations*. Comput. Math. Appl. **63** (2012), 660–668.

[38] X.B. Shu, Y. Lai, and Y. Chen, *The existence of mild solutions for impulsive fractional partial differential equations. Nonlinear Analysis: TMA* **74** (2011), 2003–2011.

[39] X. B. Shu and Y. Shi, *A study on the mild solution of impulsive fractional evolution equations*. Appl. Math. Comput. **273** (2016), 465–476.

[40] G. V. Smirnov, *Introdunction to the Theory of Differential Inclusions*. Amer. Math. Soc. Publ., Rhode Island, 2002.

[41] V. Vijayakumar, C. Ravichandran, R. Murugesu, and J.J. Trujillo, *Controllability results for a class of fractional semilinear integro-differential inclusions via resolvent operators.* Appl. Math. Comput. **247** (2014), 152–161.

[42] J.-R. Wang, M. Fečkan, and Y. Zhou, *On the new concept of solutions and existence results for impulsive fractional evolution equations*. Dyn. Partial Differ. Equ. **8** (2011), 345–361.

[43] X. Wang and X. Shu, *The existence of positive mild solutions for fractional differential evolution equations with nonlocal conditions of order* $1 < \alpha < 2$. Adv. Difference Equ. **2015** (2015), 1–15.

[44] Z.Yan, *Controllability of fractional-order partial neutral functional integro-differential inclusions with infinite delay.* J. Franklin Inst. **348** (2011), 2156–2173.

[45] Z. Yan and F. Lu, *On approximate controllability of fractional stochastic neutral integro-differential inclusions with infinite delay*. Appl. Anal.**94** (2015), 1235–1258.

[46] Z. Yan and X. Jia, *Approximate controllability of partial fractional neutral stochastic functional integro-differential inclusions with state-dependent delay.* Collect. Math. **66** (2015), 93–124.

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Indian Institute of Information Technology
Department of Mathematics
Raichur, Raichur-584135, India
alkachadda23@gmail.com

Department of Mathematics, Indian Institute of Technology Guwahati,
Guwahati-781039, India swaroop@iitg.ernet.in