$\mathscr{C}$-COTORSION MODULES

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We give some characterizations and properties of $\mathscr{C}$-cotorsion modules and strongly $\mathscr{C}$-coherent rings. Strongly $\mathscr{C}$-coherent rings and $\mathscr{C}$-semihereditary rings are characterized by $\mathscr{C}$-cotorsion modules. Moreover, we define $\mathscr{C}$-cotorsion dimensions for modules and rings respectively, these dimension have nice properties when the ring is strongly $\mathscr{C}$-coherent.

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1. INTRODUCTION

Recall that a ring R is said to be *left coherent* [2, 7] if every finitely generated left ideal of R is finitely presented, a ring R is said to be *left semihereditary* if every finitely generated left ideal of R is projective. Coherent rings, semihereditary rings and their generalizations have been studied extensively by many authors (see, for example, [2, 3, 4, 6, 9, 14, 16, 19, 20, 22]). In [23], we introduced the concepts of *$\mathscr{C}$-coherent rings* and *$\mathscr{C}$-semihereditary rings.*

Let $\mathscr{C}$ be a class of some finitely presented left R-modules. Following [23], a ring R is called $\mathscr{C}$-coherent if every $C \in \mathscr{C}$ is 2-presented; a ring R is called *$\mathscr{C}$-semihereditary*, if whenever $0 \to K \to P \to C \to 0$ is exact , where $C \in \mathscr{C}$, P is finitely generated projective, then K is projective. To characterize $\mathscr{C}$-coherent rings and $\mathscr{C}$-semihereditary rings, in [23], we also introduced the concepts of *$\mathscr{C}$-injective modules* and *$\mathscr{C}$-flat modules.* According to [23], a left R-module M is called *$\mathscr{C}$-injective* if $\mathrm{Ext}^1_R(C, M) = 0$ for every $C \in \mathscr{C}$, a right R-module M is called *$\mathscr{C}$-flat* if $\mathrm{Tor}^R_1(M, C) = 0$ for every $C \in \mathscr{C}$. In [24], we introduced the concepts of *$\mathscr{C}$-projective modules* and *strongly $\mathscr{C}$-coherent rings.* Following [24], a left R-module M is called $\mathscr{C}$-projective if $\mathrm{Ext}^1_R(M, E) = 0$ for any $\mathscr{C}$-injective module E; a ring R is called strongly $\mathscr{C}$-coherent, if whenever $0 \to K \to P \to C \to 0$ is exact, where $C \in \mathscr{C}$ and P is finitely generated projective, then K is $\mathscr{C}$-projective. We recall also that a right R-module M is called *cotorsion* [7] if $\mathrm{Ext}_R^1(F, M) = 0$ for any flat module right R-module F. In [24], we introduced the concept of *$\mathscr{C}$-cotorsion modules*, following [24], a right R-module M is called $\mathscr{C}$-cotorsion if $\mathrm{Ext}_R^1(F, M) = 0$ for any $\mathscr{C}$-flat module F.

In this article, we continues to study $\mathscr{C}$-cotorsion modules , strongly $\mathscr{C}$-coherent rings and $\mathscr{C}$-semihereditary rings. We give some characterizations and properties of $\mathscr{C}$-cotorsion modules, strongly $\mathscr{C}$-coherent rings, and $\mathscr{C}$-semihereditary rings. Strongly $\mathscr{C}$-coherent rings and $\mathscr{C}$-semihereditary rings are characterized by $\mathscr{C}$-cotorsion modules. As corollaries, some results of left semihereditary rings are obtained. Furthermore, we define $\mathscr{C}$-cotorsion dimensions of modules and rings, we show that over a strongly $\mathscr{C}$-coherent ring, these dimensions has some nice properties. As corollaries, some results of right perfect rings are given.

Next, we recall some known notions and facts needed in the sequel.

Given a class $\mathscr{L}$ of R-modules, we will denote by

$$\mathscr{L}^{\perp} = \{M : \mathrm{Ext}_R^1(L, M) = 0, L \in \mathscr{L}\}$$

the right orthogonal class of $\mathscr{L}$, and by

$$^{\perp}\mathscr{L} = \{M : \mathrm{Ext}_R^1(M, L) = 0, L \in \mathscr{L}\}$$

the left orthogonal class of $\mathscr{L}$.

Let $\mathcal{F}$ be a class of R-modules and M an R-module. Following [7], we say that a homomorphism $\varphi : M \to F$ where $F \in \mathcal{F}$ is an $\mathcal{F}$-preenvelope of M if for any morphism $f : M \to F'$ with $F' \in \mathcal{F}$, there is a $g : F \to F'$ such that $g\varphi = f$. An $\mathcal{F}$-preenvelope $\varphi : M \to F$ is said to be an $\mathcal{F}$-envelope if every endomorphism $g : F \to F$ such that $g\varphi = \varphi$ is an isomorphism. Dually, we have the definitions of $\mathcal{F}$-precovers and $\mathcal{F}$-covers. $\mathcal{F}$-envelopes ($\mathcal{F}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

Following [1], a pair $(\mathscr{A}, \mathscr{B})$ of classes of R-modules is called a *cotorsion pair* if $\mathscr{A}^{\perp} = \mathscr{B}$ and $^{\perp}\mathscr{B} = \mathscr{A}$. A cotorsion pair $(\mathscr{A}, \mathscr{B})$ is called *hereditary* [8, Definition 1.1] if whenever $0 \to A' \to A \to A'' \to 0$ is exact with $A, A'' \in \mathscr{A}$ then A' is also in $\mathscr{A}$. By [8, Proposition 1.2], a cotorsion pair $(\mathscr{A}, \mathscr{B})$ is hereditary if and only if whenever $0 \to B' \to B \to B'' \to 0$ is exact with $B', B \in \mathscr{B}$ then B'' is also in $\mathscr{B}$. A cotorsion pair $(\mathscr{A}, \mathscr{B})$ is called perfect [8] if every R-module has an $\mathscr{A}$-cover and a $\mathscr{B}$-envelope. A cotorsion pair $(\mathscr{A}, \mathscr{B})$ is called complete (see [7, Definition 7.16] and [17, Lemma 1.13]) if for any R-module M, there are exact sequences $0 \to M \to B \to A \to 0$ with $A \in \mathscr{A}$ and $B \in \mathscr{B}$, and $0 \to B' \to A' \to M \to 0$ with $A' \in \mathscr{A}$ and $B' \in \mathscr{B}$.

Throughout this paper, R is an associative ring with identity and all modules considered are unitary, $\mathscr{C}$ is a class of some finitely presented left R-modules. For any R-module M, $E(M)$ denotes the injective envelope of M.

2. $\mathscr{C}$-COTORSION MODULES

Throughout this paper, we will denote the class of $\mathscr{C}$-flat (resp., $\mathscr{C}$-cotorsion, $\mathscr{C}$-injective, $\mathscr{C}$-projective) modules by $\mathscr{C}\mathscr{F}$ (resp., $\mathscr{C}\mathcal{CT}$, $\mathscr{C}\mathcal{I}$, $\mathscr{C}\mathcal{P}$). By [23, Theorem 2.10(2)], the pair $(\mathscr{C}\mathscr{F}, \mathscr{C}\mathcal{CT})$ is a perfect cotorsion pair. We will denote the $\mathscr{C}$-flat cover of M by $\mathscr{C}\mathscr{F}(M)$ and denote the $\mathscr{C}$-cotorsion envelope of M by $\mathscr{C}\mathcal{CT}(M)$ respectively.

Recall that a right R-module F is called *min-flat* [14] if $\mathrm{Tor}_1^R(F, R/I) = 0$ for every minimal left ideal I; a right R-module M is called *min-cotorsion* [14] if $\mathrm{Ext}_R^1(F, M) = 0$ for any min-flat module right R-module F; a right R-module F is called *P-flat* [10] or $(1,1)$*-flat* [20] if $\mathrm{Tor}_1^R(F, R/Ra) = 0$ for every $a \in R$; a right R-module M is called *P-cotorsion* [9] if $\mathrm{Ext}_R^1(F, M) = 0$ for any P-flat module right R-module F. Let n be a nonnegative integer. Recall that a left R-module M is said to be n-presented in case there is an exact sequence of left R-modules $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ in which every F_i is a finitely generated free , equivalently projective left R-module. A right R-module F is called $(n,0)$*-flat* [21, 22] if $\mathrm{Tor}_1^R(F, V) = 0$ for every n-presented left R-module V; a right R-module M is called n*-flat* [12] if $\mathrm{Tor}_1^R(M, N) = 0$ for every finitely presented left R-module N with projective dimension $\leq n$. We call a right R-module M $(n,0)$-cotorsion if $\mathrm{Ext}_R^1(F, M) = 0$ for any $(n,0)$-flat module F, and we call a right R-module M n-cotorsion if $\mathrm{Ext}_R^1(F, M) = 0$ for any n-flat module F.

Example 1. (1) Let $\mathscr{C}$ be the class of all finitely presented left R-modules. Then a right R-module M is $\mathscr{C}$-cotorsion if and only if it is cotorsion.

(2) Let $\mathscr{C} = \{R/I : I \text{ is a minimal left ideal of } R\}$. Then a right R-module M is $\mathscr{C}$-cotorsion if and only if it is min-cotorsion.

(3) Let $\mathscr{C} = \{R/Ra : a \in R\}$. Then a right R-module M is $\mathscr{C}$-cotorsion if and only if it is P-cotorsion.

(4) Let $\mathscr{C}$ be the class of all n-presented left R-modules. Then a right R-module M is $\mathscr{C}$-cotorsion if and only if it is (n,0)-cotorsion.

(5) Let $\mathscr{C}$ be the class of all finitely presented left R-modules with projective dimension $\leq n$. Then a right R-module M is $\mathscr{C}$-cotorsion if and only if it is n-cotorsion.

Next, we give some characterizations of $\mathscr{C}$-cotorsion modules.

THEOREM 1. *Let M be a right R-module. Then the following statements are equivalent:*

(1) *M is $\mathscr{C}$-cotorsion.*

(2) *M is injective with respect to every exact sequence $0 \to C \to B \to F \to 0$ of right R-modules with F $\mathscr{C}$-flat.*

(3) *For any $\mathscr{C}$-flat module F, F is projective with respect to every exact sequence $0 \to M \to M' \to M'' \to 0$.*

Moreover, if $E(M)$ is $\mathscr{C}$-flat, then the above conditions are also equivalent to:

(4) *If the sequence $0 \to M \to F \to L \to 0$ is exact, where F is $\mathscr{C}$-flat, then $F \to L \to 0$ is a $\mathscr{C}$-flat precover of L.*

(5) *M is a kernel of a $\mathscr{C}$-flat precover $E \to L$ with E injective.*

Proof. (1) $\Rightarrow$ (2). By the exact sequence

$$\mathrm{Hom}(B, M) \to \mathrm{Hom}(C, M) \to \mathrm{Ext}^1_R(F, M) = 0.$$

(2) $\Rightarrow$ (1). For any $\mathscr{C}$-flat module F, there exists an exact sequence $0 \to K \to P \to F \to 0$, where P is projective. Hence we get an exact sequence $\mathrm{Hom}(P, M) \to \mathrm{Hom}(K, M) \to \mathrm{Ext}^1_R(F, M) \to \mathrm{Ext}^1_R(P, M) = 0$, and thus $\mathrm{Ext}^1_R(F, M) = 0$ by (2). Therefore, M is $\mathscr{C}$-cotorsion.

(1) $\Rightarrow$ (3). Assume (1). Then we have an exact sequence $\mathrm{Hom}(F, M') \to \mathrm{Hom}(F, M'') \to \mathrm{Ext}^1_R(F, M) = 0$ for every $\mathscr{C}$-flat module F, and so (3) follows.

(3) $\Rightarrow$ (1). Firstly, we have an exact sequence

$$0 \to M \to E \xrightarrow{f} N \to 0$$

of right R-modules, where E is injective. And so, for any $\mathscr{C}$-flat module F, we have an exact sequence $\mathrm{Hom}(F, E) \xrightarrow{f_*} \mathrm{Hom}(F, N) \to \mathrm{Ext}^1_R(F, M) \to \mathrm{Ext}^1_R(F, E) = 0$. By (3), f_* is epic, so $\mathrm{Ext}^1_R(F, M) = 0$, and (1) follows.

(3) $\Rightarrow$ (4). It is obvious.

(4) $\Rightarrow$ (5). Assume (4). Then, letting $E = E(M)$ and $L = E(M)/M$, we have (5).

(5) $\Rightarrow$ (1). By (5), we have an exact sequence $0 \to M \to E \xrightarrow{f} L \to 0$, where E is injective, and $E \xrightarrow{f} L \to 0$ is a $\mathscr{C}$-flat precover of L. Let F be any $\mathscr{C}$-flat right R-module. Then we get an exact sequence

$$\mathrm{Hom}(F, E) \xrightarrow{f_*} \mathrm{Hom}(F, L) \to \mathrm{Ext}^1_R(F, M) \to \mathrm{Ext}^1_R(F, E) = 0$$

with f_* epic, which implies that $\mathrm{Ext}^1_R(F, M) = 0$, and (1) follows. $\square$

PROPOSITION 1. *Let $\{M_i \mid i \in I\}$ be a family of right R-modules. Then the following statements are equivalent.*

(1) *Each M_i is $\mathscr{C}$-cotorsion.*

(2) *$\prod_{i\in I} M_i$ is $\mathscr{C}$-cotorsion.*

Proof. By the isomorphism $\mathrm{Ext}^1_R(F, \prod_{i\in I} M_i) \cong \prod_{i\in I} \mathrm{Ext}^1_R(F, M_i)$. □

PROPOSITION 2. *Let $\mathscr{F}$ be a class of right R-modules.*

(1) *If $P \in \mathscr{F}$ for any projective right R-module P and $\varphi : F \to M$ is an $\mathscr{F}$-precover, then φ is epic.*

(2) *If $E \in \mathscr{F}$ for any injective right R-module E and $\varphi : M \to F$ is an $\mathscr{F}$-preenvelope, then φ is monic.*

Proof. (1). At first, there is an epimorphism of right R-modules $P \overset{\alpha}{\to} M \to 0$, where P is projective. Since $\varphi : F \to M$ is an $\mathscr{F}$-precover and $P \in \mathscr{F}$, there exists a homomorphism $\beta : P \to F$ such that $\alpha = \varphi\beta$. It follows that φ is epic.

(2). It is dually to the proof of (1). □

COROLLARY 1. (1) *$\mathscr{C}$-flat covers of $\mathscr{C}$-cotorsion modules are $\mathscr{C}$-cotorsion.*

(2) *$\mathscr{C}$-cotorsion enlopes of $\mathscr{C}$-flat modules are $\mathscr{C}$-flat.*

Proof. (1). Let N be a $\mathscr{C}$-cotorsion module and $\varphi : F \to N$ a $\mathscr{C}$-flat cover of N. Then by Proposition 2(1), φ is epic. So we have an exact sequence $0 \to \mathrm{Ker}(\varphi) \to F \overset{\varphi}{\to} N \to 0$. Note that $\mathscr{C}$-flat modules are closed under extensions, by [18, Lemma 2.1.1], $\mathrm{Ext}^1_R(F', \mathrm{Ker}(\varphi)) = 0$ for any $\mathscr{C}$-flat module F', so $\mathrm{Ker}(\varphi)$ is $\mathscr{C}$-cotorsion since $(\mathscr{CF}, \mathscr{C}\mathcal{CT})$ is a cotorsion pair by [23, Theorem 2.10(2)], and thus F is $\mathscr{C}$-cotorsion.

(2). Let F be a $\mathscr{C}$-flat module and $\psi : F \to E$ an $\mathscr{C}$-cotorsion enlope of F. Then by Proposition 2(2), ψ is monic. So we have an exact sequence $0 \to F \overset{\psi}{\to} E \to E/im(\psi) \to 0$. Note that $\mathscr{C}$-cotorsion modules are closed under extensions , by [18, Lemma 2.1.2], $\mathrm{Ext}^1_R(E/im(\psi), N) = 0$ for any $\mathscr{C}$-cotorsion module N, so $E/im(\psi)$ is $\mathscr{C}$-flat since $(\mathscr{CF}, \mathscr{C}\mathcal{CT})$ is a cotorsion pair by [23, Theorem 2.10(2)], and thus E is $\mathscr{C}$-flat. □

THEOREM 2. *Let R be a ring and $\mathscr{C}$ be a class of some finitely presented left R-modules. Then $(\mathscr{CF}, \mathscr{C}\mathcal{CT})$ is a complete cotorsion pair.*

Proof. Let M be any right R-module. Since $(\mathscr{CF}, \mathscr{C}\mathcal{CT})$ is a perfect cotorsion pair by [23, Theorem 2.10(2)], M has a $\mathscr{C}$-flat cover $\varphi : F \to M$ and a $\mathscr{C}$-cotorsion enlope $\psi : M \to N$. Moreover, by Proposition 2, φ is epic and ψ is monic. So we have two exact sequence $0 \to K \to F \to M \to 0$ and $0 \to M \to N \to L \to 0$, where $K = \mathrm{Ker}(\varphi), L = N/im(\psi)$. Note that $\mathscr{C}$-flat modules and $\mathscr{C}$-cotorsion modules are closed under extensions, by [18, Lemma 2.1.1, Lemma 2.1.2], we have that K is $\mathscr{C}$-cotorsion, and L is $\mathscr{C}$-flat. This complete the proof. □

PROPOSITION 3. *The following statements are equivalent for a ring R:*

(1) *R is strongly $\mathscr{C}$-coherent.*

(2) $\mathrm{Ext}_R^2(C, N) = 0$ *for any left R-module $C \in \mathscr{C}$ and any $\mathscr{C}$-injective left R-module N.*

(3) *R is $\mathscr{C}$-coherent and $(\mathscr{CF}, \mathscr{C}\mathcal{CT})$ is a hereditary cotorsion pair.*

Proof. (1) ⇔ (2). It follows from [24, Theorem 1(6)].
(2) ⇔ (3). It follows from [23, Proposition 3.11]. □

COROLLARY 2. *Let R be a strongly $\mathscr{C}$-coherent ring, M a $\mathscr{C}$-cotorsion right R-module . Then* $\mathrm{Ext}_R^k(F, M) = 0$ *for all $\mathscr{C}$-flat module F and all positive integers k.*

Proof. Since R is a strongly $\mathscr{C}$-coherent ring, by Proposition 3, we have that $(\mathscr{CF}, \mathscr{C}\mathcal{CT})$ is a hereditary cotorsion pair, and so $\mathrm{Ext}_R^k(F, M) = 0$ for all $\mathscr{C}$-flat module F and all positive integers k by [8, Proposition 1.2]. □

3. THE $\mathscr{C}$-COTORSION DIMENSION OVER STRONGLY $\mathscr{C}$-COHERENT RINGS

The cotorsion dimension of modules and rings were defined by Mao and Ding in [13], now we define the $\mathscr{C}$-cotorsion dimension of modules and rings as following .

Definition 1.

(1) The $\mathscr{C}$-cotorsion dimension of a module M_R is defined by
$\mathscr{C}\mathcal{CT}\text{-}dim(M) = inf\{n : \mathrm{Ext}_R^{n+1}(F, M) = 0$ for every $\mathscr{C}$-flat module F$\}$

(2) The $\mathscr{C}$-cotorsion global dimension of a ring R is defined by
$\mathscr{C}\mathcal{CT}\text{-}D(R)$ =sup$\{\mathscr{C}\mathcal{CT}\text{-}dim(M)$: M is a right R-module$\}$

THEOREM 3. *Let R be a strongly $\mathscr{C}$-coherent ring, M a right R-module and n a nonnegative integer. Then the following conditions are equivalent:*

(1) $\mathscr{C}\mathcal{CT}\text{-}dim(M_R) \leq n$.

(2) $\mathrm{Ext}_R^{n+k}(F, M) = 0$ *for all $\mathscr{C}$-flat module F and all positive integers k .*

(3) $\mathrm{Ext}_R^{n+1}(F, M) = 0$ *for all* $\mathscr{C}$*-flat module* F *.*

(4) *If the sequence of right* R*-modules*

$$0 \to M \xrightarrow{\varepsilon} E_0 \xrightarrow{d_0} \cdots \to E_{n-1} \xrightarrow{d_{n-1}} E_n \to 0$$

is exact with $E_0, \cdots, E_{n-1}$ $\mathscr{C}$*-cotorsion, then* E_n *is also* $\mathscr{C}$*-cotorsion.*

(5) *There exists an exact sequence of right* R*-modules*

$$0 \to M \to E_0 \to \cdots \to E_{n-1} \to E_n \to 0$$

with $E_0, \cdots, E_{n-1}, E_n$ $\mathscr{C}$*-cotorsion.*

(6) $\mathscr{C}\mathcal{C}\mathcal{T}$*-dim*$(\mathscr{C}\mathscr{F}(M_R)) \leq n$*.*

Proof. (1) ⇒ (2). Use induction on n. If $n = 0$, then M is $\mathscr{C}$-cotorsion. Since R is a strongly $\mathscr{C}$-coherent ring, by Corollary 2, we have $\mathrm{Ext}_R^k(F, M) = 0$ for all $\mathscr{C}$-flat module F and all positive integers k. Now suppose that $\mathrm{Ext}_R^{n-1+k}(F, N) = 0$ for any $\mathscr{C}$-flat module F, any positive integer k and any right R-module N with $\mathscr{C}\mathcal{C}\mathcal{T}$-*dim*$(N) \leq n - 1$. Then for any right R-module M with $\mathscr{C}\mathcal{C}\mathcal{T}$-*dim*$(M) \leq n$. If $\mathscr{C}\mathcal{C}\mathcal{T}$-*dim*$(M) = 0$, then (2) holds by Corollary 2. If $\mathscr{C}\mathcal{C}\mathcal{T}$-*dim*$(M) > 0$, then there exists a positive integer $m \leq n$ such that $\mathrm{Ext}_R^{m+1}(F, M) = 0$ for any $\mathscr{C}$-flat module F, which implies that $\mathrm{Ext}_R^m(F, E(M)/M) = 0$ for any $\mathscr{C}$-flat module F. So $\mathscr{C}\mathcal{C}\mathcal{T}$-*dim*$(E(M)/M) \leq m - 1$, and hence $\mathscr{C}\mathcal{C}\mathcal{T}$-*dim*$(E(M)/M) \leq n - 1$. By hypothesis, we have $\mathrm{Ext}_R^{n-1+k}(F, E(M)/M) = 0$ for any $\mathscr{C}$-flat module F and any positive integer k, it follows that $\mathrm{Ext}_R^{n+k}(F, M) = 0$. Therefore, (2) holds by induction.

(2) ⇒ (3) ⇒ (1) and (4) ⇒ (5) are obvious.

(3) ⇒ (4). Since R is strongly $\mathscr{C}$-coherent and $E_0, \cdot\cdot, E_{n-1}$ are $\mathscr{C}$-cotorsion, by Corollary 2, we have $\mathrm{Ext}_R^{n+1}(F, M) \cong \mathrm{Ext}_R^n(F, im(d_0)) \cong \mathrm{Ext}_R^{n-1}(F, im(d_1)) \cong \cdots \cong \mathrm{Ext}_R^1(F, im(d_{n-1})) = \mathrm{Ext}_R^1(F, E_n)$. Therefore (4) follows from (3).

(5) ⇒ (3). It follows from the above isomorphism $\mathrm{Ext}_R^{n+1}(F, M) \cong \mathrm{Ext}_R^1(F, E_n)$.

(1) ⇔ (6). Let M be any right R-module and F be any $\mathscr{C}$-flat module. By the proof of Corollary 1(1), we have an exact sequence of right R-modules $0 \to K \to \mathscr{C}\mathscr{F}(M) \to M \to 0$, where K is $\mathscr{C}$-cotorsion. Thus, we get an exact sequence

$$\mathrm{Ext}_R^{n+1}(F, K) \to \mathrm{Ext}_R^{n+1}(F, \mathscr{C}\mathscr{F}(M)) \to \mathrm{Ext}_R^{n+1}(F, M) \to \mathrm{Ext}_R^{n+2}(F, K).$$

Since R is strongly $\mathscr{C}$-coherent, by Corollary 2, we have $\mathrm{Ext}_R^{n+1}(F, K) = \mathrm{Ext}_R^{n+2}(F, K) = 0$. So, $\mathrm{Ext}_R^{n+1}(F, \mathscr{C}\mathscr{F}(M)) \cong \mathrm{Ext}_R^{n+1}(F, M)$, and the results follows. □

By Theorem 3, we have immediately the next proposition.

Proposition 4. *Let R be a strongly $\mathscr{C}$-coherent ring and $0 \to A \to B \to C \to 0$ an exact sequence of right R-modules. If two of $\mathscr{C}\mathcal{CT}$-dim(A), $\mathscr{C}\mathcal{CT}$-dim(B), $\mathscr{C}\mathcal{CT}$-dim(C) are finite, then so does the third. Moreover,*

(1) $\mathscr{C}\mathcal{CT}\text{-}dim(B) \leq sup\ \{\mathscr{C}\mathcal{CT}\text{-}dim(A), \mathscr{C}\mathcal{CT}\text{-}dim(C)\}$.

(2) $\mathscr{C}\mathcal{CT}\text{-}dim(A) \leq sup\ \{\mathscr{C}\mathcal{CT}\text{-}dim(B), \mathscr{C}\mathcal{CT}\text{-}dim(C)+1\}$.

(3) $\mathscr{C}\mathcal{CT}\text{-}dim(C) \leq sup\ \{\mathscr{C}\mathcal{CT}\text{-}dim(B), \mathscr{C}\mathcal{CT}\text{-}dim(A)-1\}$.

Corollary 3. *Let R be a strongly $\mathscr{C}$-coherent ring and $0 \to A \to B \to C \to 0$ an exact sequence of right R-modules. Then $\mathscr{C}\mathcal{CT}\text{-}dim(B) < sup\ \{\mathscr{C}\mathcal{CT}\text{-}dim(A), \mathscr{C}\mathcal{CT}\text{-}dim(C)\}$ if and only if $\mathscr{C}\mathcal{CT}\text{-}dim(B) < \mathscr{C}\mathcal{CT}\text{-}dim(A)$. Moreover,*

(1) $\mathscr{C}\mathcal{CT}\text{-}dim(B) < sup\ \{\mathscr{C}\mathcal{CT}\text{-}dim(A), \mathscr{C}\mathcal{CT}\text{-}dim(C)\} < \infty$ *if and only if* $\mathscr{C}\mathcal{CT}\text{-}dim(B) < \mathscr{C}\mathcal{CT}\text{-}dim(A) = \mathscr{C}\mathcal{CT}\text{-}dim(C)+1 < \infty$.

(2) $\mathscr{C}\mathcal{CT}\text{-}dim(B) < sup\ \{\mathscr{C}\mathcal{CT}\text{-}dim(A), \mathscr{C}\mathcal{CT}\text{-}dim(C)\} = \infty$ *if and only if* $\mathscr{C}\mathcal{CT}\text{-}dim(B) < \mathscr{C}\mathcal{CT}\text{-}dim(A) = \mathscr{C}\mathcal{CT}\text{-}dim(C) = \infty$.

Proof. It is easy to see that if $\mathscr{C}\mathcal{CT}\text{-}dim(B) < \mathscr{C}\mathcal{CT}\text{-}dim(A)$ then $\mathscr{C}\mathcal{CT}\text{-}dim(B) < \sup\ \{\mathscr{C}\mathcal{CT}\text{-}dim(A), \mathscr{C}\mathcal{CT}\text{-}dim(C)\}$.

Now suppose that $\mathscr{C}\mathcal{CT}\text{-}dim(B) < \sup\ \{\mathscr{C}\mathcal{CT}\text{-}dim(A), \mathscr{C}\mathcal{CT}\text{-}dim(C)\}$, we will prove that $\mathscr{C}\mathcal{CT}\text{-}dim(B) < \mathscr{C}\mathcal{CT}\text{-}dim(A)$. Otherwise, if $\mathscr{C}\mathcal{CT}\text{-}dim(B) \geq \mathscr{C}\mathcal{CT}\text{-}dim(A)$, then we have $\mathscr{C}\mathcal{CT}\text{-}dim(A) \leq \mathscr{C}\mathcal{CT}\text{-}dim(B) < \mathscr{C}\mathcal{CT}\text{-}dim(C) < \infty$, this is contrary to the result of Proposition 4(3).

(1) We need only to prove that if
$\mathscr{C}\mathcal{CT}\text{-}dim(B) < \sup\ \{\mathscr{C}\mathcal{CT}\text{-}dim(A), \mathscr{C}\mathcal{CT}\text{-}dim(C)\} < \infty$, then
$\mathscr{C}\mathcal{CT}\text{-}dim(B) < \mathscr{C}\mathcal{CT}\text{-}dim(A) = \mathscr{C}\mathcal{CT}\text{-}dim(C)+1 < \infty$.

Indeed, in this case, we have $\mathscr{C}\mathcal{CT}\text{-}dim(B) < \mathscr{C}\mathcal{CT}\text{-}dim(A)$ by the above proof, and so we have $\mathscr{C}\mathcal{CT}\text{-}dim(A) \leq \mathscr{C}\mathcal{CT}\text{-}dim(C)+1 < \infty$ by Proposition 4(2), and $\mathscr{C}\mathcal{CT}\text{-}dim(C)+1 \leq \mathscr{C}\mathcal{CT}\text{-}dim(A)$ by Proposition 4(3), and hence $\mathscr{C}\mathcal{CT}\text{-}dim(B) < \mathscr{C}\mathcal{CT}\text{-}dim(A) = \mathscr{C}\mathcal{CT}\text{-}dim(C)+1 < \infty$.

(2). We need only to prove that if
$\mathscr{C}\mathcal{CT}\text{-}dim(B) < \sup\ \{\mathscr{C}\mathcal{CT}\text{-}dim(A), \mathscr{C}\mathcal{CT}\text{-}dim(C)\} = \infty$, then
$\mathscr{C}\mathcal{CT}\text{-}dim(B) < \mathscr{C}\mathcal{CT}\text{-}dim(A) = \mathscr{C}\mathcal{CT}\text{-}dim(C) = \infty$.

Indeed, in this case, we have $\mathscr{C}\mathcal{CT}\text{-}dim(B)$ is finite. Note that $\sup\ \{\mathscr{C}\mathcal{CT}\text{-}dim(A), \mathscr{C}\mathcal{CT}\text{-}dim(C)\} = \infty$, by the first party of Proposition 4, we have $\mathscr{C}\mathcal{CT}\text{-}dim(A) = \mathscr{C}\mathcal{CT}\text{-}dim(C) = \infty$. $\square$

Corollary 4. *Let R be a strongly $\mathscr{C}$-coherent ring, F a $\mathscr{C}$-flat right R-module and n a nonnegative integer. Then the following are equivalent:*

(1) $\mathscr{C}\mathcal{CT}$*-dim*$(F_R) \leq n$.

(2) *There exists an exact sequence of right R-modules $0 \to F \to E_0 \to \cdots \to E_{n-1} \to E_n \to 0$ such that $E_0, \cdots, E_{n-1}, E_n$ are $\mathscr{C}$-cotorsion envelopes of $\mathscr{C}$-flat modules.*

Proof. (1) ⇒ (2). By [23, Theorem 2.10], every right R-module has a $\mathscr{C}$-cotorsion envelope. And so, by Lemma 2.4(2), we can obtain exact sequences

$$0 \to F \xrightarrow{\varphi_0} E_0 \xrightarrow{\pi_0} E_0/im(\varphi_0) \to 0$$

$$0 \to E_0/im(\varphi_0) \xrightarrow{\varphi_1} E_1 \xrightarrow{\pi_1} E_1/im(\varphi_1) \to 0$$

$$\cdots$$

$$0 \to E_{n-2}/im(\varphi_{n-2}) \xrightarrow{\varphi_{n-1}} E_{n-1} \xrightarrow{\pi_{n-1}} E_{n-1}/im(\varphi_{n-1}) \longrightarrow 0,$$

where $F \overset{\varphi_0}{\to} E_0, E_0/im(\varphi_0) \overset{\varphi_1}{\to} E_1, \cdots, E_{n-2}/im(\varphi_{n-2}) \overset{\varphi_{n-1}}{\to} E_{n-1}$ are monic $\mathscr{C}$-cotorsion enolpes. Observing that $(\mathscr{C}\mathscr{F}, \mathscr{C}\mathcal{CT})$ is a cotorsion pair, by [18, Theorem 2.1.2], each $E_i/im(\varphi_i)$ is $\mathscr{C}$-flat. Thus, we get an exact sequence $0 \to F \overset{\varphi_0}{\to} E_0 \overset{\varphi_1\pi_0}{\to} \cdots \overset{\varphi_{n-1}\pi_{n-2}}{\to} E_{n-1} \overset{\pi_{n-1}}{\to} E_n \to 0$, where $E_0, \cdots, E_{n-1}$ are $\mathscr{C}$-cotorsion envelope of $\mathscr{C}$-flat modules , $E_n = E_{n-1}/im(d_{n-1})$ is $\mathscr{C}$-flat. Since $\mathscr{C}\mathcal{CT}$-$dim(F_R) \leq n$, by Theorem 3(4), E_n is $\mathscr{C}$-cotorsion, and so, as the $\mathscr{C}$-cotorsion envelope of itself, E_n is also a $\mathscr{C}$-cotorsion envelope of a $\mathscr{C}$-flat module. Therefore, (2) is proved.

(2) ⇒ (1). It follows immediately from Theorem 3. □

Theorem 4. *Let R be a strongly $\mathscr{C}$-coherent ring. Then*

(1) $\mathscr{C}\mathcal{CT}$*-D*(R) =*sup*$\{pd(F)$*: F is a $\mathscr{C}$-flat right R-module*$\}$
=*sup*$\{\mathscr{C}\mathcal{CT}$*-dim*$(F)$*: F is a $\mathscr{C}$-flat right R-module*$\}$.

(2) *If $\mathscr{C}\mathcal{CT}$-$D(R) < \infty$, then*
$\mathscr{C}\mathcal{CT}$*-D*(R) =*sup*$\{pd(F) : F \in \mathscr{C}\mathscr{F} \cap \mathscr{C}\mathcal{CT}\}$
=*sup*$\{pd(\mathscr{C}\mathcal{CT}(F))$*: F is a $\mathscr{C}$-flat right R-module*$\}$
=*sup*$\{pd(\mathscr{C}\mathscr{F}(E))$*: E is a $\mathscr{C}$-cotorsion right R-module*$\}$
=*sup*$\{\mathscr{C}\mathcal{CT}$*-dim*$(P)$*: P is a projective right R-module*$\}$.

Proof. (1). Write $\mathscr{C}\mathcal{CT}$-$D(R) = n$, $sup\{pd(F)$: F is a $\mathscr{C}$-flat right R-module$\} = m$, and $sup\{\mathscr{C}\mathcal{CT}$-$dim(F)$: F is a $\mathscr{C}$-flat right R-module$\} = k$. If $n = \infty$, then it is clear that $m \leq n$. If $n < \infty$, then $\mathscr{C}\mathcal{CT}$-$dim(M) \leq n$ for any right R-module M. Since R is strongly $\mathscr{C}$-coherent, by Theorem 3, we have $\mathrm{Ext}_R^{n+1}(F,M)=0$ for any $\mathscr{C}$-flat module F and any right R-module M, so $pd(F)\le n$ for any $\mathscr{C}$-flat module F, and thus $m\le n$. In a similar way we can prove that $n\le m$, and so $n=m$. It is easy to see that $n\ge k$. Now we prove that $m\le k$. If $k=\infty$, then it is clear that $m\le k$. If $k<\infty$. Let F be any $\mathscr{C}$-flat module and M be any right R-module. Then by Theorem 2, we have an exact sequence of right R-modules $0\to K\to F_1\to M\to 0$, where K is $\mathscr{C}$-cotorsion and F_1 is $\mathscr{C}$-flat. So, since $\mathscr{C}\mathcal{CT}\text{-}dim(F)\le k$ and R is strongly $\mathscr{C}$-coherent, by Theorem 3 and Corollary 2, we get the following exact sequence

$$0=\mathrm{Ext}_R^{k+1}(F,F_1)\to \mathrm{Ext}_R^{k+1}(F,M)\to \mathrm{Ext}_R^{k+2}(F,K)=0.$$

It shows that $\mathrm{Ext}_R^{k+1}(F,M)=0$, so $pd(F)\le k$, and hence $m\le k$. Therefore, $m=n=k$.

(2) Write $sup\{pd(F) : F\in \mathscr{C}\mathscr{F}\cap\mathscr{C}\mathcal{CT}\}=l_1$, $sup\{pd(\mathscr{C}\mathcal{CT}(F))$: F is a $\mathscr{C}$-flat right R-module$\}=l_2$, $sup\{pd(\mathscr{C}\mathscr{F}(E))$: E is a $\mathscr{C}$-cotorsion right R-module$\}=l_3$, $sup\{\mathscr{C}\mathcal{CT}\text{-}dim(P)$: P is a projective right R-module$\}=l_4$. Then by (1) and Corollary 1, we have $n\ge l_1=l_2=l_3$ and $n\ge l_4$.

Now we claim $n\le l_2$. In fact, for any $\mathscr{C}$-flat right R-module F, we have $\mathscr{C}\mathcal{CT}\text{-}dim(F)\le n$, so, by Corollary 4, we have an exact sequence of right R-modules $0\to F\to E_0\to\cdots\to E_{n-1}\to E_n\to 0$ such that $E_0,\cdots,E_{n-1},E_n$ are $\mathscr{C}$-cotorsion envelopes of $\mathscr{C}$-flat modules. By hypothesis, $pd(E_i)\le l_2$ for each i, so by induction on n we have $pd(F)\le l_2$, and hence $n\le l_2$ by (1).

Finally, we prove that $n\le l_4$. For any $\mathscr{C}$-flat module F, by (1), $pd(F)\le n$. So we have an exact sequence of right R-modules

$$0\to P_n\to P_{n-1}\to\cdots\to P_1\to P_0\to F\to 0,$$

where $P_0,P_1,\cdots,P_n$ are projective. But $\mathscr{C}\mathcal{CT}\text{-}dim(P_i)\le l_4, i=0,1,\cdots,n$, by induction on n, we have $\mathscr{C}\mathcal{CT}\text{-}dim(F)\le l_4$. It shows that $n\le l_4$ by (1). This complete the proof. □

Corollary 5. *Let R be a strongly $\mathscr{C}$-coherent ring and n a nonnegative integer. Then the following conditions are equivalent:*

(1) $\mathscr{C}\mathcal{CT}\text{-}D(R)\le n$.

(2) *All $\mathscr{C}$-flat right R-modules are of projective dimension $\le n$.*

(3) *All $\mathscr{C}$-flat right R-modules are of $\mathscr{C}$-cotorsion dimension $\le n$.*

(4) $\mathrm{Ext}_R^{n+1}(F,F')=0$ *for all $\mathscr{C}$-flat right R-modules F,F'.*

(5) $\mathrm{Ext}_R^{n+k}(F,F')=0$ *for all $\mathscr{C}$-flat right R-modules F,F' and positive integers k.*

(6) *$\mathscr{C}\mathcal{CT}$-$D(R) < \infty$ and all right R-modules in $\mathscr{C}\mathscr{F} \cap \mathscr{C}\mathcal{CT}$ are of projective dimension $\leq n$.*

(7) *$\mathscr{C}\mathcal{CT}$-$D(R) < \infty$ and all projective right R-modules are of $\mathscr{C}$-cotorsion dimension $\leq n$.*

Let $\mathcal{F}$ be a class of modules. According to [5], an $\mathcal{F}$-envelope $\phi : M \to F$ is said to have the unique mapping property if for any homomorphism $f : M \to F'$ with $F' \in \mathcal{F}$, there is a unique homomorphism $g : F \to F'$ such that $f = g\phi$. Dually, we can define the concept of an $\mathcal{F}$-cover with the unique mapping property .

Corollary 6. *Let R be a ring and $\mathscr{C}$ be a class of some finitely presented left R-modules. Then the following conditions are equivalent:*

(1) *$\mathscr{C}\mathcal{CT}$-$D(R) = 0$.*

(2) *All right R-modules are $\mathscr{C}$-cotorsion.*

(3) *All $\mathscr{C}$-flat right R-modules are projective.*

(4) *Every right R-module has a $\mathscr{C}$-cotorsion envelope with the unique mapping property.*

In this case, R is a right perfect ring. Moreover, if R is a strongly $\mathscr{C}$-coherent ring, then the above conditions are equivalent to:

(5) *All $\mathscr{C}$-flat right R-modules are $\mathscr{C}$-cotorsion.*

(6) *$\mathrm{Ext}^1_R(F, F') = 0$ for all $\mathscr{C}$-flat right R-modules F, F'.*

(7) *$\mathrm{Ext}^k_R(F, F') = 0$ for all $\mathscr{C}$-flat right R-modules F, F' and positive integers k.*

(8) *$\mathscr{C}\mathcal{CT}$-$D(R) < \infty$ and all right R-modules in $\mathscr{C}\mathscr{F} \cap \mathscr{C}\mathcal{CT}$ are projective.*

(9) *$\mathscr{C}\mathcal{CT}$-$D(R) < \infty$ and all projective right R-modules are $\mathscr{C}$-cotorsion.*

(10) *Every $\mathscr{C}$-flat right R-module has a projective cover with the unique mapping property.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3), (2) $\Rightarrow$ (4), and (3) $\Rightarrow$ (10) are obvious.

(4) $\Rightarrow$ (2). Let M be a right R-module. Then by (4), M has a $\mathscr{C}$-cotorsion envelope $\varphi : F \to \mathscr{C}\mathcal{CT}(F)$ with the unique mapping property. So, by the proof of Corollary 1(2), we get an exact sequence $0 \to M \xrightarrow{\varphi} \mathscr{C}\mathcal{CT}(M) \xrightarrow{\pi} F \to 0$, where F is $\mathscr{C}$-flat. Let $i : F \to \mathscr{C}\mathcal{CT}(F)$ be a $\mathscr{C}$-cotorsion envelope of F,

then i is monic by Proposition 2(2). Thus, we have $0 = i\pi\varphi : M \to \mathscr{C}\mathcal{CT}(F)$, $i\pi : \mathscr{C}\mathcal{CT}(M) \to \mathscr{C}\mathcal{CT}(F)$ such that $0 = (i\pi)\varphi$. But $0 = 0\varphi$, by the unique mapping property, we have $i\pi = 0$. It follows that $\pi = 0$, which implies that φ is an isomorphism, and thus M is $\mathscr{C}$-cotorsion.

If R is a strongly $\mathscr{C}$-coherent ring, then by Corollary 5 and the above proof, the equivalence of (1)-(9) are clear.

(10) $\Rightarrow$ (3). Let F be a $\mathscr{C}$-flat right R-module. Then by (10), F has a projective cover $\alpha : P \to F$ with the unique mapping property. By Proposition 2(1), α is epic, so we get an exact sequence $0 \to K \overset{\beta}{\to} P \overset{\alpha}{\to} F \to 0$, where $K = \mathrm{Ker}(\alpha)$. Since R is strongly $\mathscr{C}$-coherent, by Proposition 3, $(\mathscr{C}\mathscr{F}, \mathscr{C}\mathcal{CT})$ is a hereditary cotorsion pair, and so K is $\mathscr{C}$-flat. Thus, K has a epic projective cover $\gamma : P' \to K$ with the unique mapping property. Now, we have $0 = \alpha\beta\gamma : P' \to F$, $\beta\gamma : P' \to P$ such that $0 = \alpha(\beta\gamma)$. But $0 = \alpha 0$, by the unique mapping property, we have $\beta\gamma = 0$. And so $\beta = 0$, which implies that α is an isomorphism, and therefore F is projective. $\square$

Recall that a ring R is right perfect in case each right R-module has a projective cover, it is well known that a ring R is right perfect if and only if every flat right R-module is projective. Let $\mathscr{C}$ be the class of all finitely presented left R-modules. Then by Corollary 6, we have the following results.

Corollary 7. *Let R be a ring. Then the following conditions are equivalent:*

(1) *R is right perfect.*

(2) *All right R-modules are cotorsion.*

(3) *Every right R-module has a cotorsion envelope with the unique mapping property.*

Moreover, if R is a left coherent ring, then the above conditions are equivalent to:

(4) *All flat right R-modules are cotorsion .*

(5) $\mathrm{Ext}^1_R(F, F') = 0$ *for all flat right R-modules F, F'.*

(6) $\mathrm{Ext}^k_R(F, F') = 0$ *for all flat right R-modules F, F' and positive integers k.*

(7) *$\mathcal{CT}$-$D(R) < \infty$ and all flat cotorsion right R-modules are projective.*

(8) *$\mathcal{CT}$-$D(R) < \infty$ and all projective right R-modules are cotorsion .*

(9) *Every flat right R-module has a projective cover with the unique mapping property.*

We note that the equivalence of (1), (2), (4) in Corollary 7 was shown in [13, Corollary 2. 2. 7].

Let A be a submodule of the right R-module B. Recall that A is said to be a pure submodule of B if for all left R-module M, the induced map $A \otimes_R M \to B \otimes_R M$ is monic, or equivalently, if for every finitely presented left R-module V, the induced map $A \otimes_R V \to B \otimes_R V$ is monic. We call A a $\mathscr{C}$-pure submodule of B if for all $C \in \mathscr{C}$, the induced map $A \otimes_R C \to B \otimes_R C$ is monic.

THEOREM 5. *Let R be a strongly $\mathscr{C}$-coherent ring. Then the following statements are equivalent:*

(1) $\mathscr{C}\mathcal{CT}$-$D(R) \leq 1$.

(2) *All $\mathscr{C}$-flat right R-modules are of projective dimension ≤ 1.*

(3) *All $\mathscr{C}$-flat right R-modules are of $\mathscr{C}$-cotorsion dimension ≤ 1.*

(4) *Every quotient module of a $\mathscr{C}$-cotorsion right R-module is $\mathscr{C}$-cotorsion.*

(5) *Every quotient module of an injective right R-module is $\mathscr{C}$-cotorsion.*

(6) *Every $\mathscr{C}$-pure submodule of a projective right R-module is projective.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). It follows from Corollary 5.

(2) $\Rightarrow$ (4). Let E be any $\mathscr{C}$-cotorsion module and K a submodule of E. Then for any $\mathscr{C}$-flat right R-module F, by (2), we have $\mathrm{Ext}_R^2(F, E) = 0$. So, from the exact sequence $0 = \mathrm{Ext}_R^1(F, E) \to \mathrm{Ext}_R^1(F, E/K) \to \mathrm{Ext}_R^2(F, E) = 0$, we have $\mathrm{Ext}_R^1(F, E/K) = 0$, *i.e.*, E/K is $\mathscr{C}$-cotorsion.

(4) $\Rightarrow$ (5). It is obvious.

(5) $\Rightarrow$ (1). Let M be any right R-module and F be any $\mathscr{C}$-flat right R-module. By (5), $E(M)/M$ is $\mathscr{C}$-cotorsion, and so $\mathrm{Ext}_R^1(F, E(M)/M) = 0$. It follows that $\mathrm{Ext}_R^2(F, M) = 0$, and hence (1) holds.

(2) $\Rightarrow$ (6). Let P be a projective right R-module and K a $\mathscr{C}$-pure submodule of P. Then it is easy to see that P/K is $\mathscr{C}$-flat. By (2), $pd(P/K) \leq 1$, and so K is projective.

(6) $\Rightarrow$ (2). Let F be a $\mathscr{C}$-flat right R-module. Then there is an exact sequence $0 \to K \to P \to F \to 0$ with P projective. Since F is $\mathscr{C}$-flat, K is $\mathscr{C}$-pure in P. So K is projective by (6), and thus $pd(F) \leq 1$. □

Corollary 8. *Let R be a left coherent ring. Then the following statements are equivalent:*

(1) *All right R-modules are of cotorsion dimension ≤ 1.*

(2) *All flat right R-modules are of projective dimension ≤ 1.*

(3) *All flat right R-modules are of cotorsion dimension ≤ 1.*

(4) *Every quotient module of a cotorsion right R-module is cotorsion .*

(5) *Every quotient module of an injective right R-module is cotorsion .*

(6) *Every pure submodule of a projective right R-module is projective.*

The equivalence of (1) and (2) in the following theorem appeared in [23, Theorem 4.3], but we give it here for completeness and we give a new proof.

Theorem 6. *Let R be a ring and $\mathscr{C}$ be a class of some finitely presented left R-modules. Then following statements are equivalent:*

(1) *R is $\mathscr{C}$-semihereditary.*

(2) *Every right R-module has an epic $\mathscr{C}$-flat envelope.*

(3) *R is strongly $\mathscr{C}$-coherent and every $\mathscr{C}$-cotorsion right R-module has an epic $\mathscr{C}$-flat envelope.*

Proof. (1) $\Rightarrow$ (2). Let M be a right R-module and let $\{K_i\}_{i\in I}$ be the family of all submodules of M such that M/K_i is $\mathscr{C}$-flat. Let $F = M/\cap_{i\in I} K_i$ and π be the natural epimorphism of M to F. Define $\alpha : F \to \prod_{i\in I} M/K_i$ by $\alpha(m + \cap_{i\in I}K_i) = (m + K_i)$ for $m \in M$. Then α is a monomorphism. Since R is a $\mathscr{C}$-semihereditary ring, it is $\mathscr{C}$-coherent, so $\prod_{i\in I} M/K_i$ is $\mathscr{C}$-flat by [23, Theorem 3.3 (5)]. Moreover, by [23, Theorem 4.3 (2)], we have that every submodule of a $\mathscr{C}$-flat module is $\mathscr{C}$-flat, and so F is $\mathscr{C}$-flat. For any $\mathscr{C}$-flat right R-module F' and any homomorphism $f : M \to F'$. Since $M/Ker(f) \cong Im(f) \subseteq F'$, $M/Ker(f)$ is $\mathscr{C}$-flat, and thus $Ker(f) = K_j$ for some $j \in I$. Now we define $g : F \to F'; x + \cap_{i\in I}K_i \mapsto f(x)$, then g is a homomorphism such that $f = g\pi$. Thus, π is a $\mathscr{C}$-flat preenvelope of M. Note that epic preenvelope is an envelope, so $\pi : M \to F$ is an epic $\mathscr{C}$-flat envelope of M.

(2) $\Rightarrow$ (3). Assume (2), then we need only to prove that R is strongly $\mathscr{C}$-coherent. By (2), every right R-module has a $\mathscr{C}$-flat preenvelope, so, by [23, Theorem 3.2], R is $\mathscr{C}$-coherent. Now Let $0 \to L' \stackrel{\iota}{\to} L \to L'' \to 0$ be an exact sequence of right R-modules with $L, L'' \in \mathscr{C}\mathscr{F}$. Then L' has an epic $\mathscr{C}$-flat

envelope $\sigma : L' \to F$, so, there exists a homomorphism $\tau : F \to L$ such that $\iota = \tau\sigma$, which implies that σ is an isomorphism , and hence $L' \in \mathscr{C}\mathscr{F}$. It shows that $(\mathscr{C}\mathscr{F}, \mathscr{C}\mathcal{CT})$ is a hereditary cotorsion pair. Therefore, by Proposition 3, R is strongly $\mathscr{C}$-coherent.

(3) $\Rightarrow$ (1). Assume (3). Let $0 \to L \to P \to C \to 0$ be exact , where $C \in \mathscr{C}$, P is finitely generated projective. We will prove that L is projective. Let M be any right R-module. Since $(\mathscr{C}\mathscr{F}, \mathscr{C}\mathcal{CT})$ is a perfect cotorsion pair by [23, Theorem 2.10(2)], M has a $\mathscr{C}$-flat cover $f : F \to M$. Moreover, by Proposition 2, f is epic. Write $K = \mathrm{Ker}(f)$. Then by [18, Lemma 2.1.1], K is $\mathscr{C}$-cotorsion. By hypothesis, K has an epic $\mathscr{C}$-flat envelope $\phi : K \to F'$. Let $i : K \to F$ be the inclusion map. Then there exists a homomorphism $\alpha : F' \to F$ such that $i = \alpha\phi$. It follows that ϕ is an isomorphism , and so $K \cong F'$ is $\mathscr{C}$-flat. Note that R is strongly $\mathscr{C}$-coherent, by Proposition 3 and [23, Proposition 3.11], we have $\mathrm{Tor}_2(F, C) = 0$. Thus, by the following two exact sequences

$$0 = \mathrm{Tor}_2(M, P) \to \mathrm{Tor}_2(M, C) \to \mathrm{Tor}_1(M, L) \to \mathrm{Tor}_1(M, P) = 0$$

$$0 = \mathrm{Tor}_2(F, C) \to \mathrm{Tor}_2(M, C) \to \mathrm{Tor}_1(K, C) \to \mathrm{Tor}_1(F, C) = 0$$

we have $\mathrm{Tor}_1(M, L) \cong \mathrm{Tor}_2(M, C) \cong \mathrm{Tor}_1(K, C) = 0$, and so L is flat. Observing that R is $\mathscr{C}$-coherent, L is finitely presented, and hence it is projective, as required. □

Recall that: a ring R is called a *left Costa's n-coherent ring* [4] if every n-presented left R-module is $(n+1)$-presented; a ring R is called *left Lee n-coherent* [12](for integers $n > 0$ or $n = \infty$) if every finitely generated submodule of a free left R-module whose projective dimension is $\leq n-1$ is finitely presented; a ring R is called a *left n-hereditary ring* [22] if every $(n-1)$-presented submodule of a projective left R-module is projective. We call a ring R left n^*-hereditary if every finitely generated submodule of a projective left R-module whose projective dimension $\leq n-1$ is projective.

Corollary 9. *The following statements are equivalent for a ring R:*

(1) *R is n-hereditary.*

(2) *Every right R-module has an epic (n,0)-flat envelope.*

(3) *R is left Costa's n-coherent and every (n,0)-cotorsion right R-module has an epic flat envelope.*

Corollary 10. *The following statements are equivalent for a ring R:*

(1) *R is n^*-hereditary.*

(2) *Every right R-module has an epic n-flat envelope.*

(3) *R is left Lee's n-coherent and every n-cotorsion right R-module has an epic n-flat envelope.*

Corollary 11. *The following statements are equivalent for a ring R:*

(1) *R is semihereditary.*

(2) *Every right R-module has an epic flat envelope.*

(3) *R is left coherent and every cotorsion right R-module has an epic flat envelope.*

Proposition 5. *Let R be a $\mathscr{C}$-semihereditary ring. Then the class of $\mathscr{C}$-flat modules is closed under inverse limits.*

Proof. Let $\{F_i, \psi_i^j\}$ be an inverse system with index set I , where F_i is $\mathscr{C}$-flat. Since R is $\mathscr{C}$-semihereditary, $\varprojlim F_i$ has an epic $\mathscr{C}$-flat envelope $\sigma : \varprojlim F_i \to F$ by Theorem 6. Let $\alpha_i : \varprojlim F_i \to F_i$ be a family of morphisms of the inverse limit. Then we have $\alpha_i = \psi_i^j \alpha_j$. So there exists $f_i : F \to F_i$ such that $\alpha_i = f_i\sigma$ for each $i \in I$. Thus , we have $f_i\sigma = \psi_i^j f_j\sigma$ for any $i \leq j$, it follows that $f_i = \psi_i^j f_j$ for any $i \leq j$ because σ is epic. By the definition of inverse limits, there exists a morphism $\tau : F \to \varprojlim F_i$ such that $f_i = \alpha_i\tau$. Thus, $\alpha_i(\tau\sigma) = (\alpha_i\tau)\sigma = f_i\sigma = \alpha_i$, and so $\tau\sigma = 1_{\varprojlim F_i}$ by the definition of inverse limits. Consequently , we have $\varprojlim F_i \overset{\sigma}{\cong} F$ is $\mathscr{C}$-flat. □

Recall that a ring R is called a *left PP ring* [11] if every left ideal is projective. A ring R is called a *left PS ring* [15] if every minimal left ideal is projective.

Corollary 12. (1) *If R is a left semihereditary ring, then the class of flat right R-modules is closed under inverse limits.*

(2) [9, Corollary 3.10] *If R is a left PP ring, then the class of P-flat right R-modules is closed under inverse limits.*

(3) *If R is a left PS ring, then the class of min-flat right R-modules is closed under inverse limits.*

(4) *If R is a left n-hereditary ring, then the class of (n,0)-flat right R-modules is closed under inverse limits.*

(5) *If R is a left n^*-hereditary ring, then the class of n-flat right R-modules is closed under inverse limits.*

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